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The Dunford–Pettis property in C^*-algebras

by

CHO-HO CHU (Irvine, Calif., and London) and
BRUNO IOCHUM (Marseille)

Abstract. We give necessary and sufficient conditions for a C^*-algebra A to have the Dunford–Pettis property. We show that A^* has the Dunford–Pettis property if and only if A^{**} is a finite type I von Neumann algebra. We also show that a von Neumann algebra M has the Dunford–Pettis property if and only if $M = \bigoplus_k R_k$ where each R_k is a type I_{n_k} von Neumann algebra and $\sup_k n_k < \infty$.

A Banach space E is said to have the *Dunford–Pettis property* if every weakly compact operator defined on E is completely continuous, or equivalently, if whenever (x_n) and (f_n) are weakly null sequences in E and E^* respectively, then $\lim_{n\to\infty} f_n(x_n) = 0$.

The above definition was due to Grothendieck [9] and originated from a classical result of Dunford and Pettis [8] which says that all L_1-spaces have this property. In [9], Grothendieck began a study of the Dunford–Pettis property in Banach spaces and showed that $C(K)$ spaces have this property. Naturally, one asks whether these two classical results have "noncommutative" generalization, that is, whether C^*-algebras and preduals of von Neumann algebras possess the Dunford–Pettis property. The answer is easily seen to be negative. This note is intended to characterize operator algebras (or their duals) having the Dunford–Pettis property. We give necessary and sufficient conditions for a C^*-algebra to have the Dunford–Pettis property and deduce that the Dunford–Pettis property is inherited by C^*-subalgebras although it is not inherited by subspaces in general. We show that a von Neumann algebra M has the Dunford–Pettis property if and only $M = (\bigoplus_k R_k)_{l_\infty}$ where each R_k is a type I_{n_k} von Neumann algebra with $\sup_k n_k < \infty$. As for the Banach dual A^* of a C^*-algebra A, we show that A^* has the Dunford–Pettis property if and only if A^{**} is a finite type I von Neumann algebra. We also obtain conditions for the predual of a von Neumann algebra to have the Dunford–Pettis property. We refer to [6] for an excellent account of the Dunford–Pettis property.

We recall that a completely continuous operator sends weakly convergent sequences into norm convergent sequences.

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THEOREM 1. *Let A be a C^*-algebra. The following conditions are equivalent*:

(i) *A has the Dunford–Pettis property.*

(ii) *If (x_n) is a weakly null sequence in A, then so is the sequence $(x_n^* x_n)$.*

(iii) *If (x_n) is a weakly null sequence in A, then so is $(x_n x_n^*)$.*

(iv) *If (x_n) is a weakly null sequence in A, then so is $(x_n^* x_n + x_n x_n^*)$.*

Proof. (ii) and (iii) are clearly equivalent since (x_n) is weakly null if and only if (x_n^*) is weakly null. Hence we also have (iii) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (ii). Let (x_n) be weakly null and let $f \in A^*$. We show that $f(x_n^* x_n) \to 0$. Define an operator $T\colon A \to A^*$ by

$$(Ta)(x) = f(a^* x) \qquad (a, x \in A).$$

As A^* is weakly sequentially complete [12; p. 148], T is weakly compact [1]. But A has the Dunford–Pettis property, so T is completely continuous. Therefore $\|Tx_n\| \to 0$, that is, $\sup\{|f(x_n^* x)|\colon \|x\| \leqslant 1\} \to 0$, which gives $f(x_n^* x_n) \to 0$ since $(\|x_n\|)$ is bounded.

(iv) $\Rightarrow$ (i). Let (x_n) be a weakly null sequence in A and let (f_n) be a weakly null sequence in A^*. We need to show that $f_n(x_n) \to 0$. As $x_n^* x_n + x_n x_n^* \to 0$ weakly, by [12; p. 151], we have $\lim_{n\to\infty} f_k(x_n) = 0$ uniformly for $k = 1, 2, \ldots$ In particular, $\lim_{n\to\infty} f_n(x_n) = 0$.

COROLLARY 2. *Let A be a C^*-algebra with the Dunford–Pettis property. Then every C^*-subalgebra B of A has the property.*

Proof. Let (b_n) be a $\sigma(B, B^*)$-null sequence in B. Then it is $\sigma(A, A^*)$-null. By the above theorem, $(b_n^* b_n)$ is $\sigma(A, A^*)$-null and hence $\sigma(B, B^*)$-null.

Let us now consider some examples. We first note that if the dual E^* of a Banach space E has the Dunford–Pettis property, then so does E. If H is an infinite-dimensional Hilbert space, then the C^*-algebra $K(H)$ of compact operators does not have the Dunford–Pettis property. So $K(H)^*$ and the type I_∞ factor $B(H)$ do not have this property. We give in the following an example of a C^*-algebra having the Dunford–Pettis property but whose enveloping von Neumann algebra does not.

We first remark that the l_1-sum $E = (\bigoplus E_n)_{l_1}$ of a sequence of finite-dimensional Banach spaces has the Dunford–Pettis property (cf. [4; p. 19]), in fact, it has the Schur property which means that weakly convergent sequences are norm convergent. We include here a short proof and we thank Dr. C. Samuel for the following arguments. Let (X_n) be a weakly null sequence in E. If $\|X_n\| \not\to 0$, by considering a subsequence, we may assume $a = \inf_n \|X_n\| > 0$. Let $P_n\colon E \to E$ be the projection defined by $P_n(X) = (x_1, \ldots, x_n, 0, \ldots)$ for $X = (x_k)_k \in E$. Then $\lim_{n\to\infty} \|X - P_n(X)\| = 0$ and also $\lim_{m\to\infty} \|P_n(X_m)\| = 0$ because the sequence $(P_n(X_m))_m$ converges weakly to 0 in a finite-dimensional space. Define two increasing sequences $(n_k)_k$

and $(m_k)_k$ of integers as follows:

$$n_1 = 1 \quad \text{and} \quad \|P_{m_1}(X_1) - X_1\| < a/2$$

and for $k \geqslant 2$,

$$\|P_{m_{k-1}}(X_{n_k})\| < a/2^{k+1} \quad \text{and} \quad \|X_{n_k} - P_{m_k}(X_{n_k})\| < a/2^{k+1}.$$

Let $Z_1 = P_{m_1}(X_1)$ and $Z_k = (P_{m_k} - P_{m_{k-1}})(X_{n_k})$ for $k \geqslant 2$. Then $\|Z_k - X_{n_k}\| < a/2^k$ for all k. Also there exist $\alpha, \beta > 0$ such that for any scalars $\lambda_1, \ldots, \lambda_k$, we have

$$\alpha \sum_{j=1}^{k} |\lambda_j| \leqslant \Big\| \sum_{j=1}^{k} \lambda_j Z_j \Big\| \leqslant \beta \sum_{j=1}^{k} |\lambda_j|,$$

in other words, (Z_k) is equivalent to the canonical basis in l_1 and (Z_k) does not converge to 0 weakly. It follows that (X_{n_k}) does not converge to 0 weakly, which is a contradiction.

EXAMPLE. For $k = 1, 2, \ldots$, let M_{n_k} be the algebra of $n_k \times n_k$ complex matrices. Let $n_k \uparrow \infty$. Then the C^*-algebra $(\bigoplus_{k=1}^{\infty} M_{n_k})_{c_0}$ has the Dunford–Pettis property as the previous arguments show that $(\bigoplus_{k=1}^{\infty} M_{n_k}^*)_{l_1}$ is a Schur space. But the von Neumann envelope $(\bigoplus_{k=1}^{\infty} M_{n_k})_{l_\infty}$ does not have the Dunford–Pettis property since it contains $(\bigoplus_{k=1}^{\infty} l_2^{n_k})_{l_\infty}$ as a complemented subspace and the latter does not have the Dunford–Pettis property [6; p. 22].

THEOREM 3. *Let M be a von Neumann algebra. The following conditions are equivalent*:

(i) *M has the Dunford–Pettis property;*

(ii) *$M = \bigoplus_k R_k$ where each R_k is a type I_{n_k} von Neumann algebra with $\sup_k n_k < \infty$.*

Proof. (i) $\Rightarrow$ (ii). We first show that M is finite. If M is infinite, then there is a properly infinite projection e [12; p. 327] which is the sum of an infinite sequence $\{e_n\}$ of mutually orthogonal equivalent projections. Hence, by [12; p. 297], M contains an algebra isomorphic to $e_1 M e_1 \otimes B(H)$ with $\dim H = \infty$. But the latter clearly cannot have the Dunford–Pettis property, contradicting Corollary 2. So M is finite.

We now show that M does not contain a type II_1 summand. Indeed, if N is a type II_1 summand of M, then $(\bigoplus_{n=1}^{\infty} M_{2^n})_{l_\infty}$ embeds as a subalgebra of N (cf. [13; 1.4.4]), but does not have the Dunford–Pettis property by the Example, again contradicting Corollary 2. It follows that M is type I finite and has a unique decomposition $M = R_1 \oplus R_2 \oplus \ldots$ where each R_n is either 0 or of type I_n with $n < \infty$. If infinitely many R_n's are nonzero, then M contains a subalgebra $(\bigoplus_{k=1}^{\infty} M_{n_k})_{l_\infty}$ which does not have the Dunford–Pettis property by the Example, contradicting Corollary 2. So all but a finite number of R_n's are 0.

(ii) $\Rightarrow$ (i). This follows from the fact that a type I_n algebra has the form $C(K) \otimes M_n$ which has the Dunford–Pettis property [7].

We now consider the predual M_* of a von Neumann algebra M. The w^*-topology on M is the topology $\sigma(M, M_*)$.

PROPOSITION 4. *Let M be a von Neumann algebra with predual M_*. The following conditions are equivalent:*

(i) *M_* has the Dunford–Pettis property;*
(ii) *If (x_n) is a weakly null sequence in M, then $(x_n^* x_n)$ is w^*-null;*
(iii) *If (x_n) is a weakly null sequence in M, then $(x_n x_n^*)$ is w^*-null;*
(iv) *If (x_n) is a weakly null sequence in M, then $(x_n^* x_n + x_n x_n^*)$ is w^*-null, i.e., (x_n) is σ-strong*-null.*

Proof. (i)$\Rightarrow$(ii). Let (x_n) be weakly null in M. We show that $f(x_n^* x_n) \to 0$ for each $f \in M_*$. Define $g_n(\cdot) = f(x_n^* \cdot)$ on M. Then $g_n \in M_*$ and also, $g_n \to 0$ in the weak topology $\sigma(M_*, M)$ since for fixed $m \in M$, $x_n^* m \to 0$ in the w^*-topology $\sigma(M, M_*)$. Now by the Dunford–Pettis property of M_*, we have $g_n(x_n) \to 0$ which is just $f(x_n^* x_n) \to 0$.

Clearly (ii) and (iii) are equivalent and they imply (iv).

(iv)$\Rightarrow$(i). Let (f_n) be a $\sigma(M_*, M)$-null sequence and let (x_n) be a $\sigma(M, M^*)$-null sequence in M. We need to show $f_n(x_n) \to 0$. By condition (iv), (x_n) is a σ-strong*-null sequence in M, so by [12; Lemma 5.5], we have $\lim_{n\to\infty} f_n(x_n) = 0$.

COROLLARY 5. *Let M be a von Neumann algebra whose predual M_* has the Dunford–Pettis property. Let N be a von Neumann subalgebra of M. Then N_* has the Dunford–Pettis property.*

Proof. Let (x_n) be a weakly null sequence in N. Then (x_n) is weakly null in M. So $x_n^* x_n \to 0$ in the topology $\sigma(M, M_*)$. Hence $x_n^* x_n \to 0$ in the topology $\sigma(N, N_*)$ since every $f \in N_*$ extends to an $\tilde{f} \in M_*$ [11; 1.24.5].

PROPOSITION 6. *Let M be a von Neumann algebra with predual M_*.*

(i) *If M_* has the Dunford–Pettis property, then M is finite.*
(ii) *If M is type I finite, then M_* has the Dunford–Pettis property.*

Proof. (i) As in the proof of Theorem 3, if M is infinite, then M contains $B(H)$ for some infinite-dimensional Hilbert space H. Corollary 5 implies that $B(H)_*$ has the Dunford–Pettis property, which is impossible.

(ii) Consider the unique decomposition

$$M = R_1 \oplus \ldots \oplus R_k \oplus \ldots,$$

where R_k is of type I_{n_k} with $n_k < \infty$. Write $R_k = C(\Omega_k) \otimes M_{n_k}$. Then

$$M_* = l_1\text{-sum} \bigoplus_k L_1(\Sigma_k, (M_{n_k})_*).$$

Now each $(M_{n_k})_*$ is complemented in $E = (\bigoplus_k (M_{n_k})_*)_{c_0}$, so $L_1(\Sigma_k, (M_{n_k})_*)$ is complemented in $L_1(\Sigma_k, E)$. Hence the l_1-sum $\bigoplus_k L_1(\Sigma_k, (M_{n_k})_*)$ is complemented in the l_1-sum $\bigoplus_k L_1(\Sigma_k, E)$, and therefore, complemented in $L_1(\bigcup_k \Sigma_k, E)$ where the union is taken to be the disjoint union with the natural choice of measure. As $E^* = (\bigoplus_k M_{n_k})_{l_1}$ has the Schur property by the previous argument, $L_1(\bigcup_k \Sigma_k, E)$ has the Dunford–Pettis property by [2]. It follows that M_* has the Dunford–Pettis property.

Remark. The hyperfinite II_1-factor R embeds into any II_1-factor [5] and if one can show that R_* does not have the Dunford–Pettis property, then Corollary 5 and the above result would imply that M_* has the Dunford–Pettis property if and only if M is of type I finite. We do not know if R_* has the Dunford–Pettis property. However, we have the following result for the duals of C^*-algebras.

THEOREM 7. *Let A be a C^*-algebra. The following conditions are equivalent:*

(i) *A^* has the Dunford–Pettis property;*
(ii) *A^{**} is type I finite.*

Proof. (i)$\Rightarrow$(ii). By Proposition 6, we only need to prove that A^{**} is a type I von Neumann algebra. For this, we show that A is a type I C^*-algebra. If A is not a type I C^*-algebra, by [10; 6.7.4], A contains a C^*-subalgebra B and a closed two-sided ideal I in B such that B/I is isomorphic to the Fermion algebra $\otimes M_2$. By [3], $\otimes M_2$ contains a complemented copy of the compact operators $K(l_2)$. So $\otimes M_2$ does not have the Dunford–Pettis property nor does B/I. On the other hand, $(A/I)^*$ is complemented in A^*, hence it has the Dunford–Pettis property and so does A/I. But B/I is a subalgebra of A/I and by Corollary 2, B/I has the Dunford–Pettis property, which is impossible. This proves that A is a type I C^*-algebra.

(ii)$\Rightarrow$(i). By Proposition 6.

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Addendum. We thank Professor S. Watanabe for informing us, after submission of the paper, that there was considerable overlap between our results and those of M. Hamana's paper *On linear topological properties of some C^*-algebras*, Tôhoku Math. J. 29 (1977), 157–163. We also thank Professor G. K. Pedersen for showing the following lemma.

LEMMA. *Any quotient B of a separable C^*-algebra A with the Dunford–Pettis property also has the property.*

This is because weakly null sequences in B lift to weakly null sequences in A. Indeed, let $q\colon A\to B$ be the quotient map and (b_n) weakly null in B with $q(a_n)=b_n$ and (a_n) bounded in A. Let (u_n) be a countable approximate unit for $\ker q$ and put $c_n=(1-u_n)a_n$. Then $q(c_n)=b_n$ and (c_n) is weakly null in A. For the latter, let p be the support projection in A^{**} for $\ker q$, so that $p(1-u_n)\to 0$ strongly in A^{**}, which implies $p(1-u_n)a_n\to 0$ strongly. Hence, for $f\in A^*$, we have $f(c_n)=f(pc_n)+f((1-p)c_n)=f(p(1-u_n)a_n)+f((1-p)a_n)\to 0$. Now we have:

THEOREM. *A separable C^*-algebra A has the Dunford–Pettis property if and only if A^* has this property.*

If A has the property, then using the lemma and the proof of Theorem 7, A is type I. Moreover, A has only finite-dimensional irreducible representations for otherwise $K(l_2)$ shows up in a quotient of A. Hence A^{**} is type I finite (cf. Theorem 1 in Hamana's paper).

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
Irvine, California 92717, U.S.A.

GOLDSMITHS' COLLEGE
London SE14 6NW, U.K.

UNIVERSITÉ DE PROVENCE and
C.N.R.S., CENTRE DE PHYSIQUE THÉORIQUE
Luminy Case 907, F-13288 Marseille Cedex 9, France

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Interpolation of compact operators by Goulaouic procedure

by

FERNANDO COBOS (Madrid)

Abstract. We show that the classical Lions–Peetre compactness theorems for Banach spaces (which are the main tools for proving all known compactness results in interpolation theory) fail in the locally convex case. We also prove a positive result assuming compactness of the operator in both sides.

1. Setting of the problem. Motivated by certain problems in the theory of partial differential equations, Goulaouic studied in [6] and [7] a procedure for extending any interpolation functor for Banach couples to more general couples of locally convex spaces. Let us briefly review this procedure.

A (Hausdorff) locally convex space E is said to be the *strict projective limit* of the family of Banach spaces $(E_i)_{i\in I}$ if the following conditions are satisfied:

1) $E=\bigcap_{i\in I}E_i$.
2) E is equipped with the projective limit topology.
3) For each $i\in I$, E is dense in E_i.
4) The family $(E_i)_{i\in I}$ is directed, i.e. given any finite subset $J\subset I$, there exists $k\in I$ such that for all $j\in J$ the embedding $E_k\hookrightarrow E_j$ is continuous.

We then write $E=\varprojlim_{i\in I}=E_i$.

Let now (A_0, A_1) be a (compatible) couple of locally convex spaces (meaning that they are continuously embedded in a Hausdorff topological vector space). We say that (A_0, A_1) is the *strict projective limit* of the family $(A_{0,i}, A_{1,j})_{(i,j)\in I\times J}$ of Banach couples provided that the following conditions hold:

1) $A_0=\varprojlim_{i\in I}A_{0,i}$, $A_1=\varprojlim_{j\in J}A_{1,j}$.

2) All spaces $A_{0,i}$, $A_{1,j}$ are continuously embedded in a common Hausdorff topological vector space $\mathscr{A}$.

3) For each $(i,j)\in I\times J$, $A_0\cap A_1$ is dense in $A_{0,i}\cap A_{1,j}$ (the norm in $A_{0,i}\cap A_{1,j}$ being $\max\{\|a\|_{A_{0,i}}, \|a\|_{A_{1,j}}\}$).

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