STUDIA MATHEMATICA, T. LXXXI. (1985)

The rotation number of some transformation related to billiards in an ellipse

by

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Abstract. For an elliptic billiard table the rotation number of a standard section of a billiard flow related to the caustics has been found.

Introduction. Consider a plane, convex, smooth, closed curve $\mathscr{C}$ and a motion of a billiard ball (infinitely small) inside a billiard table bounded by $\mathscr{C}$. The billiard ball moves inside $\mathscr{C}$ along straight lines and rebounds according to the law stating that "the angle of incidence is equal to the angle of reflection".

We reduce this dynamical flow system to the transformation T of an annulus [1]. The annulus $\mathscr{A}$ is the set of unit vectors at points of $\mathscr{C}$ directed inside $\mathscr{C}$. Provide $\mathscr{C}$ with an orientation in the counterclockwise direction. We consider on $\mathscr{A}$ the coordinates φ, θ for a vector v at P. φ is the length of the positively oriented curve joining P with a fixed point O in $\mathscr{C}$ (Fig. 1). θ is the angle between the positive direction of the tangent to $\mathscr{C}$ at P and v.

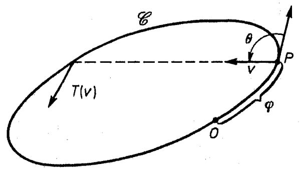

Fig. 1

Consider $T\colon \mathscr{A} \to \mathscr{A}$ defined as follows. For $v \in \mathscr{A}$ consider a trajectory which starts from v. $T(v)$ is the unit vector to which this trajectory is tangent after the first reflection from $\mathscr{C}$.

A curve $\mathscr{D}$ lying inside $\mathscr{C}$ is called a *caustic* of $\mathscr{C}$ if all segments of any trajectory are tangent to $\mathscr{D}$ or have two or no common points with $\mathscr{D}$ and there is no curve which contains $\mathscr{D}$ and satisfies this property. If a set of straight lines which contain the segments of the trajectory is considered and $\mathscr{D}$ does not lie inside $\mathscr{C}$, the curve $\mathscr{D}$ is called a *generalized caustic* (Fig. 2).

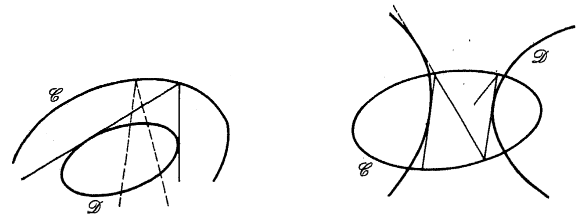

Fig. 2

Consider an orientation-preserving homeomorphism f of a circle. The *rotation number* of f is the limit of the arithmetic means of growth of the angular coordinate. It is known that this limit exists and is an invariant of topological conjugacy [2].

Remark. If there exists a measure μ invariant for f, absolutely continuous with respect to the Lebesgue measure of the circle which assumes a positive value on every nonempty open set, then we can consider the parametrization of the circle given by the arc length measured by μ. f is a rotation in this parametrization and consequently the rotation number of f is equal to $\mu[x, f(x)]$ divided by the measure μ of the circle.

The aim of this paper is to study elliptic billiard tables. G. D. Birkhoff in [1], p. 249 considered this integrable system and drew the phase portrait (we quote his picture and discussion in the next section). He proved that there exist coordinates in which T $(T\circ T)$ rotates the invariant circles but he did not give an explicit formula. In the present paper we find an invariant measure and, due to the Remark find explicit formulae for the rotation numbers of $T(T\circ T)$.

I. Billiards in an ellipse.

LEMMA 1. *Caustics and generalized caustics of an ellipse $\mathscr{E}$ are ellipses confocal with $\mathscr{E}$ lying inside it and confocal hyperbolae.*

This is a well known fact of elementary geometry (see for example [3]).

By $\mathscr{E}(\mathscr{F})$ we denote the set of all points (φ, θ) such that the billiard ball trajectory in $\mathscr{E}$ starting from (φ, θ) is tangent to the caustic or the generalized caustic $\mathscr{F}$. The sets $\mathscr{E}(\mathscr{F})$ are invariant for T. The set $\mathscr{E}(\mathscr{F})$ is the sum of two components each homeomorphic with a circle (Fig. 3).

If $\mathscr{F}$ is an ellipse, then T restricted to each of the components of $\mathscr{E}(\mathscr{F})$ is an orientation-preserving homeomorphism of the circle. But if $\mathscr{F}$ is a hyperbola, we have to consider $T\circ T$ on each component of $\mathscr{E}(\mathscr{F})$. Denote these transformations by T_1 and T_2. We shall find two transformations U and W topologically conjugate to T_1. The second one has a simple invariant measure. This will be proved in the Proposition. Finally, we shall show a

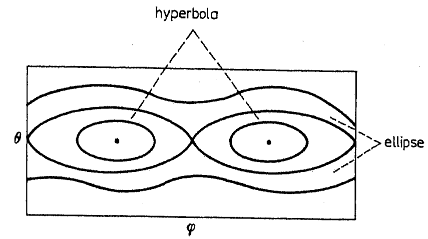

Fig. 3

conjugacy between T_2 on $\mathscr{G}(\mathscr{H})$ and T_1^2 on $\mathscr{E}(\mathscr{F})$ if for an ellipse $\mathscr{G}$ and a hyperbolic caustic $\mathscr{H}$ we take a suitable pair of ellipses $\mathscr{E}$, $\mathscr{F}$.

II. Elliptic case. Let $\mathscr{E}$ and $\mathscr{F}$ be confocal ellipses with foci E and F, respectively (Fig. 4), and eccentricities e and f (the lengths of the major axes divided by $\operatorname{dist}(E, F)$).

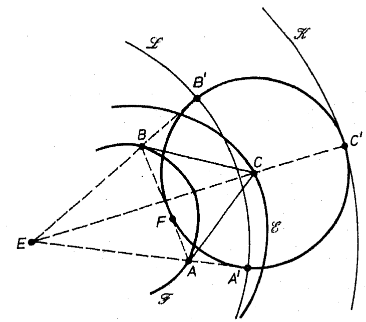

Fig. 4

THEOREM 1. *The rotation number of T_1 is equal to*

$$\frac{1}{2}\,\frac{F(\beta/2, k)}{F(\pi/2, k)}$$

where $F(\alpha, k)$ is the elliptic integral

$$F(\alpha, k) = \int_0^\alpha \frac{d\gamma}{\sqrt{1-k^2\sin^2\gamma}}$$

and

$$k = \frac{2\sqrt{f}}{1+f}, \qquad \beta = \arcsin\frac{e}{f}\cdot\frac{1-f^2}{1-e^2} \in [0, \pi/2].$$

Let C be a point of $\mathscr{E}$ and A, B the points of $\mathscr{F}$ such that the segments AC and BC are tangent to $\mathscr{F}$ (Fig. 4).

Let A', B', C' be the points of the straight lines AE, BE, CE, respectively, such that $A'E = B'E = EF/f$, $C'E = EF/e$ and the points A, B, C lie inside the respective segments $A'E$, $B'E$ and $C'E$ (in the sequel, for any two points X, Y, we denote $\operatorname{dist}(X, Y)$ by XY).

Lemma 2. *The points A', B', C' and F lie on a circle with centre C.*

Proof. The length of the segment $A'E$ is equal to the length of the major axis of the ellipse $\mathscr{F}$ and so it is equal to $AE+AF$. Thus $AF = AA'$. Furthermore, we know that the angles between two segments which join a point of an ellipse with its foci and the tangent to the ellipse at that point are equal (see [3]). Thus the points A' and F are symmetric with respect to the line AC. Consequently we have $FC = A'C$ and similarly we get $CF = B'C$ and $CF = CC'$. ∎

Let $\mathscr{L}$ and $\mathscr{K}$ be the circles with centre E and radii EF/f and EF/e, respectively (Fig. 5). We define a transformation U: $\mathscr{L} \to \mathscr{L}$ as follows. For $X \in \mathscr{L}$, $U(X)$ is such a point of $\mathscr{L}$ that the points F, X and $U(X)$ are on a suitably chosen circle tangent to $\mathscr{K}$. Lemma 2 implies that the homeomorphisms T_1 and U are topologically conjugate.

Consider the inversion with centre F with respect to the circle of radius 1. Denote by $\mathscr{K}_1$ and $\mathscr{L}_1$ the images of $\mathscr{K}$ and $\mathscr{L}$ under the inversion, respectively. The inversion restricted to $\mathscr{L}$ gives a topological conjugacy between U and W: $\mathscr{L}_1 \to \mathscr{L}_1$. W can be described in simple geometric terms: for $X \in \mathscr{L}_1$, $XW(X)$ is a chord of $\mathscr{L}_1$ tangent to $\mathscr{K}_1$ (Fig. 5).

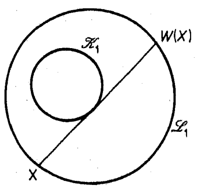

Fig. 5

For $X \in \mathscr{L}_1$ let $\varrho(X)$ be the inverse of the length of the segment tangent to $\mathscr{K}_1$ with one end at X and the second on $\mathscr{K}_1$.

Proposition. *The measure ϱds is a finite invariant measure for W (ds denotes the element of arc length of $\mathscr{L}_1$).*

Proof. Let $A, B \in \mathscr{L}_1$ and $C \in \mathscr{K}_1$ be points such that the segment $AW(A)$ is tangent to $\mathscr{K}_1$ at the point C (Fig. 6). We have to show that

$$\lim_{B\to A}\frac{AB^{\cup}}{AC} = \lim_{B\to A}\frac{(W(A)\,W(B))^{\cup}}{W(A)\,C}$$

($AB^{\cup}$ denotes the length of the arc AB of the circle $\mathscr{L}_1$).

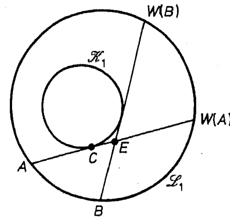

Fig. 6

Let $E = AW(A) \cap BW(B)$. We have

$$\lim_{B\to A}\frac{AB^{\cup}}{AC} = \lim_{B\to A}\frac{AB}{AE} = \lim_{B\to A}\frac{W(A)\,W(B)}{W(A)\,E} = \lim_{B\to A}\frac{(W(A)\,W(B))^{\cup}}{W(A)\,C},$$

where the second equality is the Thales Theorem for similar triangles ABE and $W(A)\,W(B)\,E$. In addition, the measure ϱds is finite because ϱ is bounded. ∎

Proof of Theorem 1. Let G, H, I and J be the points of intersection of $\mathscr{K}$ and $\mathscr{L}$ with the line EF, and K and L the centres of the circles $\mathscr{K}_1$ and $\mathscr{L}_1$, as in Fig. 7. The radius r_1 of the circle $\mathscr{L}_1$ is equal to

$$\frac{1}{2}\left(\frac{1}{FI}+\frac{1}{FH}\right) = \frac{1}{2}\left(\frac{1}{EI-EF}+\frac{1}{EH+EF}\right) = \frac{1}{EF}\cdot\frac{f}{1-f^2}.$$

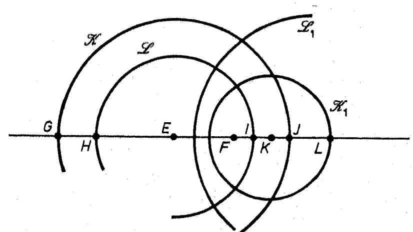

Fig. 7

Similarly we calculate the radius r_2 of the circle $\mathscr{K}_1$:

$$r_2 = \frac{1}{EF} \cdot \frac{e}{1-e^2}.$$

We have also

$$KL = \frac{1}{2}\left[\left(\frac{1}{FI} - \frac{1}{FJ}\right) - \left(\frac{1}{FH} - \frac{1}{FG}\right)\right] = \frac{1}{EF} \frac{f^2-e^2}{(1-f^2)(1-e^2)}.$$

Let α be the measure of the angle KLA (Fig. 8). The Pythagorean Theorem for the triangle ACK and the Carnot formula for the triangle AKL give

$$AC^2 = AK^2 - r_2^2 = r_1^2 + KL^2 - r_2^2 - 2KL \cdot r_1 \cdot \cos\alpha.$$

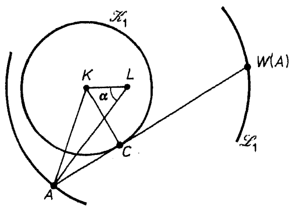

Fig. 8

Thus we have

$$AC^2 = a\left(1 - k^2 \sin\frac{\pi-\alpha}{2}\right),$$

where

$$a = \frac{1}{EF^2} \cdot \frac{f^2-e^2}{(1-f^2)(1-e^2)} \quad \text{and} \quad k = 2\frac{\sqrt{f}}{1+f}.$$

The rotation numbers of the transformations T_1 and W are equal because we have shown that they are topologically conjugate. Due to the Remark from the Introduction we obtain the rotation number of the transformation W:

$$\frac{\displaystyle\int_{\beta}^{\pi-\beta} \frac{d\alpha}{\sqrt{a}\sqrt{1-k^2\sin^2\frac{1}{2}(\pi-\alpha)}}}{\displaystyle\int_{0}^{2\pi} \frac{d\alpha}{\sqrt{a}\sqrt{1-k^2\sin^2\frac{1}{2}(\pi-\alpha)}}} = \frac{\displaystyle\int_{0}^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} - 2\int_{0}^{\beta/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}}{\displaystyle 2\int_{0}^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}}.$$

β is the angle KLA when the segment $AW(A)$ is parallel to the line KL (Fig. 9). In this case

$$\beta = \arcsin\frac{r_2}{r_1} = \arcsin\frac{e}{f} \cdot \frac{1-f^2}{1-e^2}. \quad \blacksquare$$

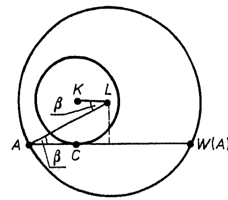

Fig. 9

III. Hyperbolic case. Let $\mathscr{G}$ and $\mathscr{H}$ be an ellipse and a hyperbola with foci I and J. Denote the eccentricities of $\mathscr{G}$ and $\mathscr{H}$ by g and h.

THEOREM 2. *There is a topological conjugacy between T_2: $\mathscr{G}(\mathscr{H}) \to \mathscr{G}(\mathscr{H})$ and T_1^2: $\mathscr{E}(\mathscr{F}) \to \mathscr{E}(\mathscr{F})$ (for suitable ellipses $\mathscr{E}$ and $\mathscr{F}$), and the rotation number of T_2 is equal to*

$$1 - 2\frac{F(\delta/2, k')}{F(\pi/2, k')},$$

where

$$\delta = \arcsin g \cdot \frac{h^2-1}{h^2-g^2} \in [0, \pi/2] \quad \textit{and} \quad k' = \frac{\sqrt{h}}{1+h}.$$

Let A and B be points of $\mathscr{E}$ such that the line AB is tangent to $\mathscr{F}$ at C (Fig. 10). Let D be the point symmetric to the point F with respect to the line AB.

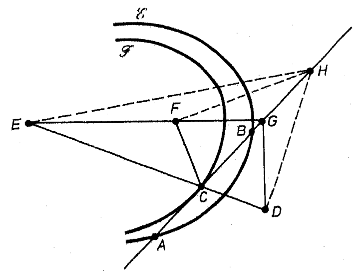

Fig. 10

LEMMA 3. *The hyperbola with foci E and D and the distance between its branches equal to EF is tangent to the line AB.*

Proof. Let G be the point of intersection of the lines AB and EF. The point G lies on the hyperbola because $EG - DG = EF$. Furthermore, for every point H of the line AB different from G we have $EH-HD < EF+ +FH-HD = EF$. ∎

Lemma 4. *The triangles ECA' and ECB' in Figure 4 are congruent.*

Proof. These triangles have one common side. We have also $EA' = EB' = EF/f$. From Lemma 2 we obtain $B'C = A'C$. ∎

Proof of Theorem 2. Suppose that $f = 1/h$, $e = g/h$ and $EF = IJ/h$. In this case $IJ = ED$, $EG+GD = IJ/g$ and the distance between the branches of $\mathscr{H}$ is equal to EF. Thus we can define two transformations V, V': $\mathscr{E}(\mathscr{F}) \to \mathscr{E}(\mathscr{F})$. Let a vector $v \in \mathscr{E}(\mathscr{F})$ at a point $A \in \mathscr{E}$ be parallel to AB. Displace the quadrilateral $EADB$ to IA_1JB_1 preserving its orientation (Fig. 11). The points A_1 and B_1 lie on the ellipse $\mathscr{G}$. From Lemma 3 we know that A_1B_1 is the segment of the billiard trajectory in $\mathscr{G}$ tangent to $\mathscr{H}$. Let $V(v)$ be the vector at A_1, belonging to $\mathscr{G}(\mathscr{H})$ and parallel to A_1B_1. $V'(v)$ is the vector symmetric to $V(v)$ with respect to the line IJ. From Lemma 4 we have $T\circ V' = V\circ T$ and $T\circ V = V'\circ T$. Hence $T\circ T\circ V = V\circ T\circ T$. Thus the transformations $T_1\circ T_1$ and T_2 are topologically conjugate. The formula of Theorem 1 for $T_1\circ T_1$ gives the formula of Theorem 2. ∎

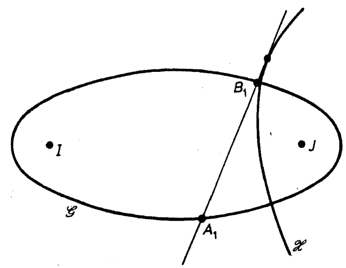

Fig. 11

IV. An application of the Lorentz transformation for finding a topological conjugacy between T_1 and W. A projective transformation of a plane preserves straight lines. We shall have another proof of Theorem 1 if the following lemma holds:

Lemma 5. *There is a projective transformation which maps the ellipses $\mathscr{E}$ and $\mathscr{F}$ into circles.*

Proof. To begin with, suppose that $\mathscr{E}$ is a circle and $\mathscr{E}$ and $\mathscr{F}$ have a common centre. Consider the sphere S with equator $\mathscr{E}$. Let α be a plane not parallel to the equator and tangent to S (Fig. 12). Let X be the antipode of the point of tangency of α and S. We project $\mathscr{E}$ and $\mathscr{F}$ from the point X

onto α. This projection transforms $\mathscr{E}$ as the stereographic projection. Therefore the image of $\mathscr{E}$ is a circle. In addition, it transforms $\mathscr{F}$ into an ellipse.

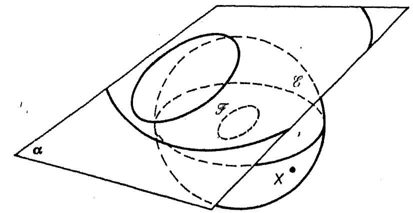

Fig. 12

If the point of tangency of α and S varies along the meridian whose plane contains the major axis of $\mathscr{F}$, the projection converts the major axis of $\mathscr{F}$ into an axis of the image. If X is the pole, the projection of the major axis is the longest chord of the image, but if it is near to the equator, this is not so.

To find an appropriate X we proceed in the following way. Introduce projective plane coordinates x, y, t; then $\mathscr{F} = \{(x, y, t) \in \mathbf{R}^3\colon x^2+y^2 = t^2\}$. A real 3×3 matrix A is called a *Lorentz transformation* if $A\Lambda A^T = \Lambda$ where

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Remark. Any Lorentz transformation preserves the light cone (suppose that the light velocity is 1), and, conversely, for every projective transformation B which preserves the light cone there exists a $\lambda \in \mathbf{R}$ such that

$$(\lambda B)\Lambda(\lambda B)^T = \Lambda.$$

The ellipse $\mathscr{F}$ in projective coordinates has a physical interpretation. Consider a two-dimensional anisotropic crystal. For the inertial observer the front of the light impulse propagating in it is an ellipse. According to the special theory of relativity the distances parallel to the motion of the observer get shorter. Thus we seek an observer moving along the major axis of the ellipse $\mathscr{F}$. This is a simple exercise in analytic geometry.

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Commuting C_0 groups and the Fuglede–Putnam theorem

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Abstract. The following generalization of the Fuglede–Putnam theorem is known (see [5]): If A, B are commuting Hermitian operators on a Banach space X and if $(A+iB)^2 x = 0$ for some $x \in X$, then $Ax = Bx = 0$. We generalize this result further, proving that if A_k $(k = 1, \ldots, n, n \geqslant 2)$ are commuting Hermitian operators on X and if $P(t_1, \ldots, t_n)$ is a complex polynomial with at most one real zero at the origin, then $P(A_1, \ldots, A_n)x = 0$ for some $x \in X$ implies $A_k x = 0$ $(k = 1, \ldots, n)$. This result holds also when iA_k are (unbounded) generators of certain one-parameter groups of operators on X. Our considerations are based on a generalization of the classical Liouville theorem for harmonic functions.

Preliminaries. Let H be a complex Hilbert space and $B(H)$ the Banach space of bounded linear operators on H. Let $a, b, c, d \in B(H)$ be self-adjoint operators such that $[a, b] = 0$, $[c, d] = 0$. The Fuglede–Putnam theorem says that if $x \in B(H)$ and $x(a+ib) = (c+id)x$, then $x(a-ib) = (c-id)x$ (see [11], § 1.6; [12], Theorem 12.16). One way to generalize this theorem is to relax the conditions $[a, b] = 0$, $[c, d] = 0$ (see [2], [10] and the references there). Another – to relax the condition $x(a+ib) = (c+id)x$ (see [1], [9] and the references there). We give here a generalization relaxing this condition and passing to a larger class of operators.

The above theorem can be reformulated as follows: Let A, B be the bounded linear operators on $B(H)$ defined by $Ax = xa - cx$, $Bx = xb - dx$, $x \in B(H)$. Then $[A, B] = 0$ and A, B are Hermitian operators in the sense of Vidav (see [3]), because the one-parameter groups e^{itA}, e^{itB} $(t \in \boldsymbol{R})$ are groups of isometries on $B(H)$ (as $e^{itA}x = e^{-itc}xe^{ita}$, $e^{itB}x = e^{-itd}xe^{itb}$, $x \in B(H)$, $t \in \boldsymbol{R}$ – see for instance [9], p. 186). The Fuglede–Putnam theorem states that if $x \in B(H)$ and $(A+iB)x = 0$, then $Ax = Bx = 0$. In this form it can be generalized to arbitrary Banach spaces, as has been done by a number of authors ([7], [8]):

(1) *Let A, B be commuting Hermitian operators on a complex Banach space X. If $x \in X$ and $(A+iB)x = 0$, then $Ax = Bx = 0$.*

Another theorem about commutation properties of Hilbert space operators is the following: If c, d are normal operators on a Hilbert space H and $T_{c,d}x = xc - dx$, $x \in B(H)$ is the generalized commutator operator on $B(H)$, then $T_{c,d}^2 x = 0$ for some $x \in B(H)$ implies $T_{c,d}x = 0$ (see [7], Corollary 6, and [1]). This result can be generalized for operators on a Banach space X