STUDIA MATHEMATICA, T. XCIV (1989)

Hausdorff dimension for piecewise monotonic maps

by

PETER RAITH (Wien)

Abstract. In this paper certain closed T-invariant subsets R of $[0, 1]$, where T is an expanding piecewise monotonic map on the interval, are considered. It is shown that the Hausdorff dimension of R is equal to the unique zero of $t \mapsto p(R, T, -t \log|T'|)$, where p denotes the pressure.

Introduction. We consider piecewise monotonic maps on the interval, i.e. $T: [0, 1] \to [0, 1]$ is a map and there exists a finite partition $\mathscr{Z} = \{Z_1, \ldots, Z_n\}$ of $[0, 1]$ into disjoint intervals such that $T|_{Z_j}$ is continuous and strictly monotone. Furthermore, we assume that T is differentiable in the interior of Z_j and that T' can be extended to a continuous function on the closure of Z_j. We also suppose that there exists some $n \in N$ with $\inf_{x \in [0,1]} |(T^n)'(x)| > 1$. Actually, we shall consider a bit more general situation.

We want to calculate the Hausdorff dimension $\mathrm{HD}_m(R)$ of perfect T-invariant subsets R of $[0, 1]$ such that $T(Z \cap R)$ is an interval in R for every $Z \in \mathscr{Z}$. The main theorem (Theorem 2) shows that $\mathrm{HD}_m(R)$ is equal to the unique zero t_R of the function $t \mapsto p(R, T, -t \log|T'|)$ defined on $\{x \in \boldsymbol{R};\ x \geqslant 0\}$, where $p(\cdot, \cdot, \cdot)$ denotes the topological pressure.

By Theorem 11 of [3] the centre of $([0, 1], T)$ can be written as $\bigcup_{\mathscr{C} \in \Gamma} L(\mathscr{C}) \cup L_\infty$, where Γ is at most countable, $L(\mathscr{C})$ and L_∞ are closed T-invariant subsets of $[0, 1]$, the sets $L(\mathscr{C})$ are topologically transitive, the topological entropy of L_∞ is zero, and the intersection of two different sets $L(\mathscr{C})$ or of some $L(\mathscr{C})$ and L_∞ is finite. If $R = L(\mathscr{C})$, then R satisfies the requirements of Theorem 2. Hence we have $\mathrm{HD}_m(R) = t_R$. If $R = L_\infty$, then Theorem 1 gives $\mathrm{HD}_m(R) = t_R = 0$.

Such a result was shown by H. McCluskey and A. Manning in [6] for a basic set of an axiom A diffeomorphism of a surface intersected with the unstable manifold of a point. In the case of a Markov map of the interval the same proof works, but it becomes simpler than in [5] and [6] (cf. also [2]). The proof of the formula for a general piecewise monotonic map T essentially relies on an approximation of T by Markov maps.

1980 *Mathematics Subject Classification* (1985 *Revision*): 28D99, 54H20.

The author would like to thank Franz Hofbauer for useful conversations and constant encouragement. Furthermore, he wishes to thank the referee for useful suggestions, in particular for those which shortened the proof of Lemma 8 essentially.

1. An upper bound for the Hausdorff dimension. In this section we prove that for any closed T-invariant subset $R \subseteq [0, 1]$ with no isolated points there exists a unique nonnegative zero of $t \mapsto p(R, T, -t\log|T'|)$, which is an upper bound for the Hausdorff dimension of R. It is easier to consider a generalization of our systems.

We introduce *piecewise monotonic systems* $(X, T, \mathscr{Z})$ (we shall use the abbreviation p.m.s.): X is a totally ordered set which is order complete and its order topology is metrisable (therefore X is a compact metric space), $\mathscr{Z}$ is a finite partition of X into disjoint intervals, T: $X \to X$ has the property that $T|_Z$ is continuous and strictly monotone and TZ is again an interval for all $Z \in \mathscr{Z}$ (hence $T|_Z$: $Z \to TZ$ is a homeomorphism). If X is connected, then the other assumptions imply that TZ is an interval for all $Z \in \mathscr{Z}$.

One gets an example of a p.m.s. by taking $X = [0, 1]$ with $\mathscr{Z}$ a finite partition of $[0, 1]$ into intervals and $T|_Z$ continuous and strictly monotone (that TZ is an interval follows from the intermediate value theorem).

Our aim is to compute the Hausdorff dimension of perfect T-invariant subsets $R \subseteq X$ such that $(R, T, \mathscr{Z}(R))$ is again a p.m.s., where $\mathscr{Z}(R) := \{Z \cap R;\ Z \in \mathscr{Z},\ Z \cap R \neq \emptyset\}$ and $(R, T, \mathscr{Z}(R))$ is an abbreviation for $(R, T|_R, \mathscr{Z}(R))$.

If $Y \subseteq X$ is T-invariant, f: $Y \to \mathbf{C}$ is a function and $n \in \mathbf{N}$, then define $S_n f := \sum_{k=0}^{n-1} f \cdot T^k$. For $n \in \mathbf{N}_0$ define

$$\mathscr{Z}_n := \{Z = Z_0 \cap T^{-1} Z_1 \cap \ldots \cap T^{-n} Z_n;\ Z_0, Z_1, \ldots, Z_n \in \mathscr{Z} \text{ and } Z \neq \emptyset\}.$$

$\mathscr{Z}$ is called a *generator* if for every sequence $V_0, V_1, \ldots$ with $V_j \in \mathscr{Z}_j$ the set $\bigcap_{j=0}^{\infty} V_j$ contains at most one point.

We call $(X, T, \mathscr{Z}, m, \varphi)$ an *expanding system* if $(X, T, \mathscr{Z})$ is a p.m.s., if $\mathscr{Z}$ is a generator, if m is a Borel probability measure on X with support X, if φ: $X \to \mathbf{R}$ has the property that $\varphi|_Z$ can be extended to a continuous function on the closure of Z for all $Z \in \mathscr{Z}$ (we call this property *piecewise continuity*), and satisfies

$$\varphi = \log\Big(\sum_{Z \in \mathscr{Z}} \frac{d(m|_Z)}{d(m \cdot T|_Z)} 1_Z\Big), \qquad \sup_{x \in X} S_n \varphi(x) < 0 \quad \text{for some } n \in \mathbf{N}.$$

For an expanding system we define the *Perron–Frobenius operator* P: $\mathscr{L}^\infty \to \mathscr{L}^\infty$ by

$$Pf(x) := \sum_{y \in T^{-1}x} e^{\varphi(y)} f(y), \tag{1.1}$$

where $\mathscr{L}^\infty$ is the set of all bounded, Borel measurable, complex-valued functions on X. The property

$$\varphi = \log\Big(\sum_{Z \in \mathscr{Z}} \frac{d(m|_Z)}{d(m \cdot T|_Z)} 1_Z\Big)$$

is equivalent to $m(Pf) = m(f)$ for every $f \in \mathscr{L}^\infty$, where $m(f) := \int_X f\,dm$ for $f \in \mathscr{L}^\infty$. If X is connected, then the other assumptions imply that $\mathscr{Z}$ is a generator.

The standard well-known example of an expanding system can be obtained in the following way. Let $X = [0, 1]$, and let $\mathscr{Z}$ be a finite partition of $[0, 1]$ into intervals such that $T|_Z$ is continuous and strictly monotone. We assume that T is differentiable in the interior of Z. Now we can take the Lebesgue measure for m and $-\log|T'|$ for φ. If φ is piecewise continuous and $\inf_{x \in [0,1]} |(T^n)'(x)| > 1$ for some n, then $(X, T, \mathscr{Z}, m, \varphi)$ is an expanding system. This also explains the name "expanding".

Now we show some properties of expanding systems which will be useful in the sequel.

LEMMA 1. *Let $(X, T, \mathscr{Z}, m, \varphi)$ be an expanding system.*

(i) $-\infty < \lim_{n\to\infty} n^{-1} \sup_{x \in X} S_n \varphi(x) = \inf_{n \in \mathbf{N}} n^{-1} \sup_{x \in X} S_n \varphi(x) < 0$.

(ii) *If $n \in \mathbf{N}$, $Z \in \mathscr{Z}_{n-1}$ and $A \subseteq Z$ is a Borel set, then*

$$\big(\inf_{x \in A} \exp S_n \varphi(x)\big) m(T^n A) \leqslant m(A) \leqslant \big(\sup_{x \in A} \exp S_n \varphi(x)\big) m(T^n A).$$

(iii) *For every $\varepsilon > 0$ there exists an n_0 such that for every integer $n \geqslant n_0$ and $Z \in \mathscr{Z}_n$ one has $m(Z) < \varepsilon$.*

(iv) *$m(\{x\}) = 0$ for all $x \in X$.*

Proof. (i) Since $\sup_{x \in X} S_{n+k} \varphi(x) \leqslant \sup_{x \in X} S_n \varphi(x) + \sup_{x \in X} S_k \varphi(x)$, it follows from Theorem 4.9 of [11] that $\lim_{n\to\infty} n^{-1} \sup_{x \in X} S_n \varphi(x)$ exists and equals $\inf_{n \in \mathbf{N}} n^{-1} \sup_{x \in X} S_n \varphi(x)$. Now the other assertions follow, since φ is bounded and there exists an n with $\sup_{x \in X} S_n \varphi(x) < 0$.

(ii)

$$m(A) = m(1_A) = m(P^n 1_A) = \sum_{\tilde{Z} \in \mathscr{Z}_{n-1}} \int_{T^n \tilde{Z}} \big((1_A \exp S_n \varphi) \cdot (T^n|_{\tilde{Z}})^{-1}\big)\,dm$$

$$= \int_{T^n Z} \big((\exp S_n \varphi) \cdot (T^n|_Z)^{-1}\big) 1_{T^n A}\,dm = \int_{T^n A} \big((\exp S_n \varphi) \cdot (T^n|_Z)^{-1}\big)\,dm.$$

Now the assertion follows, since $(T^n|_Z)^{-1} x \in A\ \ \forall x \in T^n A$.

(iii) From (ii) it follows that

$$m(Z) \leqslant \sup_{x \in X} \exp S_{n+1} \varphi(x) \qquad \forall Z \in \mathscr{Z}_n.$$

Now (i) gives the desired result.

(iv) follows from (iii) and the fact that $\mathscr{Z}_n$ is a partition of X. ∎

A p.m.s. $(X, T, \mathscr{Z})$ is called *complete* if $\mathscr{Z}$ consists of intervals which are both open and closed subsets of X. Observe that $T\colon X \to X$ is then continuous. Hence for every integer $n \geqslant 0$, $\mathscr{Z}_n$ consists of intervals which are both open and closed. If $(X, T, \mathscr{Z}, m, \varphi)$ is a complete expanding system, then φ is continuous.

We shall introduce a notion similar to the usual notion of Hausdorff dimension. This notion behaves better than the usual one under some formal operations like completion of p.m.s. And, what is really important, it coincides with the usual notion of Hausdorff dimension for subsets of $[0, 1]$.

Let $Y \subseteq X$. For $t \geqslant 0$ and $\varepsilon > 0$ define

$$m(Y, t, \varepsilon) := \inf\{\sum_{A \in \mathscr{A}} m(A)^t;\ \mathscr{A} \text{ is an at most countable cover of } Y \text{ by intervals with } m(A) < \varepsilon \text{ for all } A \in \mathscr{A}\}.$$

Clearly $\varepsilon \mapsto m(Y, t, \varepsilon)$ is decreasing. Set

$$m(Y, t) := \sup_{\varepsilon > 0} m(Y, t, \varepsilon) = \lim_{\varepsilon \to 0} m(Y, t, \varepsilon).$$

This is decreasing in t and $\inf\{t \geqslant 0;\ m(Y, t) = 0\} = \sup\{t \geqslant 0;\ m(Y, t) = \infty\}$. Then

$$\mathrm{HD}_m(Y) := \inf\{t \geqslant 0;\ m(Y, t) = 0\} = \sup\{t \geqslant 0;\ m(Y, t) = \infty\}$$

is called the *Hausdorff dimension* of Y with respect to the measure m. One easily sees that $0 \leqslant \mathrm{HD}_m(Y) \leqslant 1$ for every $Y \subseteq X$. Since $m(Y) \leqslant \sum_{A \in \mathscr{A}} m(A)$ for every measurable cover of Y which is at most countable, we have $\mathrm{HD}_m(Y) = 1$ for every Borel set $Y \subseteq X$ with $m(Y) > 0$. Thus Hausdorff dimension is a measure of size useful to distinguish between m-nullsets. HD_m behaves formally like usual Hausdorff dimension. In particular, the same proofs show that $\mathrm{HD}_m(Y_1) \leqslant \mathrm{HD}_m(Y_2)$ if $Y_1 \subseteq Y_2 \subseteq X$, and $\mathrm{HD}_m(\bigcup_{n \in N} Y_n) = \sup_{n \in N} \mathrm{HD}_m(Y_n)$ for every sequence $(Y_n)_{n \in N}$ of subsets of X.

A *topological dynamical system* (X, T) is a continuous map T of a compact metric space X into itself. The *pressure* $p(X, T, f)$ is defined for a continuous function $f\colon X \to \boldsymbol{R}$ by

$$p(X, T, f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} n^{-1} \log \sup_E \sum_{x \in E} \exp(S_n f(x)),$$

where the supremum is taken over all (n, ε)-separated subsets E of X. $E \subseteq X$ is called *(n, ε)-separated* if for any $x \neq y \in E$ there exists some $j \in \{0, 1, \ldots, n-1\}$ with $d(T^j x, T^j y) > \varepsilon$ (cf. § 9.1 of [11] for this and some alternative definitions).

Denote the set of all T-invariant Borel probability measures by $M(X, T)$ and the set of all ergodic T-invariant Borel probability measures by $E(X, T)$. The *variational principle* (see e.g. Theorem 9.10 of [11]) states that

$$p(X, T, f) = \sup_{\mu \in M(X,T)} (h_\mu(X, T) + \mu(f)) = \sup_{\mu \in E(X,T)} (h_\mu(X, T) + \mu(f)),$$

where $h_\mu(X, T)$ denotes the measure-theoretic entropy of (X, T, μ) (see e.g. § 4.4 of [11] for definition). We have $p(X, T, 0) = h_{\mathrm{top}}(X, T)$, where $h_{\mathrm{top}}(X, T)$ is the topological entropy of (X, T) (see e.g. § 7.1 and § 7.2 of [11] for definition).

Unfortunately, piecewise monotonic transformations are not continuous in general. If $(X, T, \mathscr{Z}, m, \varphi)$ is a complete expanding system, and if R is a closed T-invariant subset of X, then $T|_R\colon R \to R$ is continuous. Therefore (R, T) is a topological dynamical system ((R, T) is an abbreviation for $(R, T|_R)$). Since $\mathscr{Z}(R)$ is a generator, we have for every continuous $f\colon R \to \boldsymbol{R}$

$$p(R, T, f) = \lim_{n \to \infty} n^{-1} \log \sum_{Z \in \mathscr{Z}_{n-1}(R)} \sup_{x \in Z} \exp(S_n f(x)), \tag{1.2}$$

where $\mathscr{Z}_n(R) := \{Z \cap R;\ Z \in \mathscr{Z}_n,\ Z \cap R \neq \emptyset\}$ (cf. Theorem 9.6 of [11]).

If $(X, T, \mathscr{Z}, m, \varphi)$ is an expanding system which is not complete, then $T\colon X \to X$ need not be continuous. In order to define the pressure we extend $(X, T, \mathscr{Z}, m, \varphi)$ to a complete expanding system.

Let $Y \subseteq X$. $x \in Y$ is called an *isolated point* of Y if there exists an open $U \subseteq X$ with $U \cap Y = \{x\}$. Y is called *perfect* if Y is closed and contains no isolated points.

Let $(X, T, \mathscr{Z})$ be a p.m.s. and suppose that X contains no isolated points. Now we shall define the completion of $(X, T, \mathscr{Z})$. This construction is always used when one wants to make a p.m.s. into a topological dynamical system (cf. Walters, Hofbauer and Keller). Set

$$D := \{x \in X;\ \exists\, y \in X \text{ with either } x < y \text{ and } (x, y) = \emptyset \text{ or } y < x \text{ and } (y, x) = \emptyset\} \cup \{\inf X, \sup X\}.$$

Now define

$$E := \{\inf Z, \sup Z;\ Z \in \mathscr{Z}\}, \qquad W := \Big(\bigcup_{k=0}^{\infty} T^{-k} E\Big) \setminus D.$$

W is at most countable. Set

$$\hat{X} := (X \setminus W) \cup \{x^-, x^+;\ x \in W\}.$$

For $x \in X \setminus W$ set $x^- = x^+ = x$ and define $x^+ < y^-$ if $x < y$ in X, and $x^- < x^+$ if $x \in W$. The transformation $T|_{X \setminus (W \cup E)}$ can be uniquely extended to a continuous map $\hat{T}\colon \hat{X} \to \hat{X}$. Set

$$\hat{E} := \{x^-, x^+;\ x \in E\} \cup \{\tilde{x};\ x \in (E \cap D) \setminus \{\inf X, \sup X\}\},$$

where for $x \in D \setminus \{\inf X, \sup X\}$, $\tilde{x}$ is the unique element with either $x < \tilde{x}$ and $(x, \tilde{x}) = \emptyset$ or $\tilde{x} < x$ and $(\tilde{x}, x) = \emptyset$. Then there exists some $k \in N$ with $\hat{E} = \{x_1, \ldots, x_{2k}\}$, where $x_1 < \ldots < x_{2k}$, $(x_{2j}, x_{2j+1}) = \emptyset$ for $j = 1, \ldots, k-1$, and $(x_{2j-1}, x_{2j}) \neq \emptyset$ for $j=1, \ldots, k$. Set

$$\hat{\mathscr{Z}} := \{[x_{2j-1}, x_{2j}];\ j=1, \ldots, k\}.$$

$(\hat{X}, \hat{T}, \hat{\mathscr{Z}})$ is a complete p.m.s. If $\mathscr{Z}$ is a generator, then $\hat{\mathscr{Z}}$ is a generator.

If A is a closed subset of X, then let $\hat{A}$ be the closure of $A \setminus W$ in $\hat{X}$. If A is a closed subset of X, and $f\colon A \to \boldsymbol{R}$ is a piecewise continuous function, then there exists a unique continuous function $\hat{f}\colon (A \setminus E)^{\wedge} \to \boldsymbol{R}$ with $\hat{f}|_{A\setminus(W \cup E)} = f|_{A\setminus(W \cup E)}$. $(\hat{X}, \hat{T}, \hat{\mathscr{Z}})$ is called the *completion* of $(X, T, \mathscr{Z})$, $\hat{A}$ and $\hat{f}$ are called the *extensions* of A and f to $(\hat{X}, \hat{T}, \hat{\mathscr{Z}})$. Observe that the completion depends on the partition $\mathscr{Z}$.

Let R be a perfect T-invariant subset of X. Then $\hat{R}$ is a perfect $\hat{T}$-invariant subset of $\hat{X}$. Hence $(\hat{R}, \hat{T})$ is a topological dynamical system. If $f\colon R \to \boldsymbol{R}$ is a piecewise continuous function, then $\hat{f}\colon \hat{R} \to \boldsymbol{R}$ is a continuous function (in this case $(R \setminus E)^{\wedge} = \hat{R}$). Now we define $p(R, T, f) := p(\hat{R}, \hat{T}, \hat{f})$. We shall need the following lemma, which shows that $p(R, T, f)$ does not depend on the partition $\mathscr{Z}$.

LEMMA 2. *Suppose that $(X, T, \mathscr{Z})$ is a p.m.s. and that X contains no isolated points. Let R be a perfect T-invariant subset of X, let $f\colon R \to \boldsymbol{R}$ be a piecewise continuous function, and let $\mathscr{Y}$ be a finite partition of X into intervals which refines $\mathscr{Z}$. If $(\tilde{X}, \tilde{T}, \tilde{\mathscr{Y}})$ is the completion of the p.m.s. $(X, T, \mathscr{Y})$, and if $\tilde{R}$ and $\tilde{f}$ are the extensions of R and f to $(\tilde{X}, \tilde{T}, \tilde{\mathscr{Y}})$, then $p(\tilde{R}, \tilde{T}, \tilde{f}) = p(R, T, f)$.*

Proof. By the definition of $p(R, T, f)$ we can assume that $(X, T, \mathscr{Z})$ is complete and that $\mathscr{Y}$ is a subpartition of $\mathscr{Z}$. Now the lemma follows immediately from the variational principle. ∎

Let $(X, T, \mathscr{Z}, m, \varphi)$ be an expanding system. Let $(\hat{X}, \hat{T}, \hat{\mathscr{Z}})$ be the completion of $(X, T, \mathscr{Z})$ and let $\hat{\varphi}$ be the extension of φ to $(\hat{X}, \hat{T}, \hat{\mathscr{Z}})$. For a Borel set $A \subseteq \hat{X}$ set $\hat{m}(A) := m(A \cap (X \setminus W))$. Since $X \setminus (\hat{X} \cap X)$ is at most countable, we see by Lemma 1 (iv) that $\hat{m}$ is a Borel probability measure on $\hat{X}$ with support $\hat{X}$. Clearly there exists an n with $\sup_{x\in\hat{X}} S_n \hat{\varphi}(x) < 0$. Hence using also the fact that $\hat{m}$ is concentrated on $\hat{X} \cap X$ we find that $(\hat{X}, \hat{T}, \hat{\mathscr{Z}}, \hat{m}, \hat{\varphi})$ is a complete expanding system. We call it the *completion* of $(X, T, \mathscr{Z}, m, \varphi)$.

If A is a closed subset of X, then $A \setminus (\hat{X} \cap X)$ and $\hat{A} \setminus (\hat{X} \cap X)$ are at most countable. Since $A \cap (\hat{X} \cap X) = \hat{A} \cap (\hat{X} \cap X)$, Lemma 1 (iv) shows that $\mathrm{HD}_m(A) = \mathrm{HD}_{\hat{m}}(\hat{A})$. Hence the Hausdorff dimension does not change if one completes a set.

Now we can show the following lemma by adapting the proof given in [2] to our situation (a similar proof is given in [6]).

LEMMA 3. *Suppose that $(X, T, \mathscr{Z}, m, \varphi)$ is an expanding system and that R is a perfect T-invariant subset of X. Then the function $t \mapsto p(R, T, t\varphi)$ defined on $\boldsymbol{R}$ is continuous and strictly decreasing, and has a unique nonnegative zero t_R.*

Proof. By the definition of the pressure we can assume that $(X, T, \mathscr{Z}, m, \varphi)$ is complete, and $R \subseteq X$ is closed and T-invariant. The map $t \mapsto p(R, T, t\varphi)$ is continuous by Theorem 9.7 (iv) of [11]. Since $(X, T, \mathscr{Z}, m, \varphi)$ is an expanding system there exists an n with $\sup_{x\in X} S_n \varphi(x) < 0$. Define

$$r := -n^{-1} \sup_{x\in X} S_n \varphi(x) > 0. \tag{1.3}$$

Then we have for every $\mu \in M(R, T)$ and $t \geqslant 0$

$$\mu(t\varphi) = \frac{t}{n} \mu(S_n \varphi) \leqslant -tr.$$

By (1.3) and the variational principle this gives

$$p(R, T, t_2\varphi) \leqslant p(R, T, t_1\varphi) - (t_2 - t_1)r < p(R, T, t_1\varphi)$$

if $t_1 < t_2$. Using $p(R, T, 0) = h_{\mathrm{top}}(R, T) < \infty$ and (1.3) now gives $\lim_{t\to\infty} p(R, T, t\varphi) = -\infty$. Since $h_{\mathrm{top}}(R, T) \geqslant 0$ there exists a unique $t_R \geqslant 0$ with $p(R, T, t_R\varphi) = 0$. ∎

Let $(X, T, \mathscr{Z}, m, \varphi)$ be an expanding system, and let R be a closed T-invariant subset of X. If R is topologically transitive (i.e. R is the ω-limit set of some element of R), then either R contains no isolated points or R is a periodic orbit. If R is a periodic orbit, then $\mathrm{HD}_m(R) = t_R = 0$. Hence the conclusion of Theorem 1 is valid for a topologically transitive set R, and the conclusion of Theorem 2 remains valid if R is topologically transitive and $(R, T, \mathscr{Z}(R))$ is a p.m.s.

The next lemma will be useful in different places of this paper.

LEMMA 4. *Let $(X, T, \mathscr{Z}, m, \varphi)$ be an expanding system. Then for every $\varepsilon > 0$ there exists a finite partition $\mathscr{Y}_\varepsilon$ of X into intervals, which refines $\mathscr{Z}$, such that*

$$\sup_{x,y\in Y} |\varphi(x) - \varphi(y)| < \varepsilon \quad \textit{for all } Y \in \mathscr{Y}_\varepsilon.$$

Proof. This is an easy consequence of the piecewise continuity of φ and the compactness of X. ∎

Now we prove the main theorem of this section. The proof is similar to the proof given in [6]. As we consider a one-dimensional map we can use the fact that $\mathscr{Z}$ is a generator. Therefore our proof is a bit simpler than the proof in [6].

THEOREM 1. *Let $(X, T, \mathscr{Z}, m, \varphi)$ be an expanding system. If R is a perfect T-invariant subset of X, then* $\mathrm{HD}_m(R) \leqslant t_R$.

Proof. Let $t > t_R$. Then $p(R, T, t\varphi) < 0$ by Lemma 3. Choose $\eta > 0$ small enough that

$$p(R, T, t\varphi) + t\eta < 0. \tag{1.4}$$

Set $\mathscr{Y} := \mathscr{Y}_\eta$ as in Lemma 4. Let $(\hat{X}, \hat{T}, \hat{\mathscr{Y}}, \hat{m}, \hat{\varphi})$ be the completion of $(X, T, \mathscr{Y}, m, \varphi)$, and let $\hat{R}$ be the extension of R to $(\hat{X}, \hat{T}, \hat{\mathscr{Y}})$. By (1.2), (1.4) and Lemma 2 there exists a $\delta > 0$ such that

$$t\eta + n^{-1} \log \sum_{Y \in \hat{\mathscr{Y}}_{n-1}(\hat{R})} \sup_{x \in Y} \exp\left(tS_n \hat{\varphi}(x)\right) \leqslant -\delta \tag{1.5}$$

for n large enough. Set $\mathscr{A}_n := \{Y \in \hat{\mathscr{Y}}_n;\ Y \cap \hat{R} \neq \emptyset\}$. Then $Y \in \mathscr{A}_n$ if and only if $Y \cap \hat{R} \in \hat{\mathscr{Y}}_n(\hat{R})$. Since $\mathscr{Y}$ was chosen as in Lemma 4 we have

$$\sup_{x \in Y} \exp\left(tS_n \hat{\varphi}(x)\right) \leqslant \exp(nt\eta) \sup_{x \in Y \cap \hat{R}} \exp\left(tS_n \hat{\varphi}(x)\right) \tag{1.6}$$

for $Y \in \mathscr{A}_{n-1}$.

Let $\varepsilon > 0$. Then by Lemma 1 (iii), $\hat{m}(Y) < \varepsilon$ $\forall Y \in \mathscr{A}_{n-1}$ if n is large enough. Since $\hat{m}(\hat{T}^n Y) \leqslant 1$ it follows from the definition of the Hausdorff dimension, from Lemma 1 (ii), and from (1.6) and (1.5) that

$$\begin{aligned}\hat{m}(\hat{R}, t, \varepsilon) &\leqslant \sum_{Y \in \mathscr{A}_{n-1}} \hat{m}(Y)^t \leqslant \sum_{Y \in \mathscr{A}_{n-1}} \sup_{x \in Y} \exp\left(tS_n \hat{\varphi}(x)\right) \\ &\leqslant \exp(nt\eta) \sum_{Y \in \hat{\mathscr{Y}}_{n-1}(\hat{R})} \sup_{x \in Y} \exp\left(tS_n \hat{\varphi}(x)\right) \leqslant e^{-n\delta}\end{aligned}$$

if n is large enough. Now $n \to \infty$ gives $\hat{m}(\hat{R}, t, \varepsilon) = 0$, and $\varepsilon \to 0$ gives $\hat{m}(\hat{R}, t) = 0$. As $t > t_R$ was arbitrary, we get $\mathrm{HD}_m(R) = \mathrm{HD}_{\hat{m}}(\hat{R}) \leqslant t_R$. ∎

2. A formula for the Hausdorff dimension. In this section we show that for every perfect T-invariant subset $R \subseteq X$ with the property that $(R, T, \mathscr{Z}(R))$ is a p.m.s., the unique nonnegative real zero t_R of $t \mapsto p(R, T, t\varphi)$ is equal to the Hausdorff dimension $\mathrm{HD}_m(R)$ of R (hence $t_R \in [0, 1]$).

In order to prove the inequality $t_R \leqslant \mathrm{HD}_m(R)$ we need some facts obtained mainly by Hofbauer (see e.g. [3] and [4]). For the convenience of the reader we discuss some of them.

Let $(X, T, \mathscr{Z})$ be a complete p.m.s. and suppose that X contains no isolated points. We define an at most countable oriented graph $(\mathscr{D}, \to)$ for $(X, T, \mathscr{Z})$, called the *Markov diagram*, which describes the orbit structure of (X, T). Let $Z_0 \in \mathscr{Z}$ and let $D \neq \emptyset$ be a closed subinterval of Z_0 with no isolated points. A nonempty $C \subseteq X$ is called a *successor* of D if there is a $Z \in \mathscr{Z}$ with $C = TD \cap Z$, and we write $D \to C$. As TD is a closed interval with no isolated points, and as Z is a closed and open interval, each successor C of D is again a closed subinterval of some element of $\mathscr{Z}$ with no isolated points. Let $\mathscr{D}$ be the smallest set with $\mathscr{Z} \subseteq \mathscr{D}$ and such that $D \in \mathscr{D}$ and $D \to C$ imply $C \in \mathscr{D}$. Then $(\mathscr{D}, \to)$ is called the *Markov diagram* of $(X, T, \mathscr{Z})$ (cf. [3]). $\mathscr{D}$ is at most countable and its elements are closed intervals with no isolated points.

Let $f\colon X \to \boldsymbol{R}$ be a function with $f|_Z$ constant for all $Z \in \mathscr{Z}$. Since $(X, T, \mathscr{Z})$ is a complete p.m.s., f is continuous. As $f|_Z$ is constant, $f|_D$ is constant for every $D \in \mathscr{D}$, and we denote this constant by f_D. This gives rise to a $\mathscr{D} \times \mathscr{D}$-matrix $F(f) := (F_{CD})_{C \in \mathscr{D}, D \in \mathscr{D}}$ defined by

$$F_{CD} := \begin{cases}\exp f_C & \text{if } C \to D,\\ 0 & \text{otherwise.}\end{cases} \tag{2.1}$$

Since the number of successors of each $D \in \mathscr{D}$ is bounded by card $\mathscr{Z}$, we have $\sum_{D \in \mathscr{D}} F_{CD} \leqslant K$ for all $C \in \mathscr{D}$, where $K := (\operatorname{card} \mathscr{Z}) \|\exp f\|_\infty$.

If $\mathscr{C} \subseteq \mathscr{D}$, then denote by $F_{\mathscr{C}}(f)$ the $\mathscr{C} \times \mathscr{C}$-matrix $(F_{CD})_{C \in \mathscr{C}, D \in \mathscr{C}}$. Then for every $u \in l^1(\mathscr{C})$ we have

$$\|uF_{\mathscr{C}}(f)\|_1 = \sum_{D \in \mathscr{C}} \Big| \sum_{C \in \mathscr{C}} u_C F_{CD} \Big| \leqslant \sum_{C \in \mathscr{C}} |u_C| \sum_{D \in \mathscr{C}} F_{CD} \leqslant K \|u\|_1$$

and similarly $\|F_{\mathscr{C}}(f) v\|_\infty \leqslant K \|v\|_\infty$ for all $v \in l^\infty(\mathscr{C})$. Hence $u \mapsto uF_{\mathscr{C}}(f)$ is an $l^1(\mathscr{C})$-operator and $v \mapsto F_{\mathscr{C}}(f) v$ is an $l^\infty(\mathscr{C})$-operator. Denote by $\|F_{\mathscr{C}}(f)\|$ the norm of the former operator; it is equal to the norm of the latter and we have the formula

$$\|F_{\mathscr{C}}(f)\| = \sup_{C \in \mathscr{C}} \sum_{D \in \mathscr{C}} F_{CD}. \tag{2.2}$$

Hence

$$r(F_{\mathscr{C}}(f)) := \lim_{n \to \infty} \|F_{\mathscr{C}}(f)^n\|^{1/n} \tag{2.3}$$

is the spectral radius of both operators.

If $\mathscr{C} \subseteq \mathscr{D}$ and $C, D \in \mathscr{C}$, we say there exists a *path of length* n ($n \geqslant 2$) from C to D in $\mathscr{C}$ if there exist $C_1, \ldots, C_n \in \mathscr{C}$ with $C_1 = C$, $C_n = D$ and $C_j \to C_{j+1}$ for $j = 1, \ldots, n-1$.

A subset $\mathscr{C}$ of $\mathscr{D}$ is called *irreducible* if for any $C, D \in \mathscr{C}$ there exists a finite path from C to D in $\mathscr{C}$. $\mathscr{C}$ is called *maximal irreducible* if $\mathscr{C}$ is irreducible and if no $\mathscr{C}' \neq \mathscr{C}$ with $\mathscr{C} \subseteq \mathscr{C}' \subseteq \mathscr{D}$ is irreducible (in [3] the latter sets are called irreducible). $\mathscr{C}$ is called *finite irreducible* if $\mathscr{C}$ is irreducible and finite. Define

$$\Gamma := \{\mathscr{C} \subseteq \mathscr{D};\ \mathscr{C} \text{ is maximal irreducible}\}. \tag{2.4}$$

Suppose $\mathscr{C} \subseteq \mathscr{D}$. A sequence $C_0 C_1 \ldots$ is called an *infinite path* in $\mathscr{C}$ if $C_j \in \mathscr{C}$ and $C_j \to C_{j+1}$ for all $j \in N_0$. We say that an infinite path $C_0 C_1 \ldots$ in $\mathscr{C}$ *represents* $x \in X$ if $T^j x \in C_j$ for $j \geqslant 0$. Define

(2.5) $L(\mathscr{C}) := \{x \in X;\ \exists$ an infinite path in $\mathscr{C}$ which represents $x\}$.

LEMMA 5. *Suppose that $(X, T, \mathscr{Z})$ is a complete p.m.s. and that X contains no isolated points. Let $(\mathscr{D}, \to)$ be the Markov diagram of $(X, T, \mathscr{Z})$.*

(i) *$T(L(\mathscr{C})) \subseteq L(\mathscr{C})$ for every $\mathscr{C} \subseteq \mathscr{D}$.*

(ii) *Suppose that $\mathscr{C} \subseteq \mathscr{D}$ and $Z \in \mathscr{Z}_n$ for some $n \in N_0$ with $Z \cap L(\mathscr{C}) \neq \emptyset$. Then there exists a finite path $C_0 C_1 \dots C_n$ in $\mathscr{C}$ with $\bigcap_{j=0}^{n} T^{-j} C_j \subseteq Z$ and $C_n \subseteq T^n Z$.*

(iii) *If $\mathscr{C} \subseteq \mathscr{D}$ is finite, then $L(\mathscr{C})$ is closed.*

(iv) *If $\mathscr{Z}$ is a generator and if $\mathscr{C} \subseteq \mathscr{D}$ is maximal irreducible, then $L(\mathscr{C})$ is closed.*

Proof. (i) and (ii) are evident.

(iii) is shown by a standard diagonalization argument.

(iv) is proved in [3] (see I.§ 2 and Theorem 11 (i) in [3]). ■

The next lemma generalizes Theorem 7 of [3] and gives some approximation result (in the same way one can show that $p(X, T, f) = \log r(F(f))$).

LEMMA 6. *Let $(X, T, \mathscr{Z})$ be a complete piecewise monotonic system, and suppose that X contains no isolated points and that $\mathscr{Z}$ is a generator. Suppose that $f\colon X^* \to R$ is a function with $f|_Z$ constant for all $Z \in \mathscr{Z}$. Let $(\mathscr{D}, \to)$ be the Markov diagram of $(X, T, \mathscr{Z})$.*

(i) *If $\mathscr{C} \subseteq \mathscr{D}$ is maximal irreducible or finite irreducible, then $p(L(\mathscr{C}), T, f) = \log r(F_{\mathscr{C}}(f))$.*

(ii) *If $\mathscr{C} \subseteq \mathscr{D}$ is maximal irreducible and if*

$$r(F_{\mathscr{C}}(f)) > \lim_{n\to\infty} \|\exp S_n f\|_\infty^{1/n},$$

then for every $\varepsilon > 0$ there exists a finite irreducible $\mathscr{C}' \subseteq \mathscr{C}$ with

$$p(L(\mathscr{C}), T, f) - \varepsilon \leqslant p(L(\mathscr{C}'), T, f) \leqslant p(L(\mathscr{C}), T, f).$$

Proof. (i) The proof is analogous to the proof of Theorem 7 of [3]. We have only to substitute the notion "number of paths of length $k+1$ from some fixed C to some fixed D" by "$\sum \exp S_k f(x)$, where $x \in \bigcap_{j=0}^{k} T^{-j} C_j$ and the sum is taken over all paths $C_0 C_1 \dots C_k$ of length $k+1$ from C to D in $\mathscr{C}$". In the case of maximal irreducible $\mathscr{C}$ the corollary of Theorem 10 of [3] ensures the existence of the sets $\mathscr{F}_n$ needed in the proof.

(ii) By (i) and the continuity of log it remains to show that if the assumptions are satisfied and $\delta > 0$, then there exists a finite irreducible $\mathscr{C}' \subseteq \mathscr{C}$ with $r(F_{\mathscr{C}}(f)) - \delta \leqslant r(F_{\mathscr{C}'}(f)) \leqslant r(F_{\mathscr{C}}(f))$ (we also use the fact that log is increasing).

Define

$$\mathscr{D}_0 := \mathscr{Z}, \quad \mathscr{D}_n := \mathscr{D}_{n-1} \cup \{D \in \mathscr{D};\ \exists C \in \mathscr{D}_{n-1} \text{ with } C \to D\} \quad \text{if } n \geqslant 1.$$

Then $\mathscr{D}_0 \subseteq \mathscr{D}_1 \subseteq \dots$ and $\bigcup_{n=0}^{\infty} \mathscr{D}_n = \mathscr{D}$. Set $F_n(f) := F_{\mathscr{D}\setminus\mathscr{D}_n}(f)$. Then by Lemma 8 of [4] we have

$$\limsup_{n\to\infty} r(F_n(f)) \leqslant \lim_{n\to\infty} \|\exp S_n f\|_\infty^{1/n}.$$

Now the proof of Corollary 1 (ii) to Theorem 9 in [3] shows the existence of a nonzero nonnegative $u \in l^1(\mathscr{C})$ and a nonzero nonnegative $v \in l^\infty(\mathscr{C})$ with $uF_{\mathscr{C}}(f) = \lambda u$ and $F_{\mathscr{C}}(f) v = \lambda v$, where $\lambda = r(F_{\mathscr{C}}(f))$. By the irreducibility of $\mathscr{C}$ we have $u_C > 0$ and $v_C > 0$ for all $C \in \mathscr{C}$. Now we define for $C, D \in \mathscr{C}$

$$P_{CD} := F_{CD} v_D / \lambda v_C, \qquad \pi_C := u_C v_C.$$

Then P is an irreducible stochastic $\mathscr{C} \times \mathscr{C}$-matrix, $\pi \in l^1(\mathscr{C})$ and $\pi P = \pi$. Denote the entries of P^n by $P_{CD}^{(n)}$ and the entries of $F_{\mathscr{C}}(f)^n$ by $\tilde{F}_{CD}^{(n)}$. Since P is irreducible the number $R := \limsup_{n\to\infty} (P_{CD}^{(n)})^{1/n}$ is independent of C and D by Theorem 6.1 of [9]. As $\pi P = \pi$ and $\pi > 0$, we have a positive recurrent Markov chain, hence $P_{CC}^{(nd)} \to p_C$ for some $p_C > 0$ and some $d \in N$, and $P_{CC}^{(n)} = 0$ for every n with $d \nmid n$. Therefore $R = 1$. Since $\tilde{F}_{CD}^{(n)} = \lambda^n v_C P_{CD}^{(n)} / v_D$, we have for all $C, D \in \mathscr{C}$

$$\limsup_{n\to\infty} (\tilde{F}_{CD}^{(n)})^{1/n} = \lambda = r(F_{\mathscr{C}}(f)).$$

Now by Theorem 3 of [8] there is a sequence $\mathscr{C}_1 \subseteq \mathscr{C}_2 \subseteq \dots$ of finite irreducible subsets of $\mathscr{C}$ with $\bigcup_{n=1}^{\infty} \mathscr{C}_n = \mathscr{C}$ and $\lim_{n\to\infty} r(F_{\mathscr{C}_n}(f)) = r(F_{\mathscr{C}}(f))$, since the quantity R_n used in [8] equals $r(F_{\mathscr{C}_n}(f))^{-1}$ by the finiteness of $\mathscr{C}_n$. Now the lemma is proved, since $r(F_{\mathscr{C}'}(f)) \leqslant r(F_{\mathscr{C}}(f))$ for all $\mathscr{C}' \subseteq \mathscr{C}$. ■

Now we show a lemma which will be useful in the proof of Lemma 8.

LEMMA 7. *Suppose that $(X, T, \mathscr{Z}, m, \varphi)$ is a complete expanding system, and that $R \subseteq X$ is perfect, T-invariant and such that $(R, T, \mathscr{Z}(R))$ is a p.m.s. Let $\mathscr{C}$ be a finite subset of the Markov diagram of $(R, T, \mathscr{Z}(R))$. Then there is a $c > 0$ such that for every integer $n \geqslant 0$ and for every $Z \in \mathscr{Z}_n$ with $Z \cap L(\mathscr{C}) \neq \emptyset$ one has $c \leqslant m(T^n Z) \leqslant 1$.*

Proof. By the construction of the Markov diagram each $C \in \mathscr{C}$ has no isolated points, and hence C contains infinitely many elements. Therefore for every $C \in \mathscr{C}$ there exists a nonempty open set $U_C \subseteq X$ such that every interval $I \subseteq X$ with $C \subseteq I$ satisfies $U_C \subseteq I$. Define $c := \min\{m(U_C);\ C \in \mathscr{C}\}$. Since $\mathscr{C}$ is finite, the support of m is X, and each U_C is nonempty and open, we get $c > 0$. If $Z \in \mathscr{Z}_n$ with $Z \cap L(\mathscr{C}) \neq \emptyset$, then $Z \cap R \cap L(\mathscr{C}) \neq \emptyset$ and $Z \cap R \in \mathscr{Z}_n(R)$. Hence by Lemma 5(ii) there is a $C \in \mathscr{C}$ with $C \subseteq T^n(Z \cap R) \subseteq T^n Z$. Since $T^n Z$ is an interval, we have $U_C \subseteq T^n Z$ and therefore $c \leqslant m(T^n Z)$. ■

Let $(X, T, \mathscr{Z}, m, \varphi)$ be a complete expanding system. Let μ be an ergodic T-invariant Borel probability measure on X. By the ergodic theorem we have $\lim_{n\to\infty} n^{-1} S_n \varphi(x) = \mu(\varphi)$ for μ-a.e. x. We define

(2.6) $\chi_\mu := -\mu(\varphi)$, the *Lyapunov exponent* of μ.

By Lemma 1 (i) we have $\chi_\mu > 0$. Furthermore,

(2.7) $$h_\mu := h_\mu(X, T)$$

is the measure-theoretic entropy of (X, T, μ).

The next lemma is the basic step in proving that $t_R \leqslant \mathrm{HD}_m(R)$. Since we consider a one-dimensional map, our proof is in some sense simpler than the proof given in [5]. Instead of considerations on topological entropy for noncompact sets used in [5] we use the Shannon–McMillan–Breiman theorem.

LEMMA 8. *Let $(X, T, \mathscr{Z}, m, \varphi)$ be a complete expanding system and let $(R, T, \mathscr{Z}(R))$ be as in Lemma 7. Set*

$$s := \sup_{Z\in\mathscr{Z}} \sup_{x,y\in Z} |\varphi(x)-\varphi(y)|.$$

If $\mathscr{C}$ is a finite irreducible subset of the Markov diagram of $(R, T, \mathscr{Z}(R))$ and μ is an ergodic T-invariant Borel probability measure on $L(\mathscr{C})$, then

$$\mathrm{HD}_m(L(\mathscr{C})) \geqslant \frac{h_\mu}{\chi_\mu + s}.$$

Proof. If $h_\mu = 0$, then the lemma is obviously true.

Now suppose that $h_\mu > 0$. Fix an arbitrary $\delta \in (0, h_\mu)$. By the ergodic theorem and the Shannon–McMillan–Breiman theorem (see e.g. p. 93 of [11]) there exists a Borel set $A \subseteq L(\mathscr{C})$ with $\mu(A) > 0$ and an $n_0 \geqslant 0$ such that for every $n \geqslant n_0$

(2.8) $n^{-1} S_n \varphi(x) \geqslant -\chi_\mu - \delta$ if $x \in A$,

(2.9) $\mu(Z) \leqslant \exp(-n(h_\mu - \delta))$ if $Z \in \mathscr{Z}_n$ and $Z \cap A \neq \emptyset$.

Now let $\varepsilon_0 > 0$ be so small that every interval $I \subseteq X$ with $m(I) \leqslant \varepsilon_0$ intersects at most two elements of $\mathscr{Z}_{n_0}$. Fix an arbitrary $\varepsilon \in (0, \varepsilon_0]$.

Let $Y \subseteq X$ be an interval with $m(Y) < \varepsilon$. Let $n \geqslant 0$ be the smallest number (maybe $n = \infty$) such that $Y \cap A$ intersects at least three different intervals $Z_1, Z_2, Z_3 \in \mathscr{Z}_n$. Since $\varepsilon \leqslant \varepsilon_0$ we see by the choice of ε_0 that $n-1 \geqslant n_0$. If $k < n$, then $Y \cap A$ intersects at most two elements of $\mathscr{Z}_k$. Hence (2.9) gives

(2.10) $\mu(Y \cap A) \leqslant 2\exp(-k(h_\mu - \delta))$ if $n_0 \leqslant k < n$.

Suppose first that $n < \infty$. As $n_0 \leqslant n-1 < n$ we get by (2.10)

(2.11) $$\mu(Y \cap A) \leqslant 2\exp(-(n-1)(h_\mu - \delta)) = 2\exp(h_\mu - \delta)\exp(-n(h_\mu - \delta)).$$

Now Lemma 1 (ii) and the definition of s give for $j = 1, 2, 3$

$$m(Z_j) \geqslant m(T^n Z_j) \inf_{x\in Z_j\cap A} \exp(S_n\varphi(x) - ns).$$

Since $Z_j \cap A \neq \emptyset$ we get from (2.8) and from Lemma 7

(2.12) $$m(Z_j) \geqslant m(T^n Z_j)\exp(-n(\chi_\mu + s + \delta)) \geqslant c\exp(-n(\chi_\mu + s + \delta)).$$

Set

(2.13) $$t := (h_\mu - \delta)/(\chi_\mu + s + \delta).$$

As the interval Y intersects three different intervals Z_1, Z_2, Z_3, it contains at least one of them. Hence (2.11)–(2.13) give

(2.14) $$m(Y)^t \geqslant c^t \exp(-n(h_\mu - \delta)) \geqslant (\tfrac{1}{2} c^t \exp(\delta - h_\mu))\mu(Y \cap A).$$

If otherwise $n = \infty$, then (2.10) holds for every $k \geqslant n_0$. Therefore $\mu(Y \cap A) = 0$, and (2.14) is trivial in this case.

Now (2.14) gives $m(A, t, \varepsilon) \geqslant (\frac{1}{2} c^t \exp(\delta - h_\mu))\mu(A) > 0$ and by the choice of ε the same estimate is true for $m(A, t)$. Therefore by (2.13)

$$\mathrm{HD}_m(L(\mathscr{C})) \geqslant \mathrm{HD}_m(A) \geqslant (h_\mu - \delta)/(\chi_\mu + s + \delta),$$

since $A \subseteq L(\mathscr{C})$. Now letting $\delta \to 0$ gives $\mathrm{HD}_m(L(\mathscr{C})) \geqslant h_\mu/(\chi_\mu + s)$, which is the desired result. ∎

Remarks. 1. The above proof shows that $\mathrm{HD}_m(Y) \geqslant h_\mu/(\chi_\mu + s)$ for every Borel set $Y \subseteq L(\mathscr{C})$ with $\mu(Y) > 0$.

2. In an analogous way one can show that for every ergodic T-invariant Borel probability measure μ on X there exists a Borel set $A \subseteq X$ with $\mu(A) = 1$ and $\mathrm{HD}_m(A) \leqslant h_\mu/\chi_\mu$ (in [7] it is shown that this inequality is valid if A is the set of generic points of μ). Here we use instead of Lemma 7 the fact that $m(B) \leqslant 1$ for every Borel set $B \subseteq X$. By Lemma 4 we can choose s arbitrarily small, which gives the desired result.

Now we can prove the main result. Lemma 9 will give examples of sets which satisfy the assumptions of this theorem. Recall that t_R denotes the unique nonnegative zero of $t \mapsto p(R, T, t\varphi)$ and $\mathrm{HD}_m(R)$ denotes the Hausdorff dimension of R.

THEOREM 2. *Let $(X, T, \mathscr{Z}, m, \varphi)$ be an expanding system. If R is a perfect T-invariant subset of X such that $(R, T, \mathscr{Z}(R))$ is a piecewise monotonic system, then $\mathrm{HD}_m(R) = t_R$.*

Proof. If $t_R = 0$ the result follows from Theorem 1.

Now suppose that $t_R > 0$. Choose an arbitrary $t \in (0, t_R)$. Then Lemma 3 gives $p(R, T, t\varphi) > 0$. Define

$$r := -\lim_{n\to\infty} n^{-1} \sup_{x\in X} S_n \varphi(x), \tag{2.15}$$

which exists and satisfies $r > 0$ by Lemma 1 (i). We get for every ergodic T-invariant Borel probability measure μ on X

$$\chi_\mu \geqslant r > 0. \tag{2.16}$$

Now choose an arbitrary $\varepsilon \in (0, p(R, T, t\varphi)/(2t))$. By Lemma 4 there exists a finite partition $\mathscr{Y}$ of X into intervals which refines $\mathscr{Z}$ and satisfies

$$\sup_{Y\in\mathscr{Y}} \sup_{x,y\in Y} |\varphi(x)-\varphi(y)| \leqslant \varepsilon. \tag{2.17}$$

Let $(\hat{X}, \hat{T}, \hat{\mathscr{Y}}, \hat{m}, \hat{\varphi})$ be the completion of $(X, T, \mathscr{Y}, m, \varphi)$ and let $\hat{R}$ be the extension of R to $(\hat{X}, \hat{T}, \hat{\mathscr{Y}})$. Observe that after passing to the completion the above inequalities remain valid.

Define $\varphi_1 \colon \hat{X} \to \boldsymbol{R}$ as follows: if $x \in Y \in \hat{\mathscr{Y}}$, then set $\varphi_1(x) := \inf_{y\in Y} \hat{\varphi}(y)$. Then $\varphi_1|_Y$ is constant for all $Y \in \hat{\mathscr{Y}}$. Furthermore, by the choice of ε, by the definition of φ_1, and by (2.15)–(2.17) we have

$$\lim_{n\to\infty} n^{-1} \sup_{x\in\hat{X}} tS_n \varphi_1(x) \leqslant -tr < 0, \quad p(\hat{R}, \hat{T}, t\varphi_1) > 0. \tag{2.18}$$

By the second inequality of (2.18), by Theorem 11 of [3], and by Corollary 2.18 of [1] there exists a maximal irreducible subset $\mathscr{C}$ of the Markov diagram of $(\hat{R}, \hat{T}, \hat{\mathscr{Y}}(\hat{R}))$ such that $p(L(\mathscr{C}), \hat{T}, t\varphi_1) > 0$. Using (2.18) and Lemma 6 yields that there exists a finite irreducible $\mathscr{C}' \subseteq \mathscr{C}$ such that $p(L(\mathscr{C}'), \hat{T}, t\varphi_1) > 0$. Therefore the variational principle gives the existence of an ergodic $\hat{T}$-invariant Borel probability measure μ on $L(\mathscr{C}')$ with $h_\mu + \mu(t\varphi_1) > 0$. Hence

$$t < \frac{h_\mu}{-\mu(\varphi_1)} \leqslant \frac{h_\mu}{\chi_\mu} = \frac{h_\mu}{\chi_\mu+\varepsilon}\left(1+\frac{\varepsilon}{\chi_\mu}\right) \leqslant \frac{h_\mu}{\chi_\mu+\varepsilon}\left(1+\frac{\varepsilon}{r}\right),$$

where the second inequality follows from the definition of φ_1 and the third from (2.16). Now (2.17) and Lemma 8 give

$$t \leqslant \mathrm{HD}_{\hat{m}}(L(\mathscr{C}'))\left(1+\frac{\varepsilon}{r}\right) \leqslant \mathrm{HD}_{\hat{m}}(\hat{R})\left(1+\frac{\varepsilon}{r}\right) = \mathrm{HD}_m(R)\left(1+\frac{\varepsilon}{r}\right),$$

since $L(\mathscr{C}') \subseteq \hat{R}$. Letting first $\varepsilon \to 0$ and then $t \to t_R$ we get $t_R \leqslant \mathrm{HD}_m(R)$. Theorem 1 gives the desired result. ∎

Remark. Theorem 2 and Lemma 3 imply that $\mathrm{HD}_m(R) = 0$ is equivalent to $h_{\mathrm{top}}(R, T) = 0$ if the assumptions of Theorem 2 are satisfied.

Our restriction that $(R, T, \mathscr{Z}(R))$ is a p.m.s. is not very nice. We give examples of sets which satisfy this condition. In particular, any set $L(\mathscr{C})$ with $\mathscr{C} \in \Gamma$ does. Another family of such sets can be found in [10].

LEMMA 9. *Suppose that $(X, T, \mathscr{Z})$ is a p.m.s.*

(i) *If R is one of the sets listed in* (a), (b) *or* (c) *below, then there exists a finite partition $\mathscr{Y}$ of X into intervals such that $(R, T, \mathscr{Y}(R))$ is a p.m.s.*

(a) *$R = \bigcup_{j=1}^n F_j$, where $F_1, \ldots, F_n$ are closed intervals in X with $TR \subseteq R$.*

(b) *R is closed and satisfies $T^{-1}R = R$.*

(c) *$R = \bigcap_{j=0}^\infty \overline{X \setminus T^{-j}G}$, where G is T-invariant.*

(ii) *Suppose that $(X, T, \mathscr{Z})$ is complete, that X is perfect, and that $\mathscr{Z}$ is a generator. If $R = L(\mathscr{C})$, where $\mathscr{C}$ is a maximal irreducible subset of the Markov diagram of $(X, T, \mathscr{Z})$, then there exists a finite partition $\mathscr{Y}$ of X into intervals such that $(R, T, \mathscr{Y}(R))$ is a p.m.s.*

Proof. (i) (a) is easily shown if one considers a subpartition $\mathscr{Y}$ of $\mathscr{Z}$ which contains the nonempty sets among $F_j \cap T^{-1}F_k$.

(b) follows from $T(Z \cap R) = T(Z \cap T^{-1}R) = TZ \cap R$.

(c) One can show that $T^{-1}R = R \cup E$, where E is finite. Consider a finite partition $\mathscr{Y}$ of X into intervals which refines $\mathscr{Z}$ and contains $\{x\}$ for every $x \in E$. Now the proof of (b) gives the desired result.

(ii) It is shown in [3] that $L(\mathscr{C}) = \bigcap_{j=0}^\infty \overline{F \setminus T^{-j}G}$, where $F = \bigcup_{k=1}^n F_k$, the F_k are as in (i) (a), and $TG \subseteq G$. Now the desired result follows from (i) (a) and (c). ∎

Now we shall give some corollaries of Theorem 2. Suppose that $(X, T, \mathscr{Z}, m, \varphi)$ is a complete expanding system. Let $(\mathscr{D}, \to)$ be the Markov diagram of $(X, T, \mathscr{Z})$, and denote by Γ the set of all maximal irreducible subsets of $\mathscr{D}$. Theorem 11 of [3] describes the centre $\Omega_\infty(X, T)$ of (X, T) (see e.g. § 5.3 of [11] for definition):

$$\Omega_\infty(X, T) = \bigcup_{\mathscr{C}\in\Gamma} L(\mathscr{C}) \cup L_\infty,$$

where $L(\mathscr{C})$ and L_∞ are closed T-invariant subsets of X, the sets $L(\mathscr{C})$ are topologically transitive, $h_{\mathrm{top}}(L_\infty, T) = 0$, and the intersection of two different sets $L(\mathscr{C})$ or of some $L(\mathscr{C})$ and L_∞ is finite.

COROLLARY 2.1. *Let $(X, T, \mathscr{Z}, m, \varphi)$ be a complete expanding system.*

(i) *Suppose that $\mathscr{C} \in \Gamma$, and set $L := L(\mathscr{C})$. Then $\mathrm{HD}_m(L) = t_L$.*

(ii) *Set $L := L_\infty$. Then $\mathrm{HD}_m(L) = t_L = 0$.*

Proof. Since $h_{\mathrm{top}}(L_\infty, T) = 0$ Lemma 3 gives $t_L = 0$ if $L = L_\infty$, and (ii) follows from Theorem 1. If $L = L(\mathscr{C})$, where $\mathscr{C} \in \Gamma$, then (i) follows from Lemma 9 (ii) and from Theorem 2. ∎

Remarks. 1. Using Corollary 2.1, Lemma 6 and the Perron–Frobenius theorem one can show that $\mathrm{HD}_m(L(\mathscr{C})) > 0$ if $\mathscr{C} \in \Gamma$ and $L(\mathscr{C})$ is not a periodic orbit.

2. It is well known that L_∞ is uncountable if $L_\infty \neq \emptyset$ (see e.g. [3]). An example of an expanding system with L_∞ nonempty is given in [7].

We consider the case where $\varphi|_R$ is constant equal to $-\alpha$. By Lemma 1 (i) we have $\alpha > 0$.

Corollary 2.2. *Let $(X, T, \mathscr{Z}, m, \varphi)$ be an expanding system. Suppose that R is a perfect T-invariant subset of X such that $(R, T, \mathscr{Z}(R))$ is a p.m.s., and that $\varphi|_R$ is constant equal to $-\alpha$. Then $\mathrm{HD}_m(R) = t_R = h_{\mathrm{top}}(R, T)/\alpha$.*

Proof. By Theorem 9.7 (i), (vi) of [11] we have $p(R, T, t\varphi) = h_{\mathrm{top}}(R, T) - t\alpha$. Therefore $t_R = h_{\mathrm{top}}(R, T)/\alpha$. Theorem 2 gives the desired result. ∎

Now we give an example. Let $X = [0, 1]$, let m be the Lebesgue measure on $[0, 1]$, and let $n \in N$. Since the case $n = 1$ is trivial, we assume $n \geqslant 2$. Suppose $0 \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \ldots \leqslant a_n < b_n \leqslant 1$ with $1/p_j := b_j - a_j$. Define $T|_{(a_j, b_j)}$ either by $Tx = p_j(x - a_j)$ or by $Tx = p_j(b_j - x)$. Set

$$A := [0, 1] \setminus \bigcup_{j=1}^{n} [a_j, b_j], \qquad R := \bigcap_{k=0}^{\infty} ([0, 1] \setminus T^{-k} A).$$

If $x \in (a_j, b_j)$, then set $\varphi(x) := -\log p_j$. Clearly we can extend T and φ to an expanding system $([0, 1], T, \mathscr{Z}, m, \varphi)$, where $(a_j, b_j) \in \mathscr{Z}$ for every j. Let $(\hat{X}, \hat{T}, \hat{\mathscr{Z}}, \hat{m}, \hat{\varphi})$ be the completion of $([0, 1], T, \mathscr{Z}, m, \varphi)$, let $\hat{R}$ be the extension of R to $(\hat{X}, \hat{T}, \hat{\mathscr{Z}})$, and denote by B_j the element of $\hat{\mathscr{Z}}$ which satisfies $B_j = [a_j^+, b_j^-]$. Then $\hat{R}$ is a perfect $\hat{T}$-invariant subset of $\hat{X}$, with $\hat{T}(\hat{R} \cap B_j) = \hat{R}$ for every j and $\hat{R} \cap Z = \emptyset$ for every $Z \in \hat{\mathscr{Z}} \setminus \{B_1, \ldots, B_n\}$. By Theorem 2 we have $\mathrm{HD}_m(R) = \mathrm{HD}_{\hat{m}}(\hat{R}) = t_R$, where t_R is the unique nonnegative zero of $t \mapsto p(\hat{R}, \hat{T}, t\hat{\varphi})$.

Set $\mathscr{C} := \{B_1 \cap \hat{R}, \ldots, B_n \cap \hat{R}\}$. Then $\mathscr{C}$ is a finite irreducible subset of the Markov diagram of $(\hat{R}, \hat{T}, \hat{\mathscr{Z}}(\hat{R}))$ (this Markov diagram is exactly $(\mathscr{C}, \to)$ with $C \to D$ for any $C, D \in \mathscr{C}$) and $\hat{R} = L(\mathscr{C})$. Since $\hat{\varphi}|_{B_j} = -\log p_j$ Lemma 6 (i) gives $p(\hat{R}, \hat{T}, t\hat{\varphi}) = \log r(F_{\mathscr{C}}(t\hat{\varphi}))$, where $F_{\mathscr{C}}(t\hat{\varphi})$ is the $n \times n$-matrix $(F_{\mathscr{C}jk})_{j,k=1}^n$ with $F_{jk} = \exp(-t \log p_j) = (1/p_j)^t$. Hence by the Perron–Frobenius theorem

$$p(\hat{R}, \hat{T}, t\hat{\varphi}) = \log\Big(\sum_{j=1}^{n} (1/p_j)^t\Big).$$

This shows that the Hausdorff dimension of R is the unique number $t \in [0, 1]$ with $\sum_{j=1}^{n} (1/p_j)^t = 1$.

If $n = 2$, $p_1 = \frac{9}{4}$ and $p_2 = 9$ (e.g. if $a_1 = 0$, $b_1 = \frac{4}{9}$, $a_2 = \frac{8}{9}$, $b_2 = 1$) in the above example, we get $\mathrm{HD}_m(R) = \frac{1}{2}$.

Finally, we consider the special case of the above example where $p_1 = p_2 = \ldots = p_n =: p$. Here the above formula gives $\mathrm{HD}_m(R) = t$, where $n(1/p)^t = 1$. Therefore $\mathrm{HD}_m(R) = \log n/\log p$ (this also follows from Corollary 2.2). If $n = 2$, $p = 3$, and $a_1 = 0$, $b_1 = \frac{1}{3}$, $a_2 = \frac{2}{3}$, $b_2 = 1$, then R is the usual Cantor set, which has Hausdorff dimension $\log 2/\log 3$.

References

[1] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes in Math. 470, Springer, Berlin 1975.

[2] —, *Hausdorff dimension of quasi-circles*, Publ. Math. IHES 50 (1979), 11–25.

[3] F. Hofbauer, *Piecewise invertible dynamical systems*, Probab. Th. Rel. Fields 72 (1986), 359–386.

[4] F. Hofbauer and G. Keller, *Zeta-functions and transfer-operators for piecewise linear transformations*, J. Reine Angew. Math. 352 (1984), 100–113.

[5] A. Manning, *A relation between Lyapunov exponents, Hausdorff dimension and entropy*, Ergodic Theory Dynamical Systems 1 (1981), 451–459.

[6] H. McCluskey and A. Manning, *Hausdorff dimension for horseshoes*, ibid. 3 (1983), 251–260. Errata: ibid. 5 (1985), 319.

[7] P. Raith, *Hausdorffdimension für stückweise monotone Abbildungen*, Dissertation, Wien 1987.

[8] E. Seneta, *Finite approximations to infinite nonnegative matrices, II: refinements and applications*, Proc. Cambridge Philos. Soc. 64 (1968), 465–470.

[9] —, *Nonnegative Matrices and Markov Chains*, Springer Ser. Statist., Springer, New York 1981.

[10] M. Urbański, *Invariant subsets of expanding mappings of the circle*, Ergodic Theory Dynamical Systems 7 (1987), 627–645.

[11] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Math. 79, Springer, New York 1982.

INSTITUT FÜR MATHEMATIK
UNIVERSITÄT WIEN
Strudlhofgasse 4, A-1090 Wien, Austria

Received April 15, 1987
Revised version October 26, 1987 (2296)