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On the uniqueness of equilibrium states for piecewise monotone mappings

by

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Abstract. Our main result is: Given a piecewise monotone interval map T and a continuous function φ with $P(T, \varphi) > \sup \varphi$ satisfying an additional regularity condition, there is at most one φ-equilibrium state for T on each topologically transitive component L_n of T, and only the finitely many L_n with $h(T|_{L_n}) \geqslant P(T, \varphi) - \sup \varphi$ can support such an equilibrium state. The additional regularity assumption is: φ is of bounded variation or φ has bounded distortion under T.

1. Introduction. For a continuous transformation T of a compact metric space X and a continuous function $\varphi\colon X \to \mathbf{R}$, the *pressure* is defined as

$$P(T, \varphi) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}}\, n^{-1} \log \sup_E \sum_{x \in E} \exp\left(\varphi(x) + \ldots + \varphi(T^{n-1} x)\right)$$

where the supremum extends over all (n, ε)-separated subsets of X (recall that E is (n, ε)-*separated* if for all $x, y \in E$ with $x \neq y$ we have $d(T^i x, T^i y) > \varepsilon$ for some $i \in \{0, \ldots, n-1\}$). Walters [W1] proved the variational principle

$$P(T, \varphi) = \sup \{h_\mu(T) + \int \varphi \, d\mu\}$$

where the supremum extends over all ergodic T-invariant measures μ. If the supremum is attained for some μ, then μ is called an *equilibrium state* for φ. For some classes of transformations such as expansive maps or piecewise monotone interval maps [MS] it is known that equilibrium states exist for all continuous φ.

The uniqueness problem is more difficult. Bowen proved uniqueness for irreducible subshifts of finite type and Hölder-continuous φ [B1] and also for general expansive systems with specification property if φ satisfies a condition similar to our (2.3) below [B2]. Walters [W2] proved uniqueness for β-transformations and Lipschitz-continuous φ, and Hofbauer [H1, H2] showed that for general piecewise monotone interval maps of positive entropy and $\varphi \equiv 0$ there is a unique equilibrium state (i.e. a measure of maximal

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entropy) on each topologically transitive subset $L \subseteq X$ for which $P(T|_L, \varphi) = P(T, \varphi)$, which reduces for $\varphi \equiv 0$ to $h_{\text{top}}(T|_L) = h_{\text{top}}(T)$. He also showed that there is only a finite number of such sets L.

Under the assumption $\sup \varphi < P(T, \varphi)$ we extend Hofbauer's result in our Theorem 2 to functions φ which are of bounded variation and have only a finite number of discontinuities or which have bounded distortion under T (see (2.3) for a definition).

A general definition of Gibbs' measures has been given in [D]. Uniqueness holds in this case, too.

At this point one technical remark is in order: strictly speaking, piecewise monotone interval maps such as β-transformations are not continuous. However, by doubling all preimages of the finitely many discontinuities, one can modify the space X and its topology in such a way that T becomes continuous [W2], [H1]. Since the new space is an extension of X by at most countably many points, all properties of T involving only nonatomic measures are unchanged. The resulting space can be described as a linearly ordered, order-eomplete space, endowed with its order topology, on which T acts continuously.

Our proof of the above result uses conformal measures. Following [DU] we say that a Borel measure m on X is *f-conformal* if for every measurable set $A \subseteq X$ such that $T|_A$ is injective and TA is measurable one has

$$m(TA) = \int_A f\, dm. \tag{1.1}$$

Denker and Urbański [DU] deal with the problem of constructing conformal measures for a given function f. (For a precursor of their results see also [HK2].) In some situations, however, the measure m is given and f is defined as the m-derivative of T, so e.g. if $T\colon [0, 1] \to [0, 1]$ is piecewise C^1, m is Lebesgue measure and $f = |T'|$. In many such cases, in particular in smooth dynamics (e.g. [L], [LY]), it is known that a T-invariant measure μ is absolutely continuous with respect to the given f-conformal measure m if and only if it satisfies the Rokhlin formula

$$h_\mu(T) = \int \log f\, d\mu. \tag{1.2}$$

In Theorem 1 we prove this equivalence in a purely measure-theoretic setting. The main assumption is that $\varphi = -\log f$ has bounded distortion under T. This result is used in the proof of Theorem 2 under the bounded distortion assumption. In the case of the bounded variation assumption we use instead the spectral decomposition of the Perron–Frobenius operator and an estimate from [K].

2. Absolute continuity and the Rokhlin formula. Let $(X, \mathcal{F}, m)$ be a finite or σ-finite measure space, ξ a finite or countable $\mathcal{F}$-measurable partition of X and $T\colon X \to X$ $\mathcal{F}$-measurable with $TB \in \mathcal{F}$ for all $B \in \mathcal{F}$ and $\bigvee_{n=0}^{\infty} T^{-n}\xi = \mathcal{F}$.

We use the notation $\xi_n = \bigvee_{i=0}^{n-1} T^{-i}\xi$, and $Z_n(x)$ is that element of ξ_n which contains x.

We always assume that $m(A) > 0$ for all $A \in \xi_n$, $n \geqslant 1$, and

$$\sup\{m(A)\colon A \in \xi_n\} < \infty \quad \text{for some } n \geqslant 1. \tag{2.1}$$

We say m is *f-conformal* if $f\colon X \to \mathbf{R}_+$ is $\mathcal{F}$-measurable and

$$m(TB) = \int_B f\, dm \quad \text{for all } B \in \mathcal{F},\ B \subseteq A \in \xi, \tag{2.2}$$

and $\varphi\colon X \to \mathbf{R}$ has *bounded distortion* under T if there is a constant $C > 0$ such that for all $n \geqslant 1$, for all $A \in \xi_n$ and for all $x, y \in A$

$$|S_n\varphi(x) - S_n\varphi(y)| \leqslant C \tag{2.3}$$

where $S_n\varphi = \sum_{i=0}^{n-1} \varphi \circ T^i$.

THEOREM 1. *Let $(X, \mathcal{F}, m)$, T and ξ be as above and assume that m is f-conformal for some f for which $\log f$ has bounded distortion under T. Suppose μ is an ergodic T-invariant probability measure, $H_\mu(\xi) < \infty$ and*

$$\sup_{n \geqslant 1} \left|\int \log m(T^n Z_n(x))\, d\mu(x)\right| =: \Gamma < \infty. \tag{2.4}$$

Then $\mu \ll m$ if and only if μ satisfies the Rokhlin formula (1.2).

Proof. If μ satisfies the Rokhlin formula, then for all $n > 0$ and writing $\varphi = -\log f$ one has

$$\begin{aligned}
0 &= n\Big(h_\mu(T) + \int \varphi\, d\mu\Big) \leqslant H_\mu(\xi_n) + \int S_n\varphi\, d\mu \\
&= -\sum_{A \in \xi_n} \mu(A)\Big\{\log \mu(A) - \frac{1}{\mu(A)}\int_A S_n\varphi\, d\mu\Big\} \\
&\leqslant -\sum_{A \in \xi_n} \mu(A)\{\log \mu(A) - S_n\varphi(x_A)\} \quad \text{for a suitable } x_A \in A \\
&= -\sum_{A \in \xi_n} \mu(A) \log\{\mu(A)\exp(-S_n\varphi(x_A))\} \\
&\leqslant -\sum_{A \in \xi_n} \mu(A) \log\Big\{\mu(A)\exp(-C)\frac{1}{m(A)}\int_A \exp(-S_n\varphi)\, dm\Big\} \\
&\qquad\qquad \text{by (2.3) and (2.1)} \\
&= C - \sum_{A \in \xi_n} \mu(A) \log\Big\{\frac{\mu(A)}{m(A)} m(T^n A)\Big\} \\
&= C - \underbrace{\sum_{A \in \xi_n} \mu(A) \log\frac{\mu(A)}{m(A)}}_{\Sigma_n} - \int \log m(T^n Z_n(x)) \\
&\leqslant C - \Sigma_n + \Gamma,
\end{aligned}$$

and it suffices to show that $\Sigma_n \to \infty$ as $n \to \infty$ if μ is not m-continuous.

But in this case $\mu \perp m$ by ergodicity of μ, in particular

$$\mu\{x\colon \mu(Z_n(x))/m(Z_n(x)) \leqslant S\} \to 0 \quad \text{as } n \to \infty \tag{2.5}$$

for all $S > 0$. Let $X_{n,j} = \{x\colon e^{-j} \leqslant \mu(Z_n(x))/m(Z_n(x)) < e^{-j+1}\}$, $j \in \mathbf{Z}$. Then

$$\mu(X_{n,j}) = \int_{X_{n,j}} \mu(Z_n(x))/m(Z_n(x))\, dm < e^{-j+1}, \tag{2.6}$$

and we have for each $k = -1, -2, -3, \ldots$

$$\begin{aligned} -\Sigma_n &= -\int \log \frac{\mu(Z_n(x))}{m(Z_n(x))}\, d\mu(x) \leqslant \sum_{j\in\mathbf{Z}} j\mu(X_{n,j}) \\ &\leqslant k \sum_{j\leqslant k} \mu(X_{n,j}) + \underbrace{\sum_{j\geqslant 1} je^{-j+1}}_{=:K_0} \qquad \text{by (2.6)} \\ &= k\mu\left\{x\colon \frac{\mu(Z_n(x))}{m(Z_n(x))} \geqslant e^{-k}\right\} + K_0 \\ &\to k + K_0 \text{ as } n \to \infty \qquad \text{by (2.5).} \end{aligned}$$

Hence $-\Sigma_n \to -\infty$ as $n \to \infty$.

Suppose now that $\mu \ll m$. By the Shannon–MacMillan–Breiman theorem and the martingale theorem

$$h_\mu(T) = -\lim_{n\to\infty} n^{-1} \log \mu(Z_n(x)) = -\lim_{n\to\infty} n^{-1} \log m(Z_n(x)) \qquad \mu\text{-a.e. } x. \tag{2.7}$$

On the other hand,

$$|\log m(T^n Z_n(x)) - \log m(Z_n(x)) + S_n \varphi(x)| \leqslant C \tag{2.8}$$

for all x and $n \geqslant 1$ in view of (2.1)–(2.3). In particular,

$$S_n \varphi(x) \leqslant C + |\log m(T^n Z_n(x))| + \text{const}$$

so that $(S_n \varphi)^+ \in L^1_\mu$ (observe (2.4) and (2.1) for the constant). Hence, by the ergodic theorem,

$$\lim_{n\to\infty} n^{-1} S_n \varphi(x) = \int \varphi\, d\mu < \infty \qquad \mu\text{-a.e. } x,$$

and in view of (2.8) and (2.4)

$$\lim_{n\to\infty} n^{-1} |S_n \varphi(x) - \log m(Z_n(x))| = 0 \qquad \text{in probability } \mu.$$

Together with (2.7) this yields the Rokhlin formula $h_\mu(T) + \int \varphi\, d\mu = 0$. ∎

We discuss an application of this result to piecewise monotone transformations:

Let X be a linearly ordered, order-complete space. Endowed with its order topology, X is compact. Suppose $T\colon X \to X$ is continuous, and there is a finite partition ξ of X into closed intervals $I_1, \ldots, I_N$ such that $T|_{I_j}$ is monotone and continuous and has the Darboux property for all j. (The *Darboux property* means that if $J \subseteq I_j$ is an interval, then also TJ is an interval.) We assume that ξ *generates* under T, i.e. $\bigcup_{n\geqslant 0} T^{-n}\xi$ generates the Borel σ-algebra on X. Such a map will be called *piecewise monotone.*

In a series of papers [H1, H2, H3], Hofbauer constructed Markov extensions $(\hat{X}, \hat{T})$ for piecewise monotone transformations (X, T): Let

$$\mathcal{M} = \{T^n Z\colon Z \in \xi_n,\ n \geqslant 1\}, \qquad \hat{X} = \{(x, D)\colon x \in D \in \mathcal{M}\}.$$

$\mathcal{M}$ is an at most countable family of compact intervals. Observe that for $Z \in \xi_n$, $Z' \in \xi_m$ we have $Z \cap T^{-n} Z' \in \xi_{m+n}$ and $T^n(Z \cap T^{-n} Z') = T^n Z \cap Z'$. Hence, if $D = T^n Z \in \mathcal{M}$ and if $D \cap Z' \neq \emptyset$, $Z' \in \xi_m$, then $T^m(D \cap Z') = T^m(T^n Z \cap Z') = T^{m+n}(Z \cap T^{-n} Z') \in \mathcal{M}$, and we can define $\hat{T}\colon \hat{X} \to \hat{X}$ by

$$\hat{T}(x, D) = (Tx,\ T(D \cap Z_1(x))).$$

A simple induction argument shows that

$$\hat{T}^n(x, D) = (T^n x,\ T^n(D \cap Z_n(x))) \qquad (n \geqslant 1). \tag{2.9}$$

With $\pi(x, D) := x$ we have $\pi \circ \hat{T} = T \circ \pi$, and if for $(x, C), (x, D) \in \hat{X}$ there is $n \geqslant 0$ such that $Z_n(x) \cap C = Z_n(x) \cap D$, then $\hat{T}^n(x, C) = \hat{T}^n(x, D)$. For $(x, C) \in \hat{X}$ set $D(\hat{x}) := C$ and $\hat{C} := \{\hat{x} \in \hat{X}\colon D(\hat{x}) = C\}$. Let $\hat{\mathcal{M}} = \{\hat{C}\colon C \in \mathcal{M}\}$ and $\hat{\xi} = \hat{\mathcal{M}} \vee \pi^{-1}\xi$. Then $\hat{T}\hat{A} \in \hat{\mathcal{M}}$ for $\hat{A} \in \hat{\xi}$; in particular, $\hat{\xi}$ is a Markov partition for $(\hat{X}, \hat{T})$.

$\hat{X}$ is endowed with the Borel structure $\hat{\mathcal{F}} = \pi^{-1}\mathcal{F} \vee \hat{\mathcal{M}}$, and if m is a measure on $\mathcal{F}$, then $\hat{m}$ on $\hat{\mathcal{F}}$ can be defined by $\hat{m}(\hat{A}) = m(\pi\hat{A})$ if $\hat{A} \in \hat{\mathcal{F}}$, $\hat{A} \subseteq \hat{C} \in \hat{\mathcal{M}}$.

Hofbauer proved in Theorem 3 of [H1]:

THEOREM A. *If μ is an ergodic T-invariant Borel probability measure on X with positive entropy, then there is a unique ergodic $\hat{T}$-invariant probability measure $\hat{\mu}$ on $\hat{X}$ such that $\mu = \hat{\mu} \circ \pi^{-1}$. Also $h_{\hat{\mu}}(\hat{T}) = h_\mu(T)$.*

Observe that $\mu \ll m$ if and only if $\hat{\mu} \ll \hat{m}$.

A first application of this construction is

LEMMA 1. *If T in Theorem 1 is piecewise monotone, if m is finite, and if $h_\mu(T) > 0$, then condition (2.4) follows from*

$$\left|\int \log m(D(\hat{x}))\, d\hat{\mu}(\hat{x})\right| < \infty$$

where $\hat{\mu}$ is the lift of μ to $\hat{X}$.

Proof.

$$\begin{aligned} -\int \log m(T^n Z_n(x))\, d\mu(x) &= -\int \log m(T^n Z_n(\pi\hat{x}))\, d\hat{\mu}(\hat{x}) \\ &\leqslant -\int \log m(D(\hat{T}^n \hat{x}))\, d\hat{\mu}(\hat{x}) \qquad \text{by (2.9)} \\ &= -\int \log m(D(\hat{x}))\, d\hat{\mu}(\hat{x}). \ ∎ \end{aligned}$$

As a matter of fact, assumption (2.4) of Theorem 1 can be avoided completely if $h_\mu(T) > 0$:

Fix any $D \in \mathscr{M}$ with $\hat{\mu}(\hat{D}) > 0$ and denote by $\hat{T}_D$ the first-return map of $\hat{T}$ to $\hat{D}$. It is well known that $\hat{\tau}_D$, the first-return time, has finite expectation under $\hat{\mu}$, namely

$$\int_{\hat{D}} \hat{\tau}_D \, d\hat{\mu} = \hat{\mu}\Big(\bigcup_{n \geq 1} \hat{T}^{-n} \hat{D}\Big) = 1$$

by ergodicity of $\hat{\mu}$. Let ζ be the partition of $\hat{D}$ which coincides with $\hat{\xi}_n$ on $\hat{D} \cap \{\hat{\tau}_D = n\}$. ζ is a countable $\hat{\mathscr{F}} \cap \hat{D}$-measurable partition, and it is not hard to show that $H_{\hat{\mu}}(\zeta) < \infty$ (see e.g. [B]). Remember $\varphi = -\log f$ and let

$$\hat{\varphi}_D := \sum_{k=0}^{\hat{\tau}_D - 1} \varphi \circ \pi \circ \hat{T}^k, \quad \hat{f}_D := \exp(-\hat{\varphi}_D), \quad \hat{\mu}_D := \hat{\mu}(\hat{D})^{-1} \cdot \hat{\mu}|_{\hat{D}}.$$

Obviously $\hat{T}_D$, ζ, $\hat{m}|_{\hat{D}}$, $\hat{\varphi}_D$ and $\hat{f}_D$ satisfy (2.1)–(2.3).

By Abramov's formula and Theorem A

$$\hat{\mu}(\hat{D}) h_{\hat{\mu}_D}(\hat{T}_D) = h_{\hat{\mu}}(\hat{T}) = h_\mu(T),$$

and by definition

$$\hat{\mu}(\hat{D}) \int \hat{\varphi}_D \, d\hat{\mu}_D = \int_{\hat{D}} \hat{\varphi}_D \, d\hat{\mu} = \int_{\hat{X}} \varphi \circ \pi \, d\hat{\mu} = \int_X \varphi \, d\mu.$$

Hence μ satisfies the Rokhlin formula for T and φ if and only if $\hat{\mu}_D$ satisfies this formula for $\hat{T}_D$ and $\hat{\varphi}_D$. Now the Markov property of $\hat{T}$ implies $\hat{T}_D(A) = \hat{D}$ for each $A \in \xi$. Hence Theorem 1 shows that the Rokhlin formula is equivalent to $\hat{\mu}_{\hat{D}} \ll \hat{m}|_{\hat{D}}$ which in turn is equivalent to $\hat{\mu} \ll \hat{m}$. So we proved

PROPOSITION 1. *For piecewise monotone T with $h_\mu(T) > 0$ Theorem* 1 *holds without assumption* (2.4).

So the really crucial assumption turns out to be the distortion bound for φ under T.

We would like to mention that Markov extensions can be constructed for more general piecewise invertible systems (see [H3], [K]) and in many cases an analogous result to Theorem A can be proved (work in progress). Hence also Proposition 1 can be extended to such systems.

3. A bound on the number of equilibrium states for piecewise monotone mappings. For our further investigations we need a closer look at Hofbauer's construction: Define a relation "$\to$" on $\hat{\xi}$ by $\hat{A} \to \hat{B}$ if and only if $\hat{B} \subseteq \hat{T}(\hat{A})$. $(\hat{\xi}, \to)$ is a directed graph, and we denote its at most countably many irreducible subsets by $\mathscr{I}_1, \mathscr{I}_2, \ldots$, $\hat{X}_n := \bigcup_{\hat{A} \in \mathscr{I}_n} \hat{A}$.

We remark that $\hat{A} \in \mathscr{I} \setminus \bigcup_n \mathscr{I}_n$ is equivalent to $\hat{A} \cap \hat{T}^n \hat{A} = \emptyset$ for all $n > 0$. Hence each ergodic, $\hat{T}$-invariant finite measure on $\hat{X}$ is concentrated on one of the $\hat{X}_n$.

Let $\hat{L}_n := \bigcap_{i \geq 0} \hat{T}^{-i} \hat{X}_n$ and $\bar{\mathscr{I}}_n := \{\hat{B} \in \hat{\xi}$: there is a path in $(\hat{\xi}, \to)$ starting at some $\hat{A} \in \mathscr{I}_n$ and ending in $\hat{B}\}$. Hofbauer proved

THEOREM B ([H3], Theorem 7, Cor. on p. 382 with remark thereafter, and Cor. 1, p. 377). $\lim_{n\to\infty} h_{\mathrm{top}}(\hat{T}|_{\hat{L}_n}) = 0$.

PROPOSITION C ([H3], Cor. on p. 382). *For all $n \geq 1$ there is a finite set $\mathscr{K}_n \subseteq \bar{\mathscr{I}}_n$ such that*

$$Y_n := \bigcup_{\hat{A} \in \bar{\mathscr{I}}_n} \hat{A} = \bigcup_{\hat{A} \in \mathscr{K}_n} \hat{A}.$$

PROPOSITION D ([H3], Theorem 6). *$L_n := \pi(\hat{L}_n) \subseteq Y_n$ is compact.*

It is easy to check that $TY_n \subseteq Y_n$ and $TL_n \subseteq L_n$.

Let $E := \{a \in X$: a is an endpoint of some $D \in \mathscr{M}\}$.

LEMMA 2. $T^{-1}(L_n \setminus E) \cap Y_n \subseteq L_n$.

Proof. Let $z \in Y_n$ with $Tz \in L_n \setminus E$. There are $\hat{A} \in \bar{\mathscr{I}}_n$ and $\hat{z} \in \hat{A}$ with $\pi(\hat{z}) = z$. As $\pi \hat{T} \hat{z} = Tz \in \pi(\hat{L}_n) \setminus E$, there is $\hat{u} \in \hat{L}_n$ with $\pi \hat{T} \hat{z} = \pi \hat{u} \notin E$. Hence there is $k \geq 0$ such that $\hat{T}^{k+1} \hat{z} = \hat{T}^k \hat{u} \in \hat{L}_n$ (observe the remark after (2.9)). As $\hat{A} \in \bar{\mathscr{I}}_n$, the irreducibility of $(\mathscr{I}_n, \to)$ implies $\hat{A} \in \mathscr{I}_n$, i.e. $\hat{z} \in \hat{X}_n$ and hence $\hat{z} \in \hat{L}_n$. So $z \in L_n$. ∎

Let $\mathscr{Z}^n$ be the coarsest partition of Y_n into compact intervals that refines ξ. Because of Proposition C, $\mathscr{Z}^n$ is finite. Hence T: $Y_n \to Y_n$ has the Darboux property with respect to $\mathscr{Z}^n$.

LEMMA 3. *T: $L_n \to L_n$ has the Darboux property with respect to $\mathscr{Z}^n|_{L_n}$ if $(\mathscr{I}_n, \to)$ does not consist of a single loop only.*

Proof. For $J \subseteq Z \in \mathscr{Z}^n$ Lemma 2 implies

$$T(J \cap L_n) \supseteq T(J \cap T^{-1}(L_n \setminus E)) = TJ \cap L_n \setminus E.$$

If J is a compact interval, then $T(J \cap L_n)$ is compact and hence $T(J \cap L_n) \supseteq \mathrm{cl}(TJ \cap L_n \setminus E)$. But if $x \in TJ \cap L_n$ is not an endpoint of the interval TJ, then $Z_n(x) \subseteq TJ$ for some n, and since $(\mathscr{I}_n, \to)$ does not consist of just one loop, $Z_n(x)$ is uncountable. Hence $\mathrm{cl}(TJ \cap L_n \setminus E) = TJ \cap L_n$, except perhaps for endpoints of TJ. In any case this shows that $T(J \cap L_n)$ is an interval in L_n. ∎

Before we state our main result, notice that $d(x, y) := \exp(-\sup\{n$: $x, y \in Z \in \xi_n\})$ is a metric for the order topology on X, since ξ generates.

THEOREM 2. *Suppose T: $X \to X$ is a piecewise monotone transformation as described above and φ: $X \to \mathbf{R}$ is continuous and satisfies*

$$\delta_\varphi := P(T, \varphi) - \sup \varphi > 0.$$

If

(a) *φ has bounded distortion under T* (see (2.3)), *or if*

(b) *φ is of bounded variation,*

then there is a unique equilibrium state for φ on each set L_n for which

$$P(T|_{L_n}, \varphi) = P(T, \varphi). \tag{3.1}$$

(3.1) *is possible only on the finitely many L_n with $h_{\mathrm{top}}(T|_{L_n}) \geq \delta_\varphi > 0$, and there are no other ergodic equilibrium states for φ.*

Proof. Adding, if necessary, a constant to φ we may assume $P(T,\varphi)=0$. We have already observed that all invariant ergodic measures of positive entropy are concentrated on some L_n. So suppose μ is an ergodic equilibrium state for φ. Then

$$h_\mu(T) = P(T,\varphi) - \int \varphi\, d\mu \geqslant \delta_\varphi > 0$$

and hence μ is concentrated on some L_n. In particular,

$$h_\mu(T) + \int \varphi\, d\mu = P(T,\varphi) = 0 \tag{3.2}$$

for each ergodic equilibrium state μ and

$$P(T,\varphi) = h_\mu(T) + \int \varphi\, d\mu = h_\mu(T|_{L_n}) + \int_{L_n} \varphi\, d\mu \leqslant P(T|_{L_n},\varphi) \leqslant P(T,\varphi).$$

So we must prove existence and uniqueness of equilibrium states for φ on those L_n where $P(T|_{L_n},\varphi) = P(T,\varphi)$. Since $T|_{L_n}$ is piecewise monotone with Darboux property (Lemma 3), the existence follows e.g. from [MS], and Theorem 3.7 of [DU] implies the existence of an atomless $\exp(-\varphi)$-conformal measure m_n on L_n. How to check the assumptions of this theorem is explained at the end of the proof.

Now consider $Z\in\xi_k$ with $Z\cap L_n \neq \emptyset$, $k\geqslant 1$ arbitrary. As there is $\hat{x}\in\hat{L}_n$ with $\pi(\hat{x})\in Z\cap L_n$, the monotonicity interval $\hat{Z}_k(\hat{x})$ of $\hat{T}^k$ containing $\hat{x}$ is mapped into Z under π. Hence, by irreducibility of $(\mathcal{I}_n, \rightarrow)$,

$$L_n = \pi(\hat{L}_n) \subseteq \pi\Big(\bigcup_{j\geqslant 0} \hat{T}^j \hat{Z}_k(\hat{x})\Big) \subseteq \bigcup_{j\geqslant 0} T^j Z.$$

Since m_n is conformal, this shows $m(Z)>0$, i.e. (2.1). In particular, if there is a Borel set $S\subseteq L_n$ with $m_n(S)>0$ and $T^{-1}S = S \bmod m_n$, then also $m_n|_S$ is $\exp(-\varphi)$-conformal, and hence S is dense in L_n.

Suppose now that φ has bounded distortion under T. Then each ergodic T-invariant probability measure μ on L_n satisfying the Rokhlin formula is absolutely continuous with respect to m_n by Theorem 1 and Proposition 1. If there were two such measures μ_1 and μ_2, $\mu_1\perp\mu_2$, then there were a decomposition of L_n into Borel sets S_1 and S_2 of positive m_n-measure such that $\mu_i(S_j)=\delta_{ij}$ and $T^{-1}S_j = S_j \bmod m_n$, and the above reasoning applied to $m_n|_{S_2}$ would yield the contradiction $\mu_1 \ll m_n|_{S_2}$. Together with (3.2) this implies the uniqueness of μ in case (a).

In case (b) when φ is of bounded variation, the Perron–Frobenius operator $\mathcal{L}_\varphi: f\mapsto \sum_{Z\in\xi}(f\cdot\exp\varphi)\circ(T|_Z)^{-1}$ acts as a positive operator on the space BV_n of m_n-equivalence classes of functions of bounded variation on L_n (see [HK1], [R]). In particular, for $0\leqslant f\in BV_n$, $n^{-1}\sum_{k=0}^{n-1}\mathcal{L}_\varphi^k f$ converges uniformly to some $\mathcal{L}_\varphi$-invariant limit $\tilde{f}\geqslant 0$ with $\int\tilde{f}\,dm_n = \int f\,dm_n$, and $\mu = \tilde{f}\cdot m_n$ is T-invariant. Suppose there are two such linearly independent densities $\tilde{f}_1$ and $\tilde{f}_2$. Then there exist two T^{-1}-invariant $(\bmod m_n)$ sets S_1 and S_2 which are not dense in L_n (see [R], Theorem 3), a contradiction to the observation made above. Hence there are a function $h\in BV_n$ and a bounded linear functional $\psi\geqslant 0$ such that $\lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1}\mathcal{L}_\varphi^k f = h\cdot\psi(f)$ in supremum norm for $f\in BV_n$. This is sufficient to proceed further as in the proof of Theorem 8.3 in [K] and to show that $\mu = h\cdot m_n$ is the only equilibrium state for φ on L_n.

We still must check the assumptions of Theorem 3.7 of [DU]: For $\varepsilon>0$ we shall construct a sequence $F_k(\varepsilon)$ of maximal (k,ε)-separated sets such that $F_{k+1}(\varepsilon)\supseteq T^{-1}F_k(\varepsilon)$ and $\operatorname{card}(F_{k+1}(\varepsilon)\setminus T^{-1}F_k(\varepsilon)) \leqslant 2\operatorname{card}(\mathcal{Z}^n|_{L_n})$ for all $k\geqslant 1$. We start with a maximal $(1,\varepsilon)$-separated set $F_1(\varepsilon)$. Obviously $T^{-1}F_1(\varepsilon)$ is $(2,\varepsilon)$-separated, and in order to enlarge it to a maximal $(2,\varepsilon)$-separated set, we have to add at most one point at each end of each interval $I\in\mathcal{Z}^n|_{L_n}$. Here we use the Darboux property of T: $L_n\to L_n$ with respect to $\mathcal{Z}^n|_{L_n}$. In the same way we obtain $F_{k+1}(\varepsilon)$ from $F_k(\varepsilon)$ for $k>1$. ∎

Finally, we would like to remark that the assumption that ξ generates is not essential, for each nontrivial atom of $\bigvee_{n=0}^{\infty} T^{-n}\xi$ is either wandering or periodic. Hence these atoms neither contribute to $P(T,\varphi)$, nor can they support ergodic measures of positive entropy. However, Propositions C and D cited from Hofbauer are no longer true, and they are important for our proof. So one has to look in great detail at Hofbauer's proofs in order to figure out what is true in the non-generating case and to substitute C and D by slightly weaker assertions which still suffice for the proof of our Theorem 2.

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On transition multimeasures with values in a Banach space

by

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Abstract. In this paper we examine transition multimeasures, i.e., set-valued vector measures parametrized by a parameter in a measurable space. First we establish the existence of transition selectors. Then we define a set-valued integral with respect to a multimeasure and we show that it generates a new transition multimeasure, for which we obtain a characterization of its measure selectors. Then we allow the parameter of the transition multimeasure to vary over a Polish space and we obtain a set-valued version of Feller's property. Finally, we look at the action of the transition multimeasure on measures defined on the parameter space.

1. Introduction. The theory of multimeasures (set-valued measures) has its origins in mathematical economics and in particular in equilibrium theory for exchange economies with production, in which the coalitions and not the individual agents are the basic economic units (see Vind [25] and Hildenbrand [15]). Since then the subject of multimeasures has been developed extensively. Important contributions were made, among others, by Artstein [1], Costé [8], [9], Costé–Pallu de la Barrière [10], Drewnowski [12], Godet-Thobie [13], Hiai [14] and Pallu de la Barrière [17]. Further applications in mathematical economics can be found in Klein–Thompson [16] and Papageorgiou [19].

In this paper we study multimeasures parametrized by the elements of a measurable space (transition multimeasures). Such multimeasures turn out to be the appropriate tool to establish the existence of Markov temporary equilibrium processes in dynamic economies (see Blume [6]).

2. Preliminaries. In this section we establish our notation and terminology and we recall some basic facts from the theories of multifunctions and multimeasures that we will need in the sequel.

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X:\ \text{nonempty, closed, (convex)}\},$$
$$P_{(w)k(c)}(X) = \{A \subseteq X:\ \text{nonempty, } (w)\text{-compact, (convex)}\}.$$

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