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The l_1^n problem and degrees of non-reflexivity II*

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Abstract. The study of Banach spaces with k-structure (introduced in part I) is continued. In particular, it is shown that every space with 2-structure contains nearly isometric copies of l_1^4. High order conjugate spaces are also studied further. It is shown that if X is separable, non-reflexive, X_{ω^2+2} is non-separable. Finally, it is shown that for every $\varepsilon > 0$ there is a non-reflexive space of type $2-\varepsilon$.

1. Introduction. This paper is a continuation of [1] but it can be read independently of [1].

In [1] we introduced for every integer k the notion of (global and local) k-structure. A Banach space admits a 1-structure if and only if it is not superreflexive. A Banach space X admits a 2-structure if and only if it is finitely representable in a space X for which X^{**}/X is not reflexive (the definitions of superreflexivity and finite-representability will be recalled at the end of this introduction.) In general, a necessary and sufficient condition for the existence (up to finite-representability) of a k-structure in a space X was given in [1] and it involves the even duals of X up to order $2k$. The notion of k-structure was introduced mainly as a tool for studying the problem of existence of almost isometric copies of l_1^n in non-superreflexive Banach spaces. It was shown in [1] that if a Banach space X admits a k-structure then X contains almost isometric copies of l_1^{k+1}. (For $k = 1$ this is the, by now classical, result of James [2] which started the whole subject of superreflexivity and existence of l_1^n.) We show here that a little variant of the argument used in [1] can give stronger results. In Section 2 below we prove for example that if X admits a 2-structure then for any integer n and any $\varepsilon > 0$ there are vectors $\{x_{i,j}\}_{i,j=1}^n$ in X all of norm 1 so that for any choice of $1 \leqslant r,\ s \leqslant n$ we have

$$\Big\| \sum_i \sum_j \theta^r(i)\eta^s(j)x_{i,j} \Big\| \geqslant n^2 - \varepsilon \tag{1.1}$$

where $\theta^r(i) = 1$ if $1 \leqslant i \leqslant r$, $\theta^r(i) = -1$ if $r < i \leqslant n$ and similarly $\eta^s(j) = 1$ if $1 \leqslant j \leqslant s$, $\eta^s(j) = -1$ if $s < j \leqslant n$.

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In [1] we used just the diagonal, i.e. $\{x_{i,i}\}_{i=1}^n$ of the set $\{x_{i,j}\}_{i,j=1}^n$. This produced (by taking $n=3$) almost isometric copies of l_1^3 in a space with 2-structure. By using just the diagonal elements we did not get almost isometric copies of l_1^n with $n>3$ in a space with 2-structure. It is a simple fact (which however surprises us somewhat) that by using also non-diagonal elements we can get even almost isometric copies of l_1^4 in every space with a 2-structure. In fact, if we use (1.1) for $n=4$ and consider the vectors $x_{1,1}$, $x_{2,2}$, $x_{3,4}$, and $x_{4,3}$ then they span a subspace $1+\varepsilon$ isometric to l_1^4.

It turns out that this method of proof stops with 4 and (1.1) (for arbitrary n) does not imply directly the existence of almost isometric copies of l_1^5 in a space with 2-structure. We leave the problem of whether 2-structure implies the existence of l_1^5 open. A natural approach to attack this problem is to try to generalize the counterexample of James [3], [4], of a space with 1-structure which does not contain almost isometric copies of l_1^3. Our efforts to do this failed till now. It seems to be simpler to construct a less precise counterexample which will show at least that 2-structure does not imply the existence of almost isometric copies of l_1^k for some sufficiently large k. While we were not able even to construct such a counterexample our approach to this problem led to some information concerning 1-structure which is of some interest. We prove in Section 3 below that the examples of James (which depend on some parameter) are in a sense almost as far as possible from containing l_1^k with k large. More precisely, we prove in the terminology of [5] (cf. Section 3 for the precise definition) that for every $p<2$ there is a non-superreflexive space of type p. The question due to Rosenthal of whether a non-superreflexive space can have the best possible type, i.e. $p=2$, remains open (some positive partial results on this question are given in [6]).

In the fourth and last section of this paper we consider transfinite duals. For a Banach space X and an even ordinal α the αth conjugate X^α of X is defined inductively by the relations $X^{\alpha+2}=(X^\alpha)^{**}$ and X^α = Completion of $\bigcup_{\substack{\beta<\alpha\\ \beta \text{ even}}} X^\beta$ whenever α is a limit ordinal (we always identify a Banach space with its canonical image in its second dual and thus the definition of X^α for α a limit ordinal makes sense). In [1] it was proved that for every non-reflexive Banach space X the space $X^{\omega+2}/X^\omega$ is infinite-dimensional (where ω is the first infinite ordinal). We show here that for every such X the space $X^{\omega^2+2}/X^{\omega^2}$ is non-separable. This result gives a simple way to construct for every non-reflexive space X a separable space Y so that Y is finitely represented in X and Y^{**} is not separable. It follows e.g. from this and the result of Section 3 that for every $p<2$ there is a separable space of type p whose second dual is non-separable.

Concerning notation, let us only recall here the definition of the terms finitely represented and superreflexive which appeared already several times above. A Banach space X is said to be *finitely represented in* Y if for every finite-dimensional subspace B of X and any $\varepsilon>0$ there is a subspace C of Y and an invertible operator T from B onto C so that $\|T\|\,\|T^{-1}\|\leqslant 1+\varepsilon$ (i.e. $d(B,C)\leqslant 1+\varepsilon$). A Banach space X is called *superreflexive* if every Banach space X which is finitely represented in Y is reflexive.

2. Spaces with k-structure. Recall [1] that a Banach space X admits a local k-structure if there is a constant M such that for all n there are biorthogonal systems $\{x_{i_1,\ldots,i_k}\}\subseteq X$, $\{f_{i_1,\ldots,i_k}\}\subseteq X^*$, $1\leqslant i_1,i_2,\ldots,i_k\leqslant n$ so that

$$\|f_{i_1,\ldots,i_k}\|\leqslant M,\qquad \Big\|\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_k=1}^{r_k}x_{i_1,i_2,\ldots,i_k}\Big\|\leqslant M \tag{2.1}$$

for all $1\leqslant r_1,r_2,\ldots,r_k\leqslant n$. A space has global k-structure if there is a single infinite biortogonal system $\{x_{i_1,\ldots,i_k},f_{i_1,\ldots,i_k}\}$ so that (2.1) holds. It was observed in [1] that every space with local k-structure has, finitely represented in it, a space with global k-structure. In what follows, we assume therefore that the spaces with k-structure actually admit a global k-structure.

Besides the notion of finite representability it will be convenient to use here a weaker notion. A Banach space X is said to be *weakly finitely represented in* Y if for every finite-dimensional subspace B of X and every $\varepsilon>0$ there is a finite-dimensional subspace C of Y and a quotient space B' of C so that $d(B,B')\leqslant 1+\varepsilon$. For the question of existence of almost isometric copies of l_1^n the notion of weakly finitely representable is well suited. Indeed it is a trivial fact that if a quotient space of a space C contains isometric (or almost isometric) copies of l_1^n for some n; the same is true for C itself.

Our first theorem produces for every space X which admits a k-structure a k-dimensional matrix space which is, on the one hand, weakly finitely represented in X and, on the other hand, has certain nice regularity properties which facilitate the study of the existence or non existence of copies of l_1^n in such spaces. For the sake of simplicity of notation we restrict ourselves to the case $k=2$ even in the statement of the theorem. The reader should however find no difficulty in generalizing the statement of the theorem and its proof to arbitrary k.

THEOREM 1. *Let X be a Banach space admitting a 2-structure. Let $\mathfrak{A}$ be the linear space of matrices $A=(a_{i,j})_{i,j=1}^\infty$ with only finitely many non-zero entries. Then there is a norm $\|\cdot\|$ on $\mathfrak{A}$ so that $(\mathfrak{A},\|\ \|)$ is weakly finitely represented in X and so that*

(a) *If $A,B\in\mathfrak{A}$ and A is a submatrix of B, then $\|A\|\leqslant\|B\|$. By "submatrix" we understand the following: there are $0<p_1<p_2<\ldots<p_n$,*

$0 < q_1 < q_2 < \ldots < q_m$ *such that* $a_{i,j} = b_{p_i,q_j}$ *for* $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ *and* $a_{i,j} = 0$ *if* $i > n$ *or* $j > m$.

(b) *If* $B \in \mathfrak{A}$ *is obtained from* $A \in \mathfrak{A}$ *by repeating a row or a column, then* $\|A\| = \|B\|$. *We say, e.g., that* A *is obtained from* B *by repeating a row if there is an integer* $0 < p$ *so that* $a_{i,j} = b_{i,j}$ *if* $i \leqslant p$, $a_{i,j} = b_{i-1,j}$ *if* $i > p$.

(c) $\|A\| = 1$, *where* $a_{i,j} = 1$ *if* $i = j = 1$ *and* $a_{i,j} = 0$ *otherwise.*

Proof. First, we define a sequence of seminorms on $\mathfrak{A}$ as follows: For $A \in \mathfrak{A}$, m, $p_1 \leqslant p_2 \leqslant \ldots \leqslant p_{2m}$, $\pi_1 \leqslant \pi_2 \leqslant \ldots \leqslant \pi_{2m}$ let

$$S\big(m, (p_i), (\pi_i), A\big) = \{x \in X \mid f_{ij}(x) = a_{kl};\ p_{2k-1} \leqslant i \leqslant p_{2k},\ \pi_{2l-1} \leqslant j \leqslant \pi_{2l},\ 1 \leqslant k, l \leqslant m\} \tag{2.2}$$

and

$$K\big(m, (p_i), (\pi_i), A\big) = \inf\big\{\|x\| \mid x \in S\big(m, (p_i), (\pi_i), A\big)\big\}. \tag{2.3}$$

(It will be useful later to notice that, if $q\colon X \to X/Y$ is the quotient map with kernel $Y = \{y \mid f_{ij}(y) = 0,\ p_{2k-1} \leqslant i \leqslant p_{2k-1}, \pi_{2l-1} \leqslant j \leqslant \pi_{2l},\ 1 \leqslant k, l \leqslant m\}$, then

$$K\big(m, (p_i), (\pi_i), A\big) = \Big\| q\Big(\sum_{i,j=l}^{m} a_{ij} \sum_{k=p_{2i-1}}^{p_{2i}} \sum_{l=\pi_{2j-1}}^{\pi_{2j}} x_{k,l}\Big)\Big\|.$$

Now define

$$K(m, A) = \lim_{p_1\to\infty} \lim_{p_2\to\infty} \ldots \lim_{p_{2m}\to\infty} \lim_{\pi_1\to\infty} \ldots \lim_{\pi_{2m}\to\infty} K\big(m, (p_i), (\pi_i), A\big). \tag{2.4}$$

Remark. A similar definition occurred in [1] without verification of its existence. We remedy that situation here.

Lemma 1. *For each* m, *and each* $A \in \mathfrak{A}$, $K(m, A)$ *exists.*

Proof. By virtue of the fact that $(x_{ij}; f_{ij})$ is a 2-structure,

$$K\big(m, (p_i), (\pi_i), A\big) \leqslant \sum |a_{ij}| \Big\| \sum_{p_{2i-1}}^{p_{2i}} \sum_{\pi_{2j-1}}^{\pi_{2j}} x_{k,l} \Big\| \leqslant 4M \sum |a_{ij}|, \tag{2.5}$$

so for each m, A, it is certain that

$$\overline{K}(m, A) = \varliminf_{p_1\to\infty} \varliminf_{p_2\to\infty} \ldots \varliminf_{\pi_{2m}\to\infty} K\big(m, (p_i), (\pi_i), A\big)$$

exists. We simply show that these liminf's are actually limits. Let $\varepsilon > 0$. Pick p_1 so that

$$\Big|\overline{K}(m, A) - \varliminf_{p_2\to\infty} \ldots \varliminf_{p_{2m}\to\infty} \varliminf_{\pi_1\to\infty} \ldots \varliminf_{\pi_{2m}\to\infty} K\big(m, (p_i), (\pi_i), A\big)\Big| < \varepsilon.$$

Let $r_1 > p_1$ and then successively pick s_1, s_2, so that $r_1 < s_1 \leqslant s_2$ and

$$\Big|\overline{K}(m, A) - \varliminf_{s_3\to\infty} \varliminf_{s_4\to\infty} \ldots \varliminf_{s_{2m}\to\infty} \ldots \varliminf_{\sigma_{2m}\to\infty} K\big(m, (s_i), (\sigma_i), A\big)\Big| < \varepsilon.$$

Continue in this way selecting (p_i), (r_i), (s_i), (π_i), (ϱ_i), (σ_i) so that

$$\big|\overline{K}(m, A) - K\big(m, (p_i), (\pi_i), A\big)\big| < \varepsilon, \qquad \big|\overline{K}(m, A) - K\big(m, (s_i), (\sigma_i), A\big)\big| < \varepsilon$$

and so that for each i, j,

$$[p_{2i-1}, p_{2i}] \times [\pi_{2j-1}, \pi_{2j}] \supseteq [r_{2i-1}, r_{2i}] \times [\varrho_{2j-1}, \varrho_{2j}] \supseteq [s_{2i-1}, s_{2i}] \times [\sigma_{2j-1}, \sigma_{2j}].$$

Since $x \in S\big(m, (p_i), (\pi_i), A\big)$ implies $x \in S\big(m, (r_i), (\varrho_i), A\big)$, it follows that

$$K\big(m, (p_i), (\pi_i), A\big) \geqslant K\big(m, (r_i), (\varrho_i), A\big).$$

Similarly,

$$K\big(m, (r_i), (\varrho_i), A\big) \leqslant K\big(m, (s_i), (\sigma_i), A\big).$$

It follows that

$$\overline{K}(m, A) + \varepsilon \geqslant K\big(m, (r_i), (\varrho_i), A\big) \geqslant \overline{K}(m, A) - \varepsilon.$$

This completes the proof.

To see that each $K(m, \cdot)$ is a seminorm, first note that positive homogeneity is trivial. In the definition of $K(m, \cdot)$, since limits are used, it is possible to select $p_1 \leqslant \ldots \leqslant p_{2m}$ then $\pi_1 \leqslant \ldots \leqslant \pi_{2m}$ inductively (given $\varepsilon > 0$) so that

$$\big|K(m, A) - K\big(m, (p_i), (\pi_i), A\big)\big| < \varepsilon, \qquad \big|K(m, B) - K\big(m, (p_i), (\pi_i), B\big)\big| < \varepsilon$$

and

$$\big|K(m, A+B) - K\big(m, (p_i), (\pi_i), A+B\big)\big| < \varepsilon.$$

However, it is immediate that

$$K\big(m, (p_i), (\pi_i), A+B\big) \leqslant K\big(m, (p_i), (\pi_i), A\big) + K\big(m, (p_i), (\pi_i), B\big) \tag{2.6}$$

so that

$$K(m, A+B) \leqslant K(m, A) + K(m, B) + 3\varepsilon,$$

proving the triangle inequality.

We now define the desired norm on $\mathfrak{A}$ by $\|A\| = \varlimsup_m K(m, A)$. Since $(x_{ij}; f_{ij})$ is a 2-structure, we have

$$M^{-1} \max |a_{ij}| \leqslant \|A\| \leqslant 4M \sum |a_{ij}|. \tag{2.7}$$

Next, we note that $K(m+1, A) \geqslant K(m, A)$ (so the limsup in the definition is actually a limit). To see this, let $\varepsilon > 0$ and choose $p_1 \leqslant p_2 \leqslant \ldots$

$\ldots \leqslant p_{2m} \leqslant p_{2m+1} \leqslant p_{2m+2}$ so that both

$$(2.8)\qquad \Big|K(m, A) - \lim_{\pi_1\to\infty} \ldots \lim_{\pi_{2m}\to\infty} K\big(m, (p_i)_{i=1}^{2m}, (\pi_i), A\big)\Big| < \varepsilon$$

and

$$(2.9)\qquad \Big|K(m+1, A) - \lim_{\pi_1\to\infty} \ldots \lim_{\pi_{2m+2}\to\infty} K\big(m, (p_i)_{i=1}^{2m+2}, (\pi_i), A\big)\Big| < \varepsilon.$$

This is possible since limits are involved. Similarly, choose $\pi_1 \leqslant \ldots \leqslant \pi_{2m+2}$ so that both

$$(2.10)\qquad \big|K(m, A) - K\big(m, (p_i), (\pi_i), A\big)\big| < \varepsilon \text{ and } \big|K(m+1, A) - K\big(m+1, (p_i), (\pi_i), A\big)\big| < \varepsilon.$$

It is clear that $K\big(m+1, (p_i), (\pi_i), A\big) \geqslant K\big(m, (p_i), (\pi_i), A\big)$, proving the assertion, since ε was arbitrary.

Now let $\mathfrak{U}_n \subseteq \mathfrak{U}$ be the subspace of matrices A such that $a_{ij} = 0$ for i or $j > n$. By the compactness of the ball in $\langle \mathfrak{U}_n, \|\cdot\|\rangle$, we have, for any $\varepsilon > 0$, an m such that for all $A \in \mathfrak{U}_n$, $\|A\| - K(m, A) \leqslant \varepsilon\|A\|$. In fact, for such m, ε, $(p_i)_{i=1}^{2m}$ and $(\pi_i)_{i=1}^{2m}$ can be chosen so that for $A \in \mathfrak{U}_n$,

$$(2.11)\qquad \big|\|A\| - K\big(m, (p_i), (\pi_i), A\big)\big| \leqslant \varepsilon\,\|A\|.$$

We are now in a position to prove the promised result.

The space has been defined. Let us verify (a). Let $B = (b_{ij})$ with $b_{ij} = 0$ if $i > M$ or $j > N$. Let $1 \leqslant i_1 < i_2 < \ldots < i_m \leqslant M$ and $1 \leqslant j_1 < j_2 < \ldots \ldots < j_n \leqslant N$ and let $a_{k,l} = b_{i_k,j_l}$ for $k \leqslant m$, $l \leqslant n$, $a_{k,l} = 0$ for other k and l. Let $\varepsilon > 0$ and pick σ so that $\|A\| - K(\sigma, A) < \varepsilon$. Pick $\tau > \sigma + \max(M-m, N-n)$ so that $\|B\| - K(\tau, B) < \varepsilon$. Now successively choose $p_1 \leqslant p_2 \leqslant \ldots \leqslant p_{2i_1-1} = r_1 \leqslant p_{2i_1} = r_2 \leqslant \ldots \leqslant p_{2\tau}$ and then $\pi_1 \leqslant \ldots \leqslant \pi_{2j_1-1} = \varrho_1 \leqslant \pi_{2j_1} = \varrho_2 \leqslant \ldots \leqslant \pi_{2\tau}$ so that both

$$(2.12)\qquad \big|K(\tau, B) - K\big(\tau, (p_i), (\pi_i), B\big)\big| < \varepsilon$$

and

$$(2.13)\qquad \big|K(\tau, A) - K\big(\tau, (r_i), (\varrho_i), A\big)\big| < \varepsilon.$$

As before, these choices are possible since only limits are involved in the definition of the quantities involved. It is clear that

$$K\big(\tau, (p_i), (\pi_i), B\big) \geqslant K\big(\tau, (r_i), (\varrho_i), A\big),$$

since

$$S\big(\tau, (p_i), (\pi_i), B\big) \subseteq S\big(\tau, (r_i), (\varrho_i), A\big).$$

This proves (1).

To see that (b) is valid, observe that by (a) it is enough to show

(b′) Let A and B be given such that

$$a_{ij} = \begin{cases} b_{ij} & \text{for } i = 1, \ldots, k, \text{ and all } j, \\ b_{i-1,j} & \text{for } i = k+1, \ldots, n+1, \text{ and all } j. \end{cases}$$

Then $\|A\| \geqslant \|B\|$.

To see this, let N be given and select (p_i), (π_i) and $(\tilde p_i)$, $(\tilde\pi_i)$ so

$$K(N, A) \sim K\big(N, (p_i), (\pi_i), A\big) \quad\text{and}\quad K(N+1, B) \sim K\big(N+1, (\tilde p_i), (\tilde\pi_i), B\big)$$

as follows: Let $p_i = \tilde p_i$ for $i = 1, \ldots, 2k+1$. Then pick $\tilde p_{2k+1} \leqslant \tilde p_{2k+2} \leqslant p_{2k+3}$ for $K(N+1, B)$, and continue by picking $p_{2k+2} = \tilde p_{2k+4} \leqslant \ldots \leqslant p_{2k+j} = \tilde p_{2k+j+2} \leqslant \ldots \leqslant \tilde p_{2(N+1)}$ for both. Now simply let $\pi_j = \tilde\pi_j$ for $j \leqslant 2N$ and $\tilde\pi_{2N+1} \leqslant \tilde\pi_{2N+2}$ be good choices for $K(N, A)$ and $K(N+1, B)$. Once again, by the conditions on A and B, and the choices of (p_i), (π_i), $(\tilde p_i)$ and $(\tilde\pi_i)$,

$$S\big(N, (p_i), (\pi_i), A\big) \subseteq S\big(N+1, (\tilde p_i), (\tilde\pi_i), B\big)$$

so that

$$K\big(N, (p_i), (\pi_i), A\big) \leqslant K\big(N+1, (\tilde p_i), (\tilde\pi_i), B\big)$$

and thus $K(N, A) \leqslant K(N+1, B)$. Therefore, $\|A\| \leqslant \|B\|$. This completes the verification of (b).

Finally, to see that $\langle \mathfrak{U}, \|\cdot\|\rangle$ is weakly finitely represented in X, let $(p_i)_{i=1}^{2m}$ and $(\pi_i)_{i=1}^{2m}$ be given increasing sequences of integers. Let

$$U = \{x \in X \mid f_{k,l}(x) = f_{p_{2i},\pi_{2j}}(x) \text{ when } p_{2i-1} \leqslant k \leqslant p_{2i} \text{ and } \pi_{2j-1} \leqslant l \leqslant \pi_{2j},\ 1 \leqslant i, j \leqslant m\}$$

and let V be the subspace of U determined by $f_{p_{2i},\pi_{2j}}(x) = 0$ for $1 \leqslant i, j \leqslant m$. By the compactness arguments noted earlier, if W is a finite-dimensional subspace of $\mathfrak{U}$, and if $\varepsilon > 0$, then there exist m, (p_i) and (π_i) so that for all $A \in W$,

$$\big|\|A\| - K\big(m, (p_i), (\pi_i), A\big)\big| < \varepsilon\,\|A\|.$$

However, $K\big(m, (p_i), (\pi_i), A\big) = \|q(x)\|$ where $f_{p_{2i},\pi_{2j}}(x) = a_{ij}$ and q is the quotient map of U onto U/V. Thus, W embeds ε-isometrically into U/V, i.e. W is ε-isometric to a subspace of a quotient of a subspace of X, and is, therefore, ε-isometric to a subspace of a quotient of X. This completes the proof of the theorem.

Before we proceed we single out a special instance of requirement (b) in the theorem which we shall need soon. Let $A \subseteq \mathfrak{U}$ and let r and s be integers; we denote by $T_{r,s}A$ the element of $\mathfrak{U}$ obtained by first repeating r times the first row of A and then repeating s times the first column of A.

We have

$$(2.14)\qquad \|T_{r,s}A\| = \|A\|, \quad T_{r,s}A(i+r, j+s) = A(i,j).$$

We come now to the result mentioned already in the introduction.

THEOREM 2. *Let X admit k-structure, let n be an integer and let $\varepsilon > 0$. Then there exist n^k elements $\{x_{i_1,i_2,\ldots,i_k}\}_{1\leqslant i_j\leqslant n}$ in X all of norm 1 so that for every choice of $1 \leqslant r_1, r_2, \ldots, r_k \leqslant n$ we have*

$$(2.15)\qquad \Big\|\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\ldots\sum_{i_k=1}^{n}\theta^{r_1}(i_1)\theta^{r_2}(i_2)\ldots\theta^{r_k}(i_k)x_{i_1,i_2,\ldots,i_k}\Big\| \geqslant n^k-\varepsilon$$

where $\theta^r(i) = 1$ if $i \leqslant r$ and $\theta^r(i) = -1$ if $i > r$.

Proof. We shall present the proof only in the case $k = 2$. In view of Theorem 1 it is easily seen that it is enough to prove the present theorem for matrix spaces satisfying (a)–(c) of Theorem 1.

Let $\varepsilon > 0$ and an integer n be given. For every integer m consider the matrix A^m defined by

$$(2.16)\qquad A^m(i,j) = \begin{cases}(-1)^{i+j} & \text{if } 1 \leqslant i, j \leqslant m,\\ 0 & \text{otherwise.}\end{cases}$$

By Theorem 1 we have that the $\lambda_m = \|A^m\|$ satisfy

$$(2.17)\qquad 1 = \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m \leqslant \lambda_{m+1} \leqslant \ldots, \qquad \lambda_m \leqslant m^2.$$

Hence there is an integer m so that

$$(2.18)\qquad \lambda_{m-1}/\lambda_m \geqslant (n^2-\varepsilon)/n^2.$$

Consider now the matrix $A^{m,n}$ which is obtained from A^m by repeating each of its rows and columns n times

$$(2.19)\qquad \|A^{m,n}\| = \|A^m\| = \lambda_m,$$

$$(2.20)\qquad A^{m,n}(i,j) = (-1)^{k+l} \quad \text{if} \quad (n-1)k < i \leqslant nk,\ (n-1)l < j \leqslant nl.$$

We claim that the n^2 vectors $\{T_{ij}A^{m,n}/\lambda_m\}_{i,j=1}^n$ have the required properties. By (2.14) and (2.19) these vectors are all of norm 1. We have to check that (2.15) holds. Pick any $1 \leqslant r, s \leqslant n$ and let $\theta^r(i) = 1$ if $i \leqslant r$, $= -1$ if $i > r$ and similarly $\theta^s(j) = 1$ if $j \leqslant s$, $= -1$ if $j > s$. By (2.14) and (2.20) we get that for $1 \leqslant k \leqslant m-1$ and $1 \leqslant l \leqslant m-1$

$$(2.21)\qquad \sum_{i=1}^{n}\sum_{j=1}^{n}\theta^r(i)\theta^s(j)T_{ij}A^{m,n}(kn+r-1, ln+s-1) = (-1)^{k+l}\cdot n^2$$

since every term in this double sum is equal to $(-1)^{k+l}$. Hence by Theorem 1 (see (a) there) it follows that if we divide the left-hand side of (2.21) by λ_m we get a vector whose norm is $\geqslant \lambda_{m-1}n^2/\lambda_m$ which by (2.18) is $\geqslant n^2-\varepsilon$ as required.

COROLLARY 1. *A space with 2-structure contains almost isometric copies of l_1^4.*

Proof. Use (2.15) with $k = 2$ and $n = 4$. Consider the vectors $x_{1,1}$, $x_{2,2}$, $x_{3,4}$ and $x_{4,3}$. By letting $1 \leqslant r \leqslant 4$, $1 \leqslant s \leqslant 4$; then the 16 triples of signs

$$(2.22)\qquad \varepsilon_2 = \theta^r(2)\theta^s(2), \quad \varepsilon_3 = \theta^r(3)\theta^s(4), \quad \varepsilon_4 = \theta^r(4)\theta^s(3)$$

contain among them all the 8 possible different choices of signs $\varepsilon_i = \pm 1$. Since $\theta^r(1)\theta^s(1) = 1$ for all r and s we get this by (2.15) that for every choice of signs $\{\varepsilon_i\}_{i=2}^4$ we have

$$(2.23)\qquad \|x_{1,1}+\varepsilon_2x_{2,2}+\varepsilon_3x_{3,4}+\varepsilon_4x_{4,3}\| \geqslant 4-\varepsilon$$

and this proves our assertion (observe that in (2.15) every partial sum of say m terms of the left-hand side must, by the triangle inequality, have norm $\geqslant m-\varepsilon$).

This method of proof will not produce almost isometric copies of l_1^5 in spaces with 2-structure. In order to see this one has only to verify that there is no choice of a permutation σ of $\{1, 2, \ldots, 5\}$ so that if we put

$$(2.24)\qquad \varepsilon_i = \theta^r(i)\theta^s(\sigma(i)), \quad 1 \leqslant i \leqslant 5,$$

then the 25 possible choices of $1 \leqslant r, s \leqslant 5$ will not produce all the 16 possible different choices of $\{\varepsilon_i\}_{i=1}^5$ in which $\varepsilon_1 = 1$. We have checked all the permutations σ and verified that they do not work.

For general k we get from Theorem 2 the following

COROLLARY 2. *Every Banach space X with k-structure contains almost isometric copies of l_1^n if the following is true.*

There exist k permutations $\sigma_1, \sigma_2, \ldots, \sigma_k$ of $\{1, 2, \ldots, n\}$ (σ_k can always be taken as the identity) so that the n^k choices of $1 \leqslant r_1, r_2, \ldots, r_k \leqslant n$ produce, by putting

$$(2.25)\qquad \varepsilon_i = \theta^{r_1}(\sigma_1(i))\theta^{r_2}(\sigma_2(i))\ldots\theta^{r_k}(\sigma_k(i)); \quad 1 \leqslant i \leqslant n,$$

all the possible 2^{n-1} choices of signs $\{\varepsilon_i\}_{i=1}^n$ in which $\varepsilon_1 = 1$.

The question which l_1^n are actually insured by our method in spaces with k-structure is thus reduced to a purely combinatorial problem. We have not studied this problem in any detail simply because we have no evidence which shows that our method of proof is really the most efficient one for finding l_1^n subspaces in spaces with k-structure.

We make only two trivial comments. It is clear that an upper bound on the n obtainable by our method is given by the requirement that $n^k \geqslant 2^{n-1}$. Hence n is at most of the order of magnitude $k\log k$. By taking $\sigma_1 = \sigma_2 = \ldots = \sigma_k =$ identity we get that the condition in Corollary

2 holds for $n = k+1$ but no larger n. (This is the situation discussed in [1].)

Among the many problems related to this section which remain open, the ones of major interest to us are those connected with two structure. Here are some of them

1. Does there exist a space admitting 2-structure which does not contain almost isometric copies of l_1^5? In fact does there exist a space with 2-structure which does not contain almost isometric copies of l_1^k for some k? (this latter question is equivalent to asking the existence of a space admitting 2-structure but not 3-structure).

A particular case of problem 1 is

2. Does there exist a cross norm α and a non-reflexive space X such that $X\otimes_\alpha X$ does not admit 3-structure (as observed in [1], $X\otimes_\alpha X$ always admits 2-structure).

3. Let X be a Banach space such that any space isomorphic to X contains almost isometric copies of l_1^3. Must X admit 2-structure or at least must X contain almost isometric copies of l_1^4?

3. On the type of non-superreflexive spaces. We begin by recalling a construction of R. C. James [4], [3]. Consider the linear space of real-valued finitely supported functions on the positive integers N. By a "bump" we mean any function which is equal to some non zero constant on one interval of integers and is equal to 0 outside this interval. The absolute value of this constant is called the magnitude of the bump. Two bumps are said to be *disjoint* if the intervals on which they are non zero are disjoint. Let $1 < q < \infty$ and consider the functional ϱ defined by

$$\varrho(y) = \inf\Big(\sum_{j=1}^{n} m_j\Big[\Big(H-\sum_{i=1}^{j-1} h_i\Big)^q - \Big(H-\sum_{i=1}^{j} h_i\Big)^q\Big]\Big)^{1/q}, \tag{3.1}$$

where the infimum is taken over all the representations of y as $\sum_{j=1}^{n} z^j$ with each z^j a sum of m_j^q disjoint bumps of magnitude h_j and $H = \sum_{i=1}^{n} h_j$. The functional ϱ does not satisfy the triangle inequality and in order to work with an actual norm we define further

$$\|x\| = \inf \sum_{i=1}^{l} \varrho(y^i), \tag{3.2}$$

where the infimum is taken over all possible representations $x = \sum_{i=1}^{l} y^i$.

The space of finitely supported sequences with the norm given by (3.2) will be denoted by X (the exponent q will be fixed throughout the argument and so we do not add it to the notation of X). As observed in [3] it is very easy to check that (3.2) actually defines a norm and that the completion of X is not reflexive.

For future use we make now an observation on the possibility of writing an element $y \in X$ as $\sum_j z^j$ with each z^j being a sum of disjoint bumps. Let y be a finitely supported function on N whose values are integers (positive or negative) whose maximal absolute value is n. Then if we put

$$z^j(i) = \begin{cases} 1 & \text{if } y(i) \geqslant j, \\ -1 & \text{if } y(i) \leqslant -j, \qquad 1 \leqslant j \leqslant n, \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

we get that

$$y = \sum_{j=1}^{n} z^j \tag{3.4}$$

and each z^j is a sum of disjoint bumps of magnitude 1. It is clear that if y is constant on some interval, the same is true for every z^j. Thus if there are m intervals of integers so that on any one of them y is constant and y vanishes outside these intervals, then the number of disjoint bumps in each z^j is at most m.

Our aim in this section is to show that the space X is of type $\left(\frac{1}{2}+\frac{1}{q}\right)^{-1} - \varepsilon$ for every $\varepsilon > 0$. In other words if we put

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{q}, \tag{3.5}$$

then for every $\varepsilon > 0$ there is a constant $C(\varepsilon)$ so that for any choice of $\{x_i\}_{i=1}^{k}$ in X we have

$$\int_0^1 \|x_1 r_1(t) + x_2 r_2(t) + \ldots + x_k r_k(t)\|\, dt \leqslant C(\varepsilon)\Big(\sum_{i=1}^{k} \|x_i\|^{p-\varepsilon}\Big)^{1/(p-\varepsilon)} \tag{3.6}$$

where $r_i(t)$ denote the usual Rademacher functions on $[0, 1]$. The integral appearing on the left-hand side is nothing but the average of $\|\sum_{i=1}^{k} \theta_i x_i\|$ taken over all the possible 2^k choices of signs θ_i (i.e. $\theta_i = \pm 1$).

As observed in [5] it is enough to prove (3.6) in the case where $\|x_1\| = \|x_2\| = \ldots = \|x_k\| = 1$. In this case (3.6) takes the form

$$\int_0^1 \Big\|\sum_{i=1}^{k} x_i r_i(t)\Big\|\, dt \leqslant C(\varepsilon) k^{1/(p-\varepsilon)}. \tag{3.7}$$

Another simple reduction is obtained by remarking that it is possible to replace in the assumption $\|\ \|$ by ϱ. That is, it is enough to show that if $\{x_i\}_{i=1}^k \in X$ with $\varrho(x_i) = 1$ for all i then (3.7) holds. Indeed, if $\{x_i\}_{i=1}^k \in X$ with $\|x_i\| = 1$ for all i, then for every $\delta < 0$ there are $\{y_i^j\}_{j=1}^{n_i}$ so that $x_i = \sum_j y_i^j$ and $\sum_j \varrho(y_i^j) \leqslant 1+\delta$. Since ϱ is positively homogeneous, there is no loss of generality to assume that $n_1 = n_2 = \ldots = n_k = n$, say, and $\varrho(y_1^j) = \varrho(y_2^j) = \ldots = \varrho(y_k^j) = \varrho^j$, say. Thus if we can show that for every j

$$(3.8)\qquad \int_0^1 \Big\| \sum_{i=1}^k y_i^j r_i(t) \Big\| \, dt \leqslant C(\varepsilon)\, \varrho^j k^{1/(p-\varepsilon)}$$

we get that also (3.7) holds by the triangle inequality for $\|\cdot\|$, the fact that $\sum_j \varrho^j \leqslant 1+\delta$ and since $\delta > 0$ was arbitrary.

We pass to the proof of (3.7) under the assumption that $\varrho(x_i) = 1$ for every i. In order not to complicate the notations in the proof by using again an arbitrary $\delta > 0$ we begin by observing that the inf in (3.1) is actually attained (we can restrict all vectors to the support of y which is finite). Hence for every i we have a representation $x_i = \sum_{j=1}^{n_i} z_i^j$ in which the infimmum in (3.1) is attained. In view of the telescopic nature of the right-hand side of (3.1) this side does not change if we break up a term z^j which is a sum of m_j disjoint bumps of magnitude h_j into two sums of m_j disjoint bumps of magnitude, h_j and $(1-\lambda)h_j$, respectively, where $0<\lambda<1$. Also if we agree to allow in (3.1) dummy summands (i.e. z^j for which $m_j = 0$) then we see that without loss of generality we may assume that all the n_i are equal and that for a fixed j the magnitude of the bumps in z_i^j are all equal to h_j, say. Hence we have

$$(3.9)\qquad x_i = \sum_{j=1}^n z_i^j, \quad 1 = \sum_{j=1}^n m_i^j d_j, \quad 1 \leqslant i \leqslant k,$$

where

$$(3.10)\qquad d_j = \Big(H - \sum_{l=1}^{j-1} h_l\Big)^q - \Big(H - \sum_{l=1}^{j} h_l\Big)^q$$

and z_i^j is the sum of m_i^j disjoint bumps of magnitude h_j.

Fix now an integer $1 \leqslant j \leqslant n$ and consider the k vectors $\{z_i^j\}_{i=1}^k$. Each z_i^j determines $2m_i^j$ integers which are the end points of the bumps appearing in z_i^j. The number of all the end points we get by considering all the k vectors is thus at most $2\sum_{i=1}^k m_i^j$ (the actual number may be smaller since the same integer may serve as an end point of a bump in $z_{i_1}^j$ and $z_{i_2}^j$ with $i_1 \neq i_2$). Thus the set N of integers contains m_j disjoint intervals with

$$(3.11)\qquad m_j \leqslant 2\sum_{i=1}^k m_i^j$$

so that on each of these intervals every one of the vectors z_i^j takes a constant value (which is either $+h_j$ or $-h_j$ or 0) and outside these intervals all the $\{z_i^j\}_{i=1}^k$ vanish. Consider now one of those m_j intervals and consider the value of $\sum_{i=1}^k z_i^j r_i(t)$ on this interval. By a basic fact concerning the Bernoulli distribution we get that for every $\varepsilon > 0$ there is a function $f(k, \varepsilon)$ (depending only on k and ε) so that

$$(3.12)\qquad m\Big\{t;\ \Big|\sum_{i=1}^k z_i^j r_i(t)\Big| > h_j k^{1/2+\varepsilon} \text{ on the given interval}\Big\} \leqslant f(\varepsilon, k),$$

where

$$(3.13)\qquad \lim_{k\to\infty} f(\varepsilon, k) k^s = 0 \quad \text{for every fixed } \varepsilon > 0 \text{ and integer } s,$$

and $m\{\ \}$ denotes the Lesbegue measure on $[0,1]$ (in fact, by the central limit theorem $f(\varepsilon, k)$ is of the order of magnitude of $e^{-k^{2\varepsilon}}$).

By using a trivial (i.e. discrete) version of Fubini's theorem, we get from (3.12) that if

$$(3.14)\qquad A_j = \Big\{t;\ \text{the number of intervals, among the } m_j \text{ intervals, we singled out, on which } \Big|\sum_{i=1}^k z_i^j r_i(t)\Big| > h_j k^{1/2+\varepsilon} \text{ is larger than } m_j\sqrt{f(\varepsilon, k)}\Big\}$$

then

$$(3.15)\qquad m(A_j) \leqslant \sqrt{f(\varepsilon, k)}.$$

For each $t \in [0,1]$ we define now vectors $\{u_{s,j,t}\}_{s=1}^l$, $\{v_{i,j,t}\}_{i=1}^k$ and $\{w_{i,j,t}\}_{i=1}^k$ so that

$$(3.16)\qquad l = [k^{1/2+\varepsilon}],$$

$$(3.17)\qquad \sum_{i=1}^k z_i^j r_i(t) = \sum_{s=1}^l u_{s,j,t} + \sum_{i=1}^k v_{i,j,t} + \sum_{i=1}^k w_{i,j,t}$$

for every $1 \leqslant j \leqslant n$ and every $t \in [0,1]$.

These vectors are defined as follows. If $t \in A_j$, we put

$$u_{s,j,t} = v_{i,j,t} = 0 \quad \text{and} \quad w_{i,j,t} = z_i^j \cdot r_i(t).$$

If $t \notin A_j$, we put $w_{i,j,t} = 0$ and in order to define the other vectors we recall that by the definition of A_j we may write

$$(3.18)\qquad \sum_{i=1}^k z_j^j r_i(t) = u + v$$

where u and v are disjointly supported, at every point the value of u and v is an integer (positive or negative) multiple of h_j, the absolute value of u is at most lh_j while that of v is at most $k\cdot h_j$. Moreover, the integers contain at most m_j (resp. $\sqrt{f(k,\varepsilon)}\cdot m_j$) intervals on each of which u (resp. v) is constant and outside which u (resp. v) vanish. The vectors $u_{s,j,t}$ (resp. $v_{i,j,t}$) are now obtained by decomposing u (resp. v) into a sum of vectors, each vector being a disjoint union of bumps of magnitude h_j, by the procedure explained in (3.3) and (3.4). Summing up we have that each one of the vectors $u_{s,j,t}$, $v_{i,j,t}$ and $w_{i,j,t}$ is a disjoint union of bumps of magnitude h_j, and

$$(3.19)\qquad \begin{cases}\text{The number of bumps in } u_{s,j,t} \text{ is } \leqslant m_j,\\ \text{The number of bumps in } v_{i,j,t} \text{ is } \leqslant \sqrt{f(k,\varepsilon)}\cdot m_j,\\ \text{The number of bumps in } w_{i,j,t} \text{ is } 1_{A_j}(t)\cdot m_i^j\end{cases}$$

(1_A denotes the characteristic function of the set A).

Put now

$$(3.20)\qquad u_{s,t}=\sum_{j=1}^n u_{s,j,t},\qquad v_{i,t}=\sum_{j=1}^n v_{i,j,t},\qquad w_{i,t}=\sum_{j=1}^n w_{i,j,t}.$$

Then by (3.9), (3.17) and (3.20),

$$(3.21)\qquad \begin{aligned}\sum_{i=1}^k x_i r_i(t) &= \sum_{i=1}^k\sum_{j=1}^n z_i^j r_i(t)\\ &= \sum_{j=1}^n\Big(\sum_{s=1}^l u_{s,j,t}+\sum_{i=1}^k v_{i,j,t}+\sum_{i=1}^k w_{i,j,t}\Big)\\ &= \sum_{s=1}^l u_{s,t}+\sum_{i=1}^k v_{i,t}+\sum_{i=1}^k w_{i,t}.\end{aligned}$$

Hence

$$(3.22)\qquad \Big\|\sum_{i=1}^k x_i r_i(t)\Big\| \leqslant \sum_{j=1}^l \varrho(u_{s,t})+\sum_{i=1}^k \varrho(v_{i,t})+\sum_{i=1}^k \varrho(w_{i,t}).$$

For a fixed integer s we have by (3.1), (3.9), (3.11), (3.19) and (3.20) that

$$(3.23)\qquad \varrho(u_{s,t}) \leqslant \Big(\sum_{j=1}^n m_j d_j\Big)^{1/q} \leqslant \Big(2\sum_{i=1}^k\sum_{j=1}^n m_i^j d_j\Big)^{1/q} \leqslant (2k)^{1/q}.$$

Also for a fixed i we have by the same equations that

$$(3.24)\qquad \varrho(v_{i,t}) \leqslant \Big(\sum_{j=1}^n \sqrt{f(k,\varepsilon)}\, m_j d_j\Big)^{1/q} \leqslant f(k,\varepsilon)^{1/2q}\cdot(2k)^{1/q}.$$

Hence by (3.13), (3.16), (3.23) and (3.24) we have for every $t\in[0,1]$

$$(3.25)\qquad \sum_{s=1}^l \varrho(u_{s,t})+\sum_{i=1}^k \varrho(v_{i,t}) \leqslant (2k)^{1/q}[k^{1/2+\varepsilon}+kf(k,\varepsilon)]$$

which is of the order of magnitude required in (3.7) (recall (3.5), the ε in (3.25) is not necessarily the same as the one in (3.7)).

In order to complete the proof of (3.7) we have to show that also the third term in the right-hand side of (3.22) is small. This term is not small for every t, only its integral over $[0,1]$ is small but this is exactly what (3.7) requires. We have by (3.1), (3.9), (3.15), (3.19) and (3.20) that

$$(3.21)\qquad \int_0^1\sum_{i=1}^k \varrho(w_{i,t})\,dt \leqslant \sum_{i=1}^k\int_0^1\Big(\sum_{j=1}^n 1_{A_j}(t)\cdot m_i^j d_j\Big)^{1/q}dt$$

$$\leqslant \sum_{i=1}^k\Big(\int_0^1\sum_{j=1}^n 1_{A_j}(t)\, m_i^j d_j\,dt\Big)^{1/q} \leqslant \sum_{i=1}^k\Big(\sum_{j=1}^n m_i^j d_j\Big)^{1/q}\cdot f(k,\varepsilon)^{1/2q} \leqslant k\cdot f(k,\varepsilon)^{1/2q}$$

and this completes the proof of (3.7). Thus we proved the following

THEOREM 3. *For every $p<2$ there is a non-superreflexive Banach space of type p.*

Remarks. (1) As we mentioned in the introduction it is an open problem whether there is a non-superreflexive space of type 2 (cf. [6]).

(2) The point where the proof of the theorem fails if one tries to carry it over to the 2-structure situation is the following. Suppose we have functions $\{x_j\}_{j=1}^k$ defined on $N\times N$ each consisting of m_j disjoint rectangular bumps of magnitude 1. By considering a sum of the form $\sum_{j=1}^k \pm x_j$ and decomposing it into a sum $\sum_{j=1}^k z^j$ by using (3.3) then the number of disjoint rectangular bumps appearing in each z^j may be of the order of magnitude of $\big(\sum_{j=1}^k m_j\big)^2$ instead of $\sum_{j=1}^k m_j$. This fact will ruin the estimates in the case where $\sum_j^k m_j \gg k$. In other words, there seems to be no way to decompose economically a sum of the form $\sum_{j=1}^k \pm x_j$ into another sum of disjoint bumps even in situations where we know that $\sup\big|\sum_{j=1}^k \pm x_j\big|$ is much smaller than k (i.e. if the order of magnitude of $k^{1/2}$). The possibility of doing just this in the one-dimensional case was the key to our proof of Theorem 3. As a matter of fact, the authors seem to have convinced themselves (however without writing down a detailed proof) that if we use the space defined in this section by changing just N to $N\times N$ and replacing the word "interval" by "rectangle" then this space will contain for any q almost isometric copies of l_1^k for every k.

4. Transfinite duals. In this section we show that if X is separable but not reflexive then there is an even ordinal α such that X^α is separable but $(X^\alpha)^{**} = X^{\alpha+2}$ is no longer separable. The interest in such a result stems from the fact that by the principle of local reflexivity X^α is finitely represented in X. Thus, while the results of [1] and the example in [3] (or Section 3 above) show that in general it is not possible to effect the reflexivity of X^{**}/X by passing to spaces finitely represented in X, we can, by simply using transfinite duals, change the density character of X^{**}/X (provided of course that X is not reflexive). The definition of the transfinite duals was given already in the introduction. Let us just recall that we always consider a space X as embedded canonically in X^{**}. Hence for every pair of even ordinals $\alpha<\beta$ there is a canonical embedding of X^α in X^β. The same is true for a pair of odd ordinals (for an even ordinal α we define $X^{\alpha+1} = (X^\alpha)^*$. Limit ordinals are for our purposes even but not odd ordinals). If $x \in X^\alpha$ with α even and $f \in X^\beta$ with β odd the evaluation of f at x is well defined (i.e. in view of our identifications it depends just on x and f but not on the particular choice of α and β).

THEOREM 4. *Let X be a non-reflexive Banach space. Then X^{ω^2+2} is non-separable.*

Proof. We shall prove the theorem by assigning to every point σ in the Cantor set Δ a point x_σ in X^{ω^2+2} so that $\|x_{\sigma_1}-x_{\sigma_2}\| \geqslant 1/2$ for every $\sigma_1 \neq \sigma_2$.

The starting point of the construction is the well-known fact that for a non-reflexive space X there is a sequence $\{x_n\}_{n=1}^\infty$ in X and $\{f_n\}_{n=1}^\infty$ in X^* such that

$$\|x_n\|,\ \|f_n\| \leqslant 2 \text{ for all } n, \qquad f_n(x_m) = \begin{cases} 1, & m \geqslant n, \\ 0, & m < n. \end{cases} \tag{4.1}$$

Starting with the points $\{x_n\}$ we shall construct inductively for every pair (σ, n), where σ is an end point of the Cantor set (i.e. $\sigma \in \Delta$ and $\sigma = j/3^k$) and n an integer, a point $x_{(\sigma,n)}$ of norm $\leqslant 2$ in a certain even conjugate of X. In order to describe the properties of the points we choose, we first introduce some notation. If $\sigma = j/3^k$ is an end point of Δ and j is prime to 3, we call k the *lewel of σ* and denote this by $k = l(\sigma)$. Next we introduce a linear order relation $<$ on the sets of paris (σ, n); we consider

$$(\sigma_1, n_1) < (\sigma_2, n_2) \quad \text{if} \quad \sigma_1 < \sigma_2 \text{ and } n_1, n_2 \text{ arbitrary};$$

if $\sigma_1 = \sigma_2 = j/3^k$, then

$$(j/3^k, n_1) < (j/3^k, n_2) \begin{cases} \text{if } n_1 < n_2 \text{ and } j \text{ even}, \\ \text{if } n_1 > n_2 \text{ and } j \text{ odd}. \end{cases}$$

The points $x_{(\sigma,n)}$ will be constructed by induction on the level of σ. All the points $x_{(\sigma,n)}$ with $l(\sigma) \leqslant k$ will be contained in $X^{(k+1)\omega}$. The essential property of these points will be that for every (σ, n) with $l(\sigma) \leqslant k$ there is an $f_{(\sigma,n)} \in (X^{(k+1)\omega})^* = X^{(k+1)\omega+1}$ so that $\|f_{(\sigma,n)}\| \leqslant 2$ and

$$f_{(\sigma,n)}(x_{(\tau,m)}) = \begin{cases} 0 & \text{if} \quad (\tau, m) < (\sigma, n), \\ 1 & \text{if} \quad (\tau, m) \geqslant (\sigma, n) \end{cases} \tag{4.2}$$

for every pair (τ, m) with $l(\tau) \leqslant k$.

We start the inductive construction with level 0, i.e. with $\sigma = 0$ or $\sigma = 1$.

We put $x_{(0,n)} = x_n$ for every integer n. As $x_{(1,n)}$ we choose any point in X^{2n} which is a limit point with respect to the w^* topology induced by X^{2n-1} of the sequence $\{x_m\}_{m=1}^\infty$. By definition, the pairs $(0, n)$ and $(1, n)$ are ordered as follows: $(0, n) < (1, m)$ for all n and m, $(0, n) < (0, m)$ if $n < m$ and $(1, n) > (1, m)$ if $n < m$. It is clear that the points we constructed are all of norm $\leqslant 2$. We have to construct the functionals $f_{(\sigma,n)}$. As $f_{(\sigma,n)}$ we can simply take the given functionals f_n. As $f_{(0,n)}$ we take any limit point in X^{2n+1} in the w^* topology induced by X^{2n} of the sequence $\{f_m\}_{m=1}^\infty$. It is easily verified that with these definitions (4.2) is satisfied for $k = 0$. Next we define the points $x_{(\sigma,n)}$ with $l(\sigma) = 1$, i.e. for $\sigma = 1/3$ and $\sigma = 2/3$. We put

$$x_{(1/3,n)} = \text{a } w^* \text{ limit point in } X^{\omega+2n} \text{ of } \{x_{(0,m)}\}_{m=1}^\infty,$$
$$x_{(2/3,n)} = \text{a } w^* \text{ limit point in } X^{\omega+2n} \text{ of } \{x_{(1,m)}\}_{m=1}^\infty.$$

The limit points are with respect to the w^* topology induced by $X^{\omega+2n-1}$.

In general the definition is as follows. Let $l(\sigma) = k$, i.e. $\sigma = j/3^k$ with j prime to 3 (and as always σ belongs to the Cantor set Δ). Put

$$x_{(\sigma,n)} = \text{a } w^* \text{ limit point in } X^{k\omega+2n} \text{ of } \{x_{(\sigma^*,m)}\}_{m=1}^\infty \tag{4.3}$$

where $\sigma^* = (j-1)/3^k$ if j is odd and $\sigma^* = (j+1)/3^k$ if j is even. (Observe that with choice of σ^* we have that $\sigma^* \in \Delta$ and $l(\sigma^*) < k$ so the right-hand side of (4.3) makes sense.) It is clear that the $x_{\sigma,n}$ have all norm $\leqslant 2$. The functionals $f_{(\sigma,n)}$ are defined inductively in a similar way. We put

$$f_{(\sigma,n)} = \text{a } w^* \text{ limit point in } X^{k\omega+2n+1} \text{ of } \{f_{(\sigma^*,m)}\}_{m=1}^\infty. \tag{4.4}$$

It is trivial to verify by induction on k that with the definitions (4.3) and (4.4) the relation (4.2) holds.

Having constructed the $x_{(\sigma,n)}$ in X^{ω^2} for every end point $\sigma \in \Delta$ and every integer n we define now points x_σ in X^{ω^2+2} *for every* $\sigma \in \Delta$ as follows. For any such σ pick a sequence $\{\sigma_j\}_{j=1}^\infty$ of end points in Δ so that $\sigma_j \to \sigma$ and let x_σ be any w^* limit point (with respect to the topology induced by X^{ω^2+1}) of the sequence $\{x_{(\sigma_j,1)}\}_{j=1}^\infty$. Similarly let $f_\sigma \in X^{\omega^2+3}$ be any w^*

limit point (with respect to the topology induced by X^{ω^2+2}) of $\{f_{(\sigma_j,1)}\}_{j=1}^{\infty}$. Then

$$\|x_\sigma\|,\ \|f_\sigma\| \leqslant 2, \quad f_\sigma(x_\tau) = \begin{cases} 1 & \text{if } \tau > \sigma, \\ 0 & \text{if } \tau < \sigma. \end{cases} \tag{4.5}$$

In particular, $\|x_{\sigma_1} - x_{\sigma_2}\| \geqslant 1/2$ for every $\sigma_1 \neq \sigma_2$, and this concludes the proof of the theorem.

COROLLARY 1. *For every separable non-reflexive Banach space X there is an ordinal α ($\alpha \leqslant \omega^2$) so that X^α is separable but $X^{\alpha+2}$ is non-separable.*

Proof. Let β be the first even ordinal so that X^β is non-separable. Then $\beta \leqslant \omega^2+2$ and β cannot be a limit ordinal. Hence $\beta = \alpha+2$ and this α has the desired property.

COROLLARY 2. *For every non-reflexive Banach space X the quotient space $X^{\omega^2+2}/X^{\omega^2}$ is non-separable.*

Proof. Use Corollary 1, the fact that if $Y \subset X$ then Y^{**}/Y is isomorphic to a subspace of X^{**}/X and that every non-reflexive space has a separable non-reflexive subspace.

It was observed in [1] that if J is the classical example of James for a quasireflexive space then J^{ω^2} is separable. This shows that the ordinals appearing in Theorem 4 and its corollaries are the best possible (i.e. cannot be replaced in general by smaller ordinals).

Added in proof: J. Farahat recently extended the result of Section 3 by proving that, for every integer k and every $p<2$, there is a space with k-structure and type p. Hence, for every k, there is a space with k-structure which does not have $k+1$-structure.

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On the best constants in the Khinchin inequality*

by

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Abstract. Let (r_j) denote the sequence of Rademacher functions. It is shown that

$$\int_0^1 \Big|\sum_{j=1}^{\infty} c_j r_j(t)\Big|\,dt > \frac{1}{\sqrt{2}}\Big(\sum_{j=1}^{\infty}|c_j|^2\Big)^{1/2}$$

or every square summable sequence of scalars (c_j). The constant $1/\sqrt{2}$ is the best the largest) possible.

1. Introduction. Let r_n denote the nth Rademacher function, i.e.

$$r_n(t) = \operatorname{sign}\sin 2^n\pi t \quad \text{for} \quad 0 \leqslant t \leqslant 1 \quad (n = 1, 2, \ldots).$$

The classical Khinchin inequality states that, for every $p \in [1, \infty)$, there exist positive constants a_p and b_p such that, for every finite sequence of scalars (c_j),

$$a_p\Big(\sum_j |c_j|^2\Big)^{1/2} \leqslant \Big(\int_0^1 \Big|\sum_j c_j r_j(t)\Big|^p dt\Big)^{1/p} \leqslant b_p\Big(\sum_j |c_j|^2\Big)^{1/2}. \tag{0}$$

Let us denote by A_p and B_p, respectively, the largest a_p and the smallest b_p satisfying (0). B. Tomaszewski has observed that the values of A_p and B_p are independent of the choice of the scalar field, i.e. they are the same for real sequences as well as for complex sequences (cf. also Remark 3 in Section 3).

Therefore in the sequel we shall consider inequality (0) for real sequences only.

Obviously, $A_p = 1$ for $p \geqslant 2$ and $B_p = 1$ for $1 \leqslant p \leqslant 2$. Stečkin [6] has shown that

$$B_{2m} = \big((2m-1)!!\big)^{1/2m} \quad \text{for} \quad m = 1, 2, 3, \ldots$$

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