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An equivalent measure for some nonsingular transformations and application

by

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Abstract. Let (Ω, a, μ) be a finite measure space and φ a nonsingular transformation on (Ω, a, μ), i.e., $\mu(A) = 0 \Rightarrow \mu(\varphi^{-1}(A)) = 0$. We characterize those transformations for which the pointwise ergodic theorem holds in L^p, $1 \leqslant p < \infty$, by a condition (M_p). This extends results of C. Ryll-Nardzewski and S. Gładysz. The condition (M_p) also characterizes those invertible transformations for which Kingman's subadditive theorem holds. An example is given showing the importance of the invertibility assumption.

Introduction. Let (Ω, a, μ) be a finite measure space, and φ a nonsingular transformation on (Ω, a, μ), i.e., $\mu(A) = 0 \Rightarrow \mu(\varphi^{-1}(A)) = 0$. If φ is invertible then we will assume that $\mu(A) = 0 \Leftrightarrow \mu(\varphi^{-1}(A)) = 0$. For $1 \leqslant p < \infty$ we assume that the operator T defined by $Tf = f \circ \varphi$ maps $L^p(\mu)$ into $L^p(\mu)$. We say that T *satisfies the pointwise ergodic theorem* (P.E.T.) in $L^p(\mu)$ if for any $f \in L^p(\mu)$ there exists $f^* \in L^p(\mu)$ such that $M_n(T)f$ converges a.e. to f^* where

$$M_n(T)f = (f + Tf + \ldots + T^{n-1}f)/n.$$

We first prove that if φ satisfies the condition

(i) $\exists k > 0$ such that for any $A \in a$

$$\limsup_n \frac{\mu(A) + \mu(\varphi^{-1}(A)) + \ldots + \mu(\varphi^{-n}(A))}{n+1} \leqslant k(\mu(A))^{1/p}$$

then there exists a finite invariant measure m absolutely continuous with respect to μ. If φ is invertible then m is equivalent to μ. When φ is invertible the last result also follows from a theorem of Y. N. Dowker [3] (see also K. Jacobs [7, p. 99]).

We will use our proof to show that (for invertible φ) T satisfies the P.E.T. in $L^p(\mu)$ if and only if

$$(M_p) \quad \begin{cases} \text{(i) as above,} \\ \text{(ii) } \liminf_n M_n(T^*)h \in L^q \text{ for all } h \in L^q_+(\mu) \end{cases}$$

(where $1/p + 1/q = 1$).

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The condition (M_1) is equivalent to the condition (H) of S. Hartman discussed in [12] ((i) $\Rightarrow$ (ii) in this case). The example given in [12] can also be used to prove that (M_p) does not imply that T is mean bounded in $L^p(\mu)$, i.e., $\sup_n \|M_n(T)\|_p < \infty$. It is not difficult to see that if T is mean bounded in $L^p(\mu)$ then T satisfies (M_p). Hence our characterization extends Theorem 1 in [12] and generalizes an earlier result obtained in [1], in the invertible case. Examples 1 and 2 of S. Gładysz [4] show that conditions (i) or (ii) cannot be dropped. We also improve part of S. Gładysz's result in Theorem 4.

Under the condition (M_p) the system $(\Omega, a, \mu, \varphi)$ is not only A.M.S. (asymptotically mean stationary in the terminology of [5]), i.e.,

$$m(A) = \lim_n \frac{\mu(A)+\mu(\varphi^{-1}(A))+ \ldots +\mu(\varphi^{-n}(A))}{n+1}$$

exists for all $A \in a$, but also the Radon–Nikodym derivative $dm/d\mu = v_0^*$ belongs to $L^p(\mu)$.

We give an application of this. It deals with the subadditive ergodic theorem of J. F. C. Kingman [8], [9]. We show that (M_p) characterizes also those invertible nonsingular transformations for which for any subadditive sequence (f_n), i.e. $f_{n+m} \leqslant f_n + f_m \circ \varphi^n$ for all positive integers n, m, we have the pointwise convergence of f_n/n in $L^p(\mu)$.

At the end of this paper we give an example of a continuous function φ on [0, 1] which induces a noninvertible transformation such that the operator T, $Tf = f\circ\varphi$, satisfies $\sup_{n\geqslant 0} \|T_n\|_1 \leqslant 2$. This transformation does not have a finite invariant measure ν equivalent to μ, and the operator T satisfies (M_p) for all $1 \leqslant p < \infty$. This example is also used to show that there exists a subadditive sequence f_n such that $\sup_n \|f_n/n\| < \infty$ and f_n/n does not converge a.e.

Remarks. W. Rechard [11, p. 486] and Y. Ito [6, p. 178] had also examples showing the importance of the invertibility assumption in Theorem 1 for the existence of a finite invariant equivalent measure.

We would like to thank F. J. Martín-Reyes for helping us simplify our original example for which we had the same conclusions but the power boundedness of T was more delicate to prove.

The results

THEOREM 1. *Let (Ω, a, μ) be a finite measure space and φ a nonsingular transformation for which the operator $Tf = f\circ\varphi$ maps $L^p(\mu)$ into $L^p(\mu)$, $1 < p < \infty$. If φ satisfies the condition*

(i) *$\exists k > 0$ such that for any $A \in a$*

$$\limsup_n \frac{\mu(A)+\mu(\varphi^{-1}(A))+ \ldots +\mu(\varphi^{-n}(A))}{n+1} \leqslant k(\mu(A))^{1/p}$$

then there exists a finite invariant measure m absolutely continuous with respect

to μ. If φ is invertible then m is given by

$$m(A) = \lim_n \frac{\mu(A)+\mu(\varphi^{-1}(A))+ \ldots +\mu(\varphi^{-n}(A))}{n+1}.$$

Proof. Let LIM be a Banach limit, i.e., a functional on a set of bounded sequences such that if $\{a_k\}$ and $\{b_k\}$ are bounded sequences we have

$$\liminf_k a_k \leqslant \mathrm{LIM}(\{a_k\}) \leqslant \limsup_k a_k, \tag{1}$$

$$\text{if } \{v_k\} = \{a_{k+1}\} \text{ then } \mathrm{LIM}(\{v_k\}) = \mathrm{LIM}(\{a_k\}), \tag{2}$$

$$\mathrm{LIM}(\alpha\{a_k\}+\beta\{b_k\}) = \alpha\,\mathrm{LIM}(\{a_k\})+\beta\,\mathrm{LIM}(\{b_k\}). \tag{3}$$

We then define for each set $A \in a$

$$m(A) = \mathrm{LIM}\frac{\mu(A)+ \ldots +\mu(\varphi^{-n}(A))}{n+1}.$$

By (1)–(3) and (i) we can see that m is a finite invariant countably additive measure absolutely continuous with respect to μ.

Let $v_0^* = dm/d\mu$ and $C = \operatorname{supp} v_0^*$, $D = \Omega \setminus C$. As in [1] we have *a priori* the following property:

(a) for almost all $\omega \in C$, $\varphi(\omega) \in C$,

(b) for almost all $\omega \in D$, $\exists n(\omega)$ such that $\varphi^{n(\omega)}(\omega) \in C$.

This is because if on a set $A \subset C$ with $\mu(A) > 0$ (and so $m(A) > 0$) we have $\varphi(A) \subset D$, then we would have $A \subset \varphi^{-1}(D)$ and $0 < m(A) \leqslant m(\varphi^{-1}(D)) = 0$. Next, if (b) were false there would be a set $A \subset \varphi^{-1}(D)$ with $\mu(A) > 0$ such that $\varphi^n(A) \subset D$ for all positive integers n. Then $\mu(A) \leqslant \mu(\varphi^{-n}(D))$ and

$$0 < \mu(A) \leqslant \liminf \frac{\mu(D)+ \ldots +\mu(\varphi^{-n}(D))}{n+1} \leqslant m(D) = 0.$$

So (a) and (b) are proved.

By using Hopf's decomposition (see [10], p. 17), Ω can be decomposed in two disjoint parts C_1 and D_1: on C_1, φ is conservative and $D_1 = \bigcup_{i=-\infty}^{\infty} \varphi^i(w)$ where w is a wandering set. A measure preserving transformation being conservative, we must have $C_1 \subset C$ (μ-a.e.) and so $D_1 \supset D$. But by (b) after some iterates all points of D should return to C. This is a contradiction as all points of D_1 stay in D_1. So we have proved that $\mu(D) = 0$ and m is equivalent to μ. It remains to show that

$$\lim_n \frac{\mu(A)+ \ldots +\mu(\varphi^{-n}(A))}{n+1}$$

exists and equals $m(A)$. We observe that T^* is an L^1-positive contraction, i.e., $T^*f \geqslant 0$ if $f \geqslant 0$ and $\|T^*\|_1 \leqslant 1$. As $T^*(v_0^*) = v_0^*$, $v_0^* > 0$, m being invariant,

T^* satisfies the mean ergodic theorem in $L^1(\mu)$. (The set of functions $f\in L^1(\mu)$ such that $f\leqslant Mv_0^*$, $M>0$, $M\in\mathbf{R}$, is dense in $L^1(\mu)$ and the sequence $M_n(T^*)f$ is then uniformly integrable.) Hence $\lim M_n(T^*)1=h_0^*$ exists in L^1 norm. So for all $A\in a$ we have

$$\lim_n \int 1_A M_n(T^*)1\,d\mu=\lim_n\int\frac{1_A+\ldots+1_A\circ\varphi_n}{n+1}\,d\mu=\int 1_A h_0^*\,d\mu.$$

In particular,

$$\liminf\frac{\mu(A)+\ldots+\mu(\varphi^{-n}(A))}{n+1}=\limsup\frac{\mu(A)+\ldots+\mu(\varphi^{-n}(A))}{n+1}$$
$$=m(A)\quad\text{by (1)}$$
$$=\int_A v_0^*\,d\mu.$$

Hence $v_0^*=h_0^*$. ■

Remark. The assumption $T\colon L^p\to L^p$ is not necessary in the proof: we can use the fact that T^* can be extended to all nonnegative functions.

THEOREM 2. *Let (Ω, a, μ) be a finite measure space and φ an invertible nonsingular transformation for which the operator $Tf=f\circ\varphi$ maps $L^p(\mu)$ into $L^p(\mu)$, $1\leqslant p<\infty$. The following are equivalent:*

(A) *T satisfies the pointwise ergodic theorem in $L^p(\mu)$.*

(B) *φ satisfies the following condition (M_p):*

$$(M_p)\quad\begin{cases}\text{(i) as in Theorem 1,}\\ \text{(ii) } \liminf\limits_n M_n(T^*)(g)\in L^q(\mu)\quad\text{for all } g\in L^q_+(\mu)\end{cases}$$

(where $1/p+1/q=1$).

Proof. We can assume that $\mu(\Omega)=1$.

(A)$\Rightarrow$(B). If $f^*=$ a.e. $\lim M_n(T)f$ then the operator $R\colon f\to f^*$ is linear, positive and maps $L^p(\mu)$ into $L^p(\mu)$. So there exists a constant K such that $\|f^*\|_p\leqslant K\|f\|_p$ for all $f\in L^p(\mu)$. (i) of (B) follows by taking $f=1_A$; we have

$$\limsup\int\frac{1_A+\ldots+1_A\circ\varphi^n}{n+1}\,d\mu\leqslant\int\limsup\frac{1_A+\ldots+1_A\circ\varphi^n}{n+1}\,d\mu$$
$$\left(\text{because }\frac{1_A+\ldots+1_A\circ\varphi^n}{n+1}\leqslant 1\text{ a.e.}\right)$$
$$\leqslant\left(\int(\limsup\ldots)^p\,d\mu\right)^{1/p}\quad(\text{because }\mu(\Omega)=1)$$
$$\leqslant K\left(\int 1_A\,d\mu\right)^{1/p}.$$

To get (ii), take $f\in L^\infty_+$. We have for $g\in L^q_+(\mu)$

$$\int f\liminf M_n(T^*)(g)\,d\mu\leqslant\liminf\int fM_n(T^*)g\,d\mu\quad\text{(by Fatou's lemma)}$$
$$=\liminf\int gM_n(T)f\,d\mu\leqslant\limsup\int gM_n(T)f\,d\mu$$
$$\leqslant\int\limsup gM_n(T)f\,d\mu\quad(\text{because }\|M_n(T)f\|_\infty\leqslant\|f\|_\infty)$$
$$\leqslant K\|f\|_p\|g\|_q.$$

So if we set $g_0^*=\liminf M_n(T^*)g$ we have $\int fg_0^*\,d\mu\leqslant K\|f\|_p\|g\|_q$. By approximating $f\in L^p(\mu)$ by functions in L^∞ we deduce that $\|g_0^*\|_q\leqslant K\|g\|_q$.

(B)$\Rightarrow$(A). We know by Theorem 1 that because of (i) there exists an equivalent finite invariant measure m given by

$$m(A)=\lim_n\frac{\mu(A)+\ldots+\mu(\varphi^{-n}(A))}{n+1}.$$

Let $v_0^*=dm/d\mu$. We already observed in the proof of Theorem 1 that $v_0^*=\lim M_n(T^*)1$ in L^1 norm. By [2], T^*, satisfying the mean ergodic theorem in L^1, also satisfies the pointwise ergodic theorem. We have $v_0^*=\liminf M_n(T^*)1=\lim M_n(T^*)1\in L^q(\mu)$ by (ii).

Hence for $f\in L^p(\mu)$ we have $f\in L^1(m)$ and by Birkhoff's classical ergodic theorem $M_n(T)f$ converges a.e. to $f^*\in L^1(m)$. It remains to show that $f^*\in L^p(\mu)$.

Let $g\in L^q_+(\mu)$, $f\in L^p_+(\mu)$. We set $A_N=\{\omega\colon\limsup M_n(T^*)g\leqslant N\}$ and $f_k=f\wedge k$. We have

$$\int g\liminf M_n(T)(1_{A_N}f_k)\,d\mu\leqslant\liminf\int gM_n(T)(1_{A_N}f_k)\,d\mu$$
$$\leqslant\limsup\int M_n(T^*)g\cdot 1_{A_N}f_k\,d\mu$$
$$\leqslant\int\limsup M_n(T^*)g\cdot 1_{A_N}g_k\,d\mu$$
$$\leqslant\int g_0^*\cdot 1_{A_N}f_kf\mu.$$

We noticed that $M_n(T^*)g$ converges a.e. for all $g\in L^1(\mu)$. By (ii) we have $\lim M_n(T^*)g\in L^q(\mu)$. So we have, as in the proof of (A)$\Rightarrow$(B), $\|g_0^*\|_q\leqslant M\|g\|_q$ for all $g\in L^q(\mu)$. Thus the last inequality gives us

$$\int g\liminf M_n(T)(1_{A_N}f_k)\,d\mu\leqslant M\|g\|_q\|1_{A_N}f_k\|_p.$$

We denote by $(1_{A_N}f_k)^*$ the pointwise limit of $M_n(T)(1_{A_N}f_k)$. The map $f\in L^p(\mu)\to f^*\in L^1(m)$ being continuous, there exists K such that for all N and k

$$\int v_0^*(1_{A_N}f_k)^*\,d\mu\leqslant K\|1_{A_N}f_k\|_p.$$

We deduce that

$$(1_{A_N}f_k)^*\underset{k}{\to}(1_{A_N}f)\underset{N}{\to}f^*,$$

the convergence being achieved by increasing sequences of functions. So $\int gf^*\,d\mu\leqslant M\|g\|_q\|f\|_p$ and $\|f^*\|_p\leqslant M\|f\|_p$. ■

COROLLARY 3. (I) *If T satisfies the stochastic ergodic theorem in $L^p(\mu)$ then* (i) *holds and T^* satisfies the pointwise ergodic theorem in $L^q(\mu)$.*

(II) *Conversely, if* (i) *holds and if T^* satisfies the stochastic ergodic theorem in $L^q(\mu)$, then T satisfies the pointwise ergodic theorem in $L^p(\mu)$.*

Proof. (I) We can easily see from the proof of Theorem 2 that in (I) "pointwise" can be replaced by "stochastic".

(II) It is known that for all $f \in L^1(\mu)$, $M_n(T^*)f$ converges stochastically and the limit is equal to $\liminf M_n(T^*)f$ for $f \geqslant 0$ (see U. Krengel [10]).

Remark. S. Gładysz gave two examples (1 and 2 in [3]) from which we can conclude that in Theorem 2 part (ii) cannot be dropped in condition (M_p) (for $p = 2$). Moreover, in his Example 1, $dm/d\mu = v_0^* \in L^p(\mu)$ so that $v_0^* \in L^p(\mu)$ does not imply the P.E.T. in $L^p(\mu)$. Furthermore, Example 2 shows that condition (i) itself does not even imply that $\varphi \in L^p$ for $p = 2$.

A part of Gładysz's result can be improved as follows. We do not assume that T maps L^p into L^p.

THEOREM 4. *Let $1 \leqslant p < \infty$ and let φ be a nonsingular transformation on (Ω, a, μ). Then the following are equivalent*:

(i) *The operator $Tf = f \circ \varphi$ satisfies the pointwise ergodic theorem in $L^p(\mu)$.*

(ii) *The system $(\Omega, a, \mu, \varphi)$ is A.M.S. (asymptotically mean stationary), i.e.,*

$$\lim_{n\to\infty} \frac{\mu(A) + \ldots + \mu(\varphi^{-n}(A))}{n+1} = m(A)$$

exists for all $A \in a$, and m is absolutely continuous with respect to μ and $dm/d\mu = v_0^* \in L^q(\mu)$.

(iii) *There exists a constant K such that*

(a) $\limsup\limits_{n} n^{-1} \sum\limits_{k=0}^{n-1} \mu(\varphi^{-k}(A)) \leqslant K(\mu(A))^{1/p}$ *for all* $A \in a$,

(b) $\liminf\limits_{n} M_n(T^*)1 \in L^q(\mu)$.

(iv) *There exists K such that $\limsup n^{-1} \sum_{k=0}^{n-1} \mu(\varphi^{-k}(A)) \leqslant K(\mu(A))^{1/p}$ for all $A \in a$ and T^* satisfies the stochastic (pointwise) ergodic theorem in $L^q(\mu)$.*

Proof. (i) $\Rightarrow$ (ii). If $f^* = \lim M_n(T)f$ a.e. then there exists a constant k such that $\|f^*\|_p \leqslant k\|f\|_p$ for all $f \in L^p(\mu)$.

As $\|T\|_\infty \leqslant 1$ it follows that

$$m(A) = \lim_{n\to\infty} \frac{\mu(A) + \ldots + \mu(\varphi^{-n}(A))}{n+1}$$

exists for all $A \in a$ and $m(A) = \int (1_A)^* d\mu \leqslant k(\mu(A))^{1/p}$. This shows that m is a finite measure absolutely continuous with respect to μ. As $m(A) = m(\varphi^{-1}(A))$, m is invariant and equals μ on φ-invariant sets (i.e., $1_B = 1_{\varphi^{-1}(B)}$). We are going to prove that $M_n(T^*)1$ converges a.e. First, if $C = \operatorname{supp} v_0^*$ we can see as in Theorem 1 that

(α) if $\omega \in C$ then $\varphi(\omega) \in C$,

(β) if $\omega \in \Omega \setminus C = D$ then there exists $n(\omega)$ such that $\varphi^{n(\omega)}(\omega) \in C$.

This implies that $\varphi^{-1}(D) \subset D$ and so for all $g \in L^1(\mu)$, $1_D M_n(T^*)g$ converges a.e. (On the set $B_k = \{\omega\colon 1_D - T^k 1_D > 0\}$ we have

$$\int \sum_{n=0}^{\infty} (1_D - T^k 1_D)(T^{*n} f)\, d\mu < +\infty,$$

see for instance [5]). Now, v_0^* being a fixed point of T^*, the convergence on C also follows (see [10]). Hence $M_n(T^*)1$ converges a.e. and as in Theorem 1 we conclude that $v_0^* = \lim M_n(T^*)1$ a.e.

It remains to prove that $v_0^* \in L^q(\mu)$. This follows by using analogous arguments to those used to establish (A) $\Rightarrow$ (B) in Theorem 2. Starting with $f \in L_+^\infty(\mu)$ we have

$$\int f \lim M_n(T^*)1\, d\mu \leqslant \liminf \int f M_n(T^*)1\, d\mu \leqslant \int \limsup M_n(T) f\, d\mu \leqslant k\|f\|_p.$$

Approximating $f \in L^p(\mu)$ by functions in $L^\infty(\mu)$ we deduce that $v_0^* \in L^q(\mu)$.

(ii) $\Rightarrow$ (iii). (a) is clear because we have $m(A) = \int_A v_0^*\, d\mu \leqslant \|v_0^*\|_q (\mu(A))^{1/p} \leqslant k(\mu(A))^{1/p}$.

(b) For $C = \operatorname{supp} v_0^*$ we have (α) and (β) as in the proof of (i) $\Rightarrow$ (ii), and hence the a.e. convergence of $M_n(T^*)g$ for all $g \in L^1(\mu)$. Therefore $M_n(T^*)1$ converges a.e. to h_0^*. We have for all $A \in a$

$$\int 1_A h_0^*\, d\mu = \int 1_A \liminf M_n(T^*)1\, d\mu \leqslant \liminf \int 1_A M_n(T^*)1\, d\mu = m(A) = \int 1_A v_0^*\, d\mu.$$

So

$$h_0^* \leqslant v_0^*. \tag{1}$$

If $A_N = \{\omega\colon \limsup M_n(T^*)1 \leqslant N\}$ we have $A_N \to \Omega$ and

$$\int 1_{A_N} h_0^*\, d\mu = \int 1_{A_N} \limsup_n M_n(T^*)1\, d\mu \geqslant \limsup_n \int 1_{A_N} M_n(T^*)1\, d\mu = \lim_n \int M_n(T) 1_{A_N}\, d\mu = \int 1_{A_N} v_0^*\, d\mu.$$

So

$$\int h_0^*\, d\mu \geqslant \int 1_{A_N} v_0^*\, d\mu \quad \text{for all } N. \tag{2}$$

(1) and (2) imply that $h_0^* = v_0^*$.

(iii) ⇒ (ii). (a) implies by Theorem 1 that

$$m(A) = \mathrm{LIM}\frac{\mu(A)+\ldots+\mu(\varphi^{-n}(A))}{n+1}$$

is a finite invariant measure such that $m \ll \mu$ and $m = \mu$ on $\mathscr{P}\varphi$, the φ-invariant sets. Again we have a decomposition of the space Ω with the properties (α) and (β). This implies that $M_n(T^*)g$ converges a.e. for all $g \in L^1(\mu)$. If $v_0^* = \lim M_n(T^*)1$ a.e., $h_0^* = dm/d\mu$, $A_N = \{\omega\colon \limsup M_n(T^*)1 \leqslant N\}$ we have $A_n \nearrow \Omega$. First, for all A we have

$$\int 1_A v_0^*\, d\mu \leqslant \liminf \int M_n(T)1_A\, d\mu \leqslant \mathrm{LIM}\frac{\mu(A)+\ldots+\mu(\varphi^{-n}(A))}{n+1} \leqslant \int 1_A h_0^*\, d\mu$$

and $v_0^* \leqslant h_0^*$ a.e.

Then as in the proof of (ii) ⇒ (iii) we have

$$\int 1_{A_N} v_0^*\, d\mu = \int 1_{A_N} \lim_n M_n(T^*)1\, d\mu \geqslant \limsup_n \int 1_{A_N} M_n(T^*)1\, d\mu \geqslant m(A_N) = \int 1_{A_N} h_0^*\, d\mu$$

and so $v_0^* = h_0^* \in L^q(\mu)$.

It remains to show that $(\Omega, a, \mu, \varphi)$ is A.M.S. We have $m(A) = \int 1_A v_0^*\, d\mu \leqslant \liminf \int M_n(T)1_A\, d\mu \leqslant m(A)$ because

$$\liminf \frac{\mu(A)+\ldots+\mu(\varphi^{-n}(A))}{n+1} \leqslant m(A) \leqslant \limsup \ldots$$

Also for all $A \in a$ and $\varepsilon > 0$ there exist $A_\varepsilon \subset A$ with $\mu(A\setminus A_\varepsilon) < \varepsilon$ and $\limsup M_n(T^*)1 \leqslant K_\varepsilon$ on A_ε. Then

$$m(A_\varepsilon) \geqslant \limsup \frac{\mu(A_\varepsilon)+\ldots+\mu(\varphi^{-n}(A_\varepsilon))}{n+1}$$

by analogous arguments to those used previously. So

$$m(A_\varepsilon) = \limsup \frac{\mu(A_\varepsilon)+\ldots+\mu(\varphi^{-n}(A_\varepsilon))}{n+1} \quad \forall \varepsilon$$

$$= \liminf \ldots, \quad A \text{ being fixed.}$$

As $m(A - A_\varepsilon) \underset{\varepsilon}{\to} 0$ and

$$\limsup \frac{\mu(A)+\ldots+\mu(\varphi^{-n}(A))}{n+1}$$

$$= \limsup \left(\frac{\mu(A\setminus A_\varepsilon)+\ldots+\mu(\varphi^{-n}(A\setminus A_\varepsilon))}{n+1} + \frac{\mu(A_\varepsilon)+\ldots+\mu(\varphi^{-n}(A_\varepsilon))}{n+1} \right)$$

$$\leqslant k(\mu(A\setminus A_\varepsilon))^{1/p} + m(A_\varepsilon)$$

we have

$$m(A) \geqslant \limsup \frac{\mu(A)+\ldots+\mu(\varphi^{-n}(A))}{n+1}$$

and hence $(\Omega, a, \mu, \varphi)$ is A.M.S.

(iii) ⇒ (i). (iii) implies (implicitly) that T^* satisfies the pointwise ergodic theorem and $v_0^* = \lim M_n(T^*)1 \in L^q(\mu)$. Hence if $f \in L^p(\mu)$ then $fv_0^* \in L^1(m)$ and Birkhoff's ergodic theorem implies that $M_n(T)f$ converges on $C = \operatorname{supp} v_0^*$. Then we use the decomposition of the space (see (α) and (β)) to deduce that $M_n(T)f$ converges a.e. on D.

(i) ⇒ (iv) is clear from the proof of (i) ⇒ (ii) because $M_n(T^*)g$ converges a.e. in $L^q(\mu)$.

(iv) ⇒ (iii) is also clear. ∎

We now give one application of Theorem 2.

Theorem 5. *Let (Ω, a, μ) be a finite measure space and φ an invertible nonsingular transformation such that the operator defined by $Tf = f\circ\varphi$ maps $L^p(\mu)$ into $L^p(\mu)$, $1 \leqslant p \leqslant \infty$. The following are equivalent:*

(C) *For any subadditive sequence (f_n) (i.e., $f_{n+m} \leqslant f_n + f_m\circ\varphi^n$ for all positive integers n, m) with $f_1^+ \in L^p(\mu)$ there exists a measurable function $f^*\colon \Omega \to \mathbf{R}\cup\{-\infty\}$ such that $(f^*)^+ \in L^p(\mu)$, $Tf^* = f^*$ a.e. and f_n/n converges a.e. to f^*.*

(D) *φ satisfies (M_p).*

Proof. (C) ⇒ (D). We just have to take an additive sequence. Then $f_n = f_1 + f_1\circ\varphi + \ldots + f_1\circ\varphi^n$ $(f_1 \in L^p_+(\mu))$ and f_n/n converges a.e. in $L^p(\mu)$.

(D) ⇒ (C). Let f_n be a subadditive sequence with $(f_1)^+ \in L^p(\mu)$. By Theorem 1 there exists an equivalent measure m, $m(A) = \int_A v_0^*\, d\mu$ with $v_0^* \in L^q(\mu)$.

As $f_1^+ \in L^p(\mu)$ we have $f_1^+ \in L^1(m)$ and by J. F. C. Kingman's theorem ([8], [9]) there exists f^* such that $f^{*+} \in L^1(m)$ and f_n/n converges a.e. to f^*. It remains to show that $f^{*+} \in L^p(\mu)$. This follows from the fact that

$$f_n^+ \leqslant f_1^+ + f_1^+\circ\varphi + \ldots + f_1^+\circ\varphi^n$$

and so $f^{*+} \leqslant \lim M_n(T)(f_1^+)$ which belongs to $L^p(\mu)$ by Theorem 2.

An example. Let φ be the continuous map on $[0, 1]$ defined by

$$\varphi(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ 3/2 - x & \text{if } \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

We then consider the transformation T, $Tf = f\circ\varphi$. We claim that T is power bounded in $L^1(\mu)$, i.e. $\sup_{n\geqslant 0}\|T^n\|_1 < \infty$). In fact, for any $A \in \mathscr{B}[0, 1] \cap [0, \frac{1}{2}]$ we have $\mu(\varphi^{-1}(A)) = \frac{1}{2}\mu(A)$ and so

$$\mu(\varphi^{-n}(A)) = \frac{1}{2^n}\mu(A)$$

for all positive integers n. If $A \subset [\frac{1}{2}, 1]$ then $\varphi^{-1}(A) = B_1 \cup C_1$ where

$B_1 \subset [0, \frac{1}{2}]$, $C_1 \subset [\frac{1}{2}, 1]$ and $\mu(B_1) = \frac{1}{2}\mu(A)$, $\mu(C_1) = \mu(A)$. Then in this case
$$\mu(\varphi^{-1}(A)) = \tfrac{1}{2}\mu(A) + \mu(A).$$
We have $\varphi^{-2}(A) = \varphi^{-1}(B_1) \cup \varphi^{-1}(C_1)$ and $\varphi^{-1}(C_1) = B_2 \cup C_2$ with $\mu(B_2) = \frac{1}{2}\mu(C_1)$ and $\mu(C_2) = \mu(C_1)$. Hence
$$\mu(\varphi^{-2}(A)) = \frac{1}{2^2}\mu(A) + \frac{1}{2}\mu(A) + \mu(A).$$
By recurrence on n we see that
$$\mu(\varphi^{-n}(A)) = \frac{1}{2^n}\mu(A) + \frac{1}{2^{n-1}}\mu(A) + \ldots + \frac{1}{2}\mu(A) + \mu(A) \leqslant 2\mu(A).$$
We conclude that T is power bounded by 2 in $L^1(\mu)$. The mean ergodic theorem (easy to obtain in this case) implies now that
$$m(A) = \int_A v_0^* \, d\mu = \lim \frac{\mu(A) + \ldots + \mu(\varphi^{-n}(A))}{n+1}$$
exists for all A in $\mathscr{B}[0, 1]$. The measure m is not equivalent to μ (Lebesgue measure). Actually, if we take a set A with $\mu(A) > 0$, $A \subset [0, \frac{1}{2}]$, then $\mu(\varphi^{-n}(A)) = 2^{-n}\mu(A) \to 0$. Thus $m(A) = 0$.

In fact, φ does not have a finite invariant measure ν equivalent to μ. This is because the set $E = (\frac{1}{4}, \frac{1}{2})$ is a wandering set: we have $\varphi^{-k}(E) = (1/2^{k+2}, 1/2^{k+1})$,
$$\bigcup_{k=0}^{\infty} \varphi^{-k}(E) = (0, \tfrac{1}{2}), \quad \varphi^{-j}(E) \cap \varphi^{-i}(E) = \emptyset \quad \text{for } i \neq j.$$

The operator T satisfies (M_p) because of the Riesz interpolation theorem. T being also an L^∞-contraction is power bounded in all $L^p(\mu)$, $1 \leqslant p \leqslant \infty$.

Remark. This example shows also that there exist nonconservative endomorphisms φ on Lebesgue measure spaces even if the induced operator T is power bounded in L^1. The conservative part here is $[\frac{1}{2}, 1]$, the remaining part is $(0, \frac{1}{2})$, a union of wandering sets. We will use this remark to get our next result which shows the importance of the invertibility assumption in Theorem 5.

THEOREM 6. *For every $\varepsilon > 0$ there exists a continuous function φ on* $[0, 1]$ *such that the operator T defined by $Tf = f \circ \varphi$ satisfies*

(γ) $\sup \|T^n\|_1 \leqslant 1 + \varepsilon$ *(hence T satisfies (M_p) for all p),*

(δ) *there exists a subadditive sequence (f_n) such that $\sup_n \|f_n/n\|_p < \infty$ and f_n/n does not converge a.e.*

Proof. The transformation is a slight modification of the previous example. We take $1 < a < \infty$ and
$$\varphi(x) = \begin{cases} ax & \text{if } 0 \leqslant x \leqslant 1/a, \\ (1+1/a) - x & \text{if } 1/a \leqslant x \leqslant 1. \end{cases}$$

Simple computations show that $\sup_n \|T^n\|_1 \leqslant a/(a-1)$. Hence it is always possible to select a such that $\sup \|T^n\|_1 \leqslant 1+\varepsilon$. The set $V = (1/a, 1/a^2)$ is wandering and $(0, 1/a) = \bigcup_{i=0}^{\infty} \varphi^{-i}(V)$.

Now we proceed similarly to [5, p. 240]. We take
$$f_n(x) = \begin{cases} -3^{p_n} & \text{if } x \in V \text{ and } 3^{p_n} \leqslant n < 3^{p_n+1}, \\ f_n(\varphi^j(x)) - 3^n & \text{if } x \in \varphi^{-j}(V), \\ 0 & \text{otherwise}. \end{cases}$$

We have
$$(*) \qquad f_{n+k}(x) \leqslant f_n(x) + f_k(\varphi^n(x))$$
for all strictly positive integers n, m and all $x \in (0, 1)$. To see that we distinguish three cases.

1) If $x \notin \bigcup_{i=0}^{\infty} \varphi^{-i}(V)$ then $\varphi^n(x) \notin \bigcup_{i=0}^{\infty} \varphi^{-i}(V)$ for all n since otherwise
$$x \in \bigcup_{i=0}^{\infty} \varphi^{-n-i}(V) \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(V),$$
a contradiction. So $f_n(x) = 0 = f_{n+k}(x) = f_k(\varphi^n(x))$.

2) If $x \in V$ then
$$f_n(x) = -3^{p_n}, \quad f_{n+k}(x) = -3^{p_{n+k}}.$$
Using the fact that $\varphi^{-j}(V) \cap \varphi^{-l}(V) = \emptyset$ if $j \neq l$ we deduce that $\varphi^n(x) \notin \bigcup_{i=0}^{\infty} \varphi^{-i}(V)$ for all positive n. Hence $f_k(\varphi^n(x)) = 0$ and $(*)$ is satisfied.

3) If $x \in \varphi^{-j_0}(V)$ for some j_0 then
$$f_{n+k}(x) = -3^{p_{n+k}} - 3^{n+k}, \quad f_n(x) = f_n(\varphi^{j_0}x) - 3^n = -3^{p_n} - 3^n.$$

- If $n > j_0$ then $f_k(\varphi^n(x)) = 0$ and it is clear that
$$-3^{p_{n+k}} - 3^{n+k} \leqslant -3^{p_n} - 3^n.$$
- If $n \leqslant j_0$ then $f_k(\varphi^n(x)) = -3^{p_k} - 3^k$ and the inequality $(*)$ means
$$3^{p_{n+k}} + 3^{n+k} - 3^{p_n} - 3^n - 3^{p_k} - 3^k \geqslant 0.$$

But $3^{p_n} + 3^n + 3^{p_k} + 3^k \leqslant 2(3^{p_{n^*}} + 3^{n^*})$ where $n^* = \max(n, k)$, and
$$2(3^{p_{n^*}} + 3^{n^*}) \leqslant 2n^* + 2 \cdot 3n^* \leqslant 3^{n^*} + 2 \cdot 3^{n^*} \leqslant 3^{n^*+1} \leqslant 3^{n+k}.$$

Now it remains to show that $\sup_n \|f_n/n\|_p < \infty$ and f_n/n does not converge a.e.

The divergence is clear from the following limits: if $x \in V$ then
$$\lim_n f_{3^n}(x)/3^n = \lim_n -3^n/3^n = -1,$$
$$\lim_n f_{3^n-1}(x)/(3^n-1) = \lim_n -3^{n-1}/(3^n-1) = -1/3.$$
The fact that $\sup_n \|f_n/n\|_p < \infty$ for all p follows from $\|f_n/n\|_\infty \leqslant 3$. ∎

QUESTIONS. 1) Can we drop (i) in the condition (M_p)?

2) What condition on the Radon–Nikodym derivative v_0^* is equivalent to (M_p)?

Janusz Woś is unfortunately no longer among us. This is one of the last papers on which he worked.

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Cross-sections of solution funnels in Banach spaces

by

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Abstract. The present paper applies negligibility theory (a part of infinite-dimensional topology) to study the geometry of the failure of Kneser's theorem in infinite-dimensional Banach spaces. In particular, it turns out that arbitrary compact subsets of the infinite-dimensional separable Hilbert space can be represented as cross-sections of solution funnels. For general infinite-dimensional Banach spaces, the existence of initial value problems with exactly two solutions is proved.

1. Introduction. Let X and Y be Banach spaces. If $U \subset X$ is open and $V \subset Y$, then $C^p(U, V)$ denotes the set of all mappings $f\colon U \to V$ (with domain U) having continuous pth Fréchet derivative, $p = 0, 1, 2, \ldots$ ($C^0(U, V)$ is simply the set of all continuous mappings.) We also let $C^\infty(U, V) = \bigcap \{C^p(U, V) \mid p \in \mathbf{N}\}$. The derivative of $f \in C^p(U, V)$ at $u \in U$ is denoted by $D_x f(u)$. The origin of X is denoted by 0_X.

For $F \in C^0(\mathbf{R} \times X, X)$, consider the ordinary differential equation (ODE)

$$D_t x = F(t, x). \tag{1}$$

For $(t_0, x_0) \in \mathbf{R} \times X$, a function $x \in C^1(I_x, X)$ is called a *solution of* (1) *through* (t_0, x_0) if I_x is an open interval in $\mathbf{R}$ containing t_0, $x(t_0) = x_0$ and $D_t x(u) = F(u, x(u))$ for all $u \in I_x$. Solutions with domain $\mathbf{R}$ are called *global*.

Let $\mathscr{F}(X)$ denote the class of functions $F \in C^0(\mathbf{R} \times X, X)$ satisfying the following conditions:

(2) for each $(t_0, x_0) \in \mathbf{R} \times X$, the ODE (1) has at least one solution through (t_0, x_0);

(3) all solutions of (1) extend to global solutions.

The well-known Peano theorem states that all $F \in C^0(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$ satisfy (2).

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