On the other hand, from Corollary 1 and (5.7) we have

$$\inf_{g\in H^1(v)} \|k-g\|_{L^1(v)} \leqslant \liminf_{n\to\infty} \sup\left\{\left|\int_{\mathbf{T}} k(z)f(z)\frac{dz}{2\pi i}\right| : f\in H^\infty(v_n^{-1}),\ \|f\|_{H^\infty(v_n^{-1})} = 1\right\}$$

$$\leqslant \sup\left\{\left|\int_{\mathbf{T}} k(z)f(z)\frac{dz}{2\pi i}\right| : f\in H^\infty(v^{-1}),\ \|f\|_{H^\infty(v^{-1})} = 1\right\}$$

since $v_n \leqslant Cv$ a.e. implies $H^\infty(v_n^{-1}) \subset H^\infty(v^{-1})$ $(n = 1, 2, \ldots)$.

Now (5.5) in the case $p = 1$ follows from Corollary 2 with Φ_n replacing k.

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Received April 5, 1988
Revised version April 14, 1989 (2426)

STUDIA MATHEMATICA, T. XCVI (1990)

A multiplier characterization of analytic UMD spaces

by

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Abstract. We prove that the Banach spaces X for which analytic martingales converge unconditionally are precisely those for which certain multipliers are bounded on the Hardy space $H^1_X(T)$.

1. Introduction. The purpose of this paper is to characterize the complex Banach spaces X for which analytic martingales converge unconditionally in terms of boundedness of certain translation-invariant operators on the vector-valued Hardy spaces $H^1_X(T)$.

Bourgain [2] and Burkholder [3] have shown that the so-called UMD Banach spaces X, defined to be those in which Walsh–Paley martingales converge unconditionally, are precisely those for which the conjugate function operator is bounded from $L^2_X(T)$ to itself. Their methods are based on transference and we use a refinement of such arguments here.

We remark that the class of Banach spaces for which analytic martingales converge unconditionally is strictly larger than the class UMD and includes such spaces as $L^1(T)$, which do not even enjoy the Radon–Nikodym property [6].

The rest of this paper is arranged as follows. In the second section we introduce some basic definitions and provide a formal statement of the result given in the abstract. We also sketch the proof of the easy half of the theorem.

In the next section we reformulate the problem in probabilistic terms, following where possible an argument of McConnell [8]. In the penultimate section we establish the multiplier theorem. Our argument uses a result of Edgar [5] which allows us to approximate certain Brownian martingales by discrete-parameter analytic martingales. In the final section of this paper we mention some other properties of analytic UMD spaces.

Garling has introduced a more general class of martingales, termed Hardy martingales, which may be used to prove renorming theorems [6]. It is known that the Banach spaces for which analytic martingales converge unconditionally are those for which Hardy martingales converge unconditionally. Indeed, this follows from the techniques of this paper.

The author acknowledges the financial assistance of the Science and Engineering Research Council while this work was being undertaken. He also expresses his thanks to Dr. R. G. Haydon of Brasenose College, Oxford for encouragement.

1980 *Mathematics Subject Classification* (1985 *Revision*): Primary 46B99; Secondary 42A45, 42B30.

2. The main result. We introduce several basic definitions.

Let X be a complex Banach space, T^∞ the infinite polydisc endowed with Haar measure P.

DEFINITION 1. (i) An *L^1-bounded analytic martingale* f is a sequence (f_n) of functions in the Bochner–Lebesgue space $L^1_X(T^\infty)$ having the form

$$f_n(\Theta) = \sum_{k=1}^{n} \beta_k(\theta_1, \ldots, \theta_{k-1}) e^{i\theta_k} \tag{1}$$

with $\|f\|_1 = \sup_n \|f_n\|_1 < \infty$.

(ii) We say that X is an *analytic unconditional martingale difference* (*AUMD*) *space* if there is a constant C, depending only on the space X, such that if $\tilde{f} = (\tilde{f}_n)$, where

$$\tilde{f}_n(\Theta) = \sum_{k=1}^{n} \varepsilon_k \beta_k(\theta_1, \ldots, \theta_{k-1}) e^{i\theta_k} \tag{2}$$

is the transform of $f = (f_n)$ by a sequence of constants ε_k bounded in modulus by 1, then

$$\|\tilde{f}\|_1 \leqslant C\|f\|_1. \tag{3}$$

To state the theorem we need the following:

DEFINITION 2. We say that the multiplier operator T_m associated with the distribution $m = \sum m_n e^{in\theta}$ satisfies *strong Hörmander–Mikhlin conditions* if there is a constant C such that

(i) $|m_n| \leqslant C$, (ii) $n|\Delta m_n| = n|m_n - m_{n-1}| \leqslant C$,

(iii) $n^2|\Delta^2 m_n| = n^2|m_{n+1} - 2m_n + m_{n-1}| \leqslant C$.

We remark that condition (ii) is a consequence of (i) and (iii), but we include it since it is needed in the proof of Lemma 4.

Our main result is the following:

THEOREM 3. *The complex Banach space X belongs to AUMD if and only if T gives a bounded operator on $H^1_X(T)$ whenever T is a multiplier satisfying strong Hörmander–Mikhlin conditions.*

Proof. The backward implication follows from an application of a well-known method of transference. By an approximation argument one reduces to the case of

$$f = \sum_{k=1}^{n} \beta_k(\theta_1, \ldots, \theta_{k-1}) e^{i\theta_k}, \tag{4}$$

where the β_k are trigonometric polynomials. Consider the family of measure-preserving transformations of T^∞ given by $(\theta_k) \to (\theta_k + n_k\zeta)$, where ζ is a parameter and the n_k are positive integers chosen recursively so that four times the greatest member of the spectrum of

$$\zeta \to \sum_{j=0}^{k-1} \beta_j(\theta_1 + n_1\zeta, \ldots, \theta_{j-1} + n_{j-1}\zeta) e^{i\theta_j + in_j\zeta}$$

is less than the least member of the spectrum of

$$\gamma_k(\zeta) = \beta_k(\theta_1 + n_1\zeta, \ldots, \theta_{k-1} + n_{k-1}\zeta) e^{i\theta_k + in_k\zeta}. \tag{5}$$

By considering smooth partitions of unity on $[0, 1]$, one can find multipliers T given by $\sum m_n e^{i\theta}$, satisfying strong Hörmander–Mikhlin conditions, where the m_n are constant on long stretches of integers. For a given sequence of constants ε_n, bounded in modulus by 1, we can find such a T so that it multiplies γ_k by ε_k $(k = 1, \ldots, n)$. Since T is bounded on $H^1_X(T)$, one can complete the proof as follows:

$$\begin{aligned} \|f\|_1 &= \iint \Big|\sum \beta_k(\theta_1 + n_1\zeta, \ldots, \theta_{k-1} + n_{k-1}\zeta) e^{i\theta_k + in_k\zeta}\Big|\, d\zeta\, dP \\ &\geqslant C \iint \Big|\sum \varepsilon_k \beta_k(\theta_1 + n_1\zeta, \ldots, \theta_{k-1} + n_{k-1}\zeta) e^{i\theta_k + in_k\zeta}\Big|\, d\zeta\, dP = C\|\tilde{f}\|_1. \end{aligned} \tag{6}$$

3. Probabilistic formulation of the problem. In this section it will be convenient to use the following notation. Let f be an analytic trigonometric polynomial valued in an AUMD space X, and let $m = \sum m_n e^{in\theta}$, with T the convolution operator associated with this distribution. Let u, v, w denote the Poisson integrals of f, m, Tf respectively. We may suppose without loss that the first few derivatives of u and of v vanish at the origin and that v is a trigonometric polynomial.

By the semigroup property of the Poisson integrals $(p_r)_{0<r<1}$, we have

$$w(r_1 r_2 e^{i\theta}) = \int v(r_1 e^{i\varphi}) u(r_2 e^{i(\theta-\varphi)})\, d\varphi (2\pi)^{-1}. \tag{7}$$

We differentiate (7) twice with respect to r_1, once with respect to r_2 and set $r_1 = s$, $r_2 = s^3$. This gives us, after a little reduction,

$$\begin{aligned} s^7 w_{,rrr}(s^4 e^{i\theta}) &= \int (v_{,rr}(se^{i\varphi}) - 2s^{-1} v_{,r}(se^{i\varphi})) u_{,r}(s^3 e^{i(\theta-\varphi)})\, d\varphi (2\pi)^{-1} \\ &\quad + 2\int s^{-4} v_{,r}(se^{i\varphi}) u(s^3 e^{i(\theta-\varphi)})\, d\varphi (2\pi)^{-1}. \end{aligned} \tag{8}$$

We now integrate the following expression twice, and change variables. This gives us

$$\begin{aligned} Tf(e^{i\theta}) &= \int_0^1 w_{,r}(re^{i\theta})\, dr = 2^{-1} \int_0^1 (1-r)^2 w_{,rrr}(re^{i\theta})\, dr \\ &= 2\int_0^1 (1-s^4)^2 w_{,rrr}(s^4 e^{i\theta}) s^3\, ds \\ &= 2\iint (1-s^4)^2 (v_{,rr}(se^{i\varphi}) - 2s^{-1} v_{,r}(se^{i\varphi})) u_{,r}(s^3 e^{i(\theta-\varphi)}) s^{-4}\, d\varphi (2\pi)^{-1}\, ds \\ &\quad + 4\int_0^1 \int (1-s^4)^2 v_{,r}(se^{i\varphi}) u(s^3 e^{i(\theta-\varphi)}) s^{-8}\, d\varphi (2\pi)^{-1}\, ds \end{aligned} \tag{9}$$

where we have used (8) at the last step.

The $L^1_X(T)$ norm of the last term in equation (9) is estimated as follows. Since $s^{-8}(1-s^4)^2 v_{,r}(se^{i\varphi})$ is bounded we have

$$(10)\quad \int \Big|\int_0^1 \int (1-s^4)^2 v_{,r}(se^{i\varphi}) u(s^3 e^{i(\theta-\varphi)}) s^{-8}\, d\varphi\, ds \Big| d\theta$$
$$\leqslant \int_0^1 \int (1-s^4)^2 |v_{,r}(se^{i\varphi})| s^{-8} \int |u(s^3 e^{i(\theta-\varphi)})|\, d\theta\, ds\, d\varphi \leqslant C \int |f(\theta)|\, d\theta.$$

By the smoothness assumptions on f, we can approximate the integral with respect to s in the other term in (9) by a Riemann sum. The following approximants converge boundedly to Tf:

$$(11)\quad g_N(e^{i\theta}) = \sum_{j=0}^{N-1} \int (1-r_j^4)^2 r_j^{-6} \big(v_{,rr}(r_j e^{i\varphi}) - 2r_j^{-1} v_{,r}(r_j e^{i\varphi})\big)\big[u(r_{j+1}^2 r_j e^{i(\varphi-\theta)}) - u(r_j^3 e^{i(\varphi-\theta)})\big]\, d\varphi/2\pi$$

where $0 < A = r_0 < r_1 < \ldots < r_N = 1$ and A is chosen sufficiently small. We need the following elementary lemma, which can be proved using Abel summation.

LEMMA 4. *The function*

$$(12)\quad h(r, \varphi) = (1-r^4)^2 \big(v_{,rr}(re^{i\varphi}) - 2r^{-1} v_{,r}(re^{i\varphi})\big) p_r^{-1}(\varphi) r^{-6}$$

is bounded ($1/2 < r < 1$, $\varphi \in R$) *when T is a multiplier of strong Hörmander–Mikhlin type.*

The basic idea of the following proof is that the image of a Brownian motion in the unit disc under a vector-valued analytic polynomial function behaves like an analytic martingale. The situation is complicated by the fact that it seems necessary to introduce auxiliary Brownian motions.

We let z_t be Brownian motion in D and consider various conditional probability measures on the Wiener space. Let P^θ denote the probability measure given by conditioning z_t to begin at the origin and to exit the unit disc at θ, and let P_z denote the measure given by conditioning the motion to begin at z. The following result may be obtained from Durrett's book [4].

LEMMA 5. *Let z_t be Brownian motion conditioned to begin at $z_0 = 0$ and to exit D at θ. Then there is a strong Markov process $X(r)$ $(0 < r \leqslant 1)$ valued in T with sample paths continuous a.s. such that $z_{S_r} = re^{iX(r)}$ where $S_r = \inf\{t\colon |z_t| = r\}$. Further, this process has independent increments.*

We introduce a Brownian motion $\tilde{z}_t$ independent of z_t and denote its expectations, probabilities and associated X-process with a tilde. For convenience we denote

$$(13)\quad d_j = u\big(r_{j+1} r_j,\ X(r_{j+1}) - \tilde{X}(r_j)\big) - u\big(r_j^2,\ X(r_j) - \tilde{X}(r_j)\big).$$

LEMMA 6. *We have the following representation for g_N:*

$$(14)\quad g_N(e^{i\theta}) = \tilde{E}^0 E^\theta \sum_{j=0}^{N-1} h\big(r_j, \tilde{X}(r_j)\big) d_j.$$

Proof. We recall that the function u is harmonic on D and that the transition densities of the Markov process $X(r)$ are given by Poisson kernels. Hence, applying Fubini's Theorem we have

$$(15)\quad E^\theta \tilde{E}^0\big(h(r_j, \tilde{X}(r_j)) d_j\big)$$
$$= E^\theta \int p_{r_j}(\varphi) h(r_j, \varphi)\big[u\big(r_{j+1} r_j, X(r_{j+1}) - \varphi\big) - u\big(r_j^2, X(r_j)\big)\big]\, d\varphi/2\pi$$
$$= E^\theta \int (1-r_j^4)^2 r_j^{-6}\big(v_{,rr}(r_j e^{i\varphi}) - 2r_j^{-1} v_{,r}(r_j e^{i\varphi})\big)\big[u\big(r_{j+1} r_j, X(r_{j+1}) - \varphi\big) - u\big(r_j^2, X(r_j) - \varphi\big)\big]\, d\varphi/2\pi$$
$$= \int (1-r_j^4)^2 r_j^{-6}\big(v_{,rr}(r_j e^{i\varphi}) - 2r_j^{-1} v_{,r}(r_j e^{i\varphi})\big) E^\theta\big[u\big(r_{j+1} r_j, X(r_j) - \varphi\big) - u\big(r_j^2, X(r_j) - \varphi\big)\big]\, d\varphi/2\pi$$
$$= \int (1-r_j^4)^2 r_j^{-6}\big(v_{,rr}(r_j e^{i\varphi}) - 2r_j^{-1} v_{,r}(r_j e^{i\varphi})\big)\big[u(r_{j+1}^2 r_j, \theta - \varphi) - u(r_j^3, \theta - \varphi)\big]\, d\varphi/2\pi.$$

On summing over j we obtain the desired representation.

4. An approximation argument. In the following section we wish to apply martingale transforms to martingales constructed by applying analytic functions to Brownian motion. In order to justify this, we now show how to approximate such a martingale by an analytic martingale.

There is no loss in generality in supposing that X is finite-dimensional, since AUMD is evidently a local property. This serves to simplify the construction of martingales. We note that under the conditional probability $P_{Ae^{i\varphi}}$, $\sum d_j$ is a martingale for fixed $\tilde{\omega}$. We aim to approximate this by an analytic martingale.

At the jth stage of construction we consider the process

$$(16)\quad u\big((z_t + r_j e^{iX(r_j)})(r_j e^{-i\tilde{X}(r_j)})\big) \quad (t \geqslant 0),$$

where z_t is a Brownian motion starting at the origin. By a result of Edgar [5], we can find an analytic martingale

$$(17)\quad \sum_{n_j+1}^{n_{j+1}} \beta_k\big(\theta_{n_j+1}, \ldots, \theta_{k-1}, X(r_j), \tilde{\omega}\big) e^{i\theta_k}$$

depending measurably on $\tilde{\omega}$, $X(r_j)$ and such that

$$(18)\quad E\Big|u\big((z_{\tau_{j+1}} + r_j e^{iX(r_j)}) r_j e^{-i\tilde{X}(r_j)}\big) - u\big(r_j^2 e^{i(X(r_j) - \tilde{X}(r_j))}\big) - \sum_{n_j+1}^{n_{j+1}} \beta_k\big(\theta_{n_j+1}, \ldots, \theta_{k-1}, X(r_j), \tilde{\omega}\big) e^{i\theta_k}\Big| < \varepsilon/N \qquad \forall \tilde{\omega},$$

where $\tau_{j+1} = \inf\{t\colon |z_t + r_j e^{iX(r_j)}| = r_{j+1}\}$.

In this way we construct a discrete-parameter analytic martingale $\sum \beta_k(\theta_1, \ldots, \theta_{k-1}, \tilde{\omega})e^{i\theta_k}$ which approximates $\sum d_k$ in the sense of (18).

5. Conclusion of the proof of the theorem. We are now in a position to apply the previous observations to estimate the $L^1_X(T)$ norm of g_N. Lemma 6 gives

$$(19)\qquad \int |g_N(e^{i\theta})|\, d\theta \leqslant \int \tilde{E}^0 E^\theta \Big|\sum_{j=0}^{N-1} h(r_j, \tilde{X}(r_j)) d_j\Big|\, d\theta$$

$$= \int \tilde{E}^0 \int p_A(\theta-\varphi) E^\theta_{Ae^{i\varphi}} \Big|\sum_{j=0}^{N-1} h(r_j, \tilde{X}(r_j)) d_j\Big|\, d\theta\, d\varphi$$

$$= \int \tilde{E}^0 E_{Ae^{i\varphi}} \Big|\sum_{j=0}^{N-1} h(r_j, \tilde{X}(r_j)) d_j\Big|\, d\varphi.$$

By the approximation argument of the previous section and the hypothesis that X is in AUMD, this is bounded by

$$(20)\qquad C\int \tilde{E}^0 E_{Ae^{i\varphi}} \Big|\sum_{j=0}^{N-1} d_j\Big|\, d\varphi + \varepsilon \leqslant C\int \tilde{E}^0 E^\theta \Big|\sum_{j=0}^{N-1} d_j\Big|\, d\theta + \varepsilon.$$

We now let $\eta_{2j} = r_j^2$, $\eta_{2j+1} = r_j r_{j+1}$, $\alpha_{2j} = 1$, $\alpha_{2j+1} = 0$ for $j = 0, 1, \ldots, N-1$.

By the properties of $X(r)$ mentioned in Lemma 5 of Section 3 of this paper, the sequence of random variables $X(r_0) - \tilde{X}(r_0)$, $X(r_1) - \tilde{X}(r_0), \ldots, X(r_{N-1}) - \tilde{X}(r_{N-1})$ has the same joint distributions under $P^\theta \otimes P^0$ as $X(\eta_0), X(\eta_1), \ldots, X(\eta_{2N-1})$ under P^θ. Hence we have

$$(21)\qquad \int \tilde{E}^0 E^\theta \Big|\sum_{j=0}^{N-1} d_j\Big|\, d\theta = \int E^\theta \Big|\sum_{j=0}^{2N-2} \alpha_j \big(u(\eta_j, X(\eta_j)) - u(\eta_{j+1}, X(\eta_{j+1}))\big)\Big|\, d\theta$$

$$= \iint p_A(\varphi-\theta) E^\theta_{Ae^{i\varphi}} \Big|\sum_{j=0}^{2N-2} \alpha_j \big(u(\eta_j, X(\eta_j)) - u(\eta_{j+1}, X(\eta_{j+1}))\big)\Big|\, d\varphi\, d\theta.$$

We are now able to use the fact that this latest expression is the L^1_X norm of a martingale which we can approximate by an analytic martingale. This gives us

$$(22)\qquad \int |g_N(e^{i\theta})|\, d\theta \leqslant C\int E_{Ae^{i\varphi}} \Big|\sum_{j=0}^{2N-2} \big(u(\eta_j, X(\eta_j)) - u(\eta_{j+1}, X(\eta_{j+1}))\big)\Big|\, d\varphi.$$

Since the norm on the Banach space X is subharmonic and $X(r)$ has density $p_A(\cdot - \varphi)$ under $P_{Ae^{i\varphi}}$, we can complete the calculation as follows:

$$(23)\qquad \int |g_N(e^{i\theta})|\, d\theta \leqslant C\int E_{Ae^{i\varphi}} |u(\eta_{2N-2}, X(\eta_{2N-2}))|\, d\varphi$$

$$\leqslant C\int E_{Ae^{i\varphi}} |u(1, X(1))|\, d\varphi = C\iint p_A(\varphi-\theta) |f(e^{i\theta})|\, d\theta\, d\varphi = C\int |f(e^{i\theta})|\, d\theta.$$

6. Concluding remarks. It is a consequence of Theorem 3 that if X is an analytic UMD space with cotype 2, then X satisfies Paley's Theorem, i.e. there is a positive constant C, depending only on X, such that

$$(24)\qquad \int \Big|\sum e^{in\theta} x_n\Big|\, d\theta \geqslant C\Big(\sum |x_{2^n}|^2\Big)^{1/2}$$

for all analytic trigonometric polynomials $\sum e^{in\theta} x_n$.

To see this, note that there exists a sequence (T_k) of strong Hörmander–Mikhlin multipliers such that $T_k(e^{i2^k\theta}) = e^{i2^k\theta}$ and for every sequence of signs $\varepsilon = (\varepsilon_k)$, $T_\varepsilon = \sum \varepsilon_k T_k$ is a strong Hörmander–Mikhlin multiplier with norm bounded by a constant independent of ε. Then the stated properties of X give positive constants C such that

$$(25)\qquad \int \Big|\sum e^{in\theta} x_n\Big|\, d\theta \geqslant C E_\varepsilon \int \Big|T_\varepsilon\Big(\sum_n e^{in\cdot} x_n\Big)\Big|\, d\theta$$

$$\geqslant C\int E_\varepsilon \Big|\sum_k \varepsilon_k T_k\Big(\sum_n e^{in\cdot} x_n\Big)\Big|\, d\theta$$

$$\geqslant C\int \Big(\sum_k \Big|T_k\Big(\sum_n e^{in\cdot} x_n\Big)\Big|^2\Big)^{1/2} d\theta \geqslant C\Big(\sum_k \Big(\int \Big|T_k\Big(\sum_n e^{in\cdot} x_n\Big)\Big|\, d\theta\Big)^2\Big)^{1/2}$$

$$\geqslant C\Big(\sum_k |x_{2^k}|^2\Big)^{1/2}.$$

The hypothesis that X has cotype two is necessary, as can be seen from a transfer argument between the Sidon set $(2^n)_{n\geqslant 0}$ in $\mathbf{Z}$ and the subset of the Cantor group on which the Rademacher functions "live".

The validity of the inequality (24) does not imply that X is in AUMD, however. In [7] the authors show that the space c_1 of trace-class operators does not belong to AUMD, whereas c_1 satisfies Paley's Theorem [1].

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Received July 4, 1988 (2454)
Revised version February 13 and April 13, 1989

STUDIA MATHEMATICA, T. XCVI (1990)

Weighted Lorentz norm inequalities for integral operators

by

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Abstract. Conditions depending on the kernel $K(x, y)$ are given for weight functions w and v so that the integral operator $Kf(x) = \int_{-\infty}^{x} K(x, y) f(y)\, dy$, where $K(x, y) \geqslant 0$ is defined on $\Delta = \{(x, y)\colon y < x\}$, is bounded from a Lorentz space $L^{r,s}((-\infty, \infty), v\,dx)$ into another Lorentz space $L^{p,q}((-\infty, \infty), w\,dx)$. In Theorem 1 the kernel $K(x, y)$ is supposed to be nonincreasing in x. In Theorem 2 the kernel is supposed to be nondecreasing in y. Dual results for the dual operators are given. Finally, it is shown that the stated conditions on the kernels are not always necessary.

1. Introduction. Our purpose is to find conditions that imply weighted Lorentz norm inequalities for the integral operators K and K^* defined by

$$Kf(x) = \int_{-\infty}^{x} K(x, y) f(y)\, dy, \tag{1.1}$$

$$K^*f(x) = \int_{x}^{\infty} K(y, x) f(y)\, dy, \tag{1.2}$$

where $K(x, y)$ is defined on $\Delta = \{(x, y) \in \mathbf{R}^2\colon y < x\}$ and it is nonnegative. Two kinds of kernels $K(x, y)$, either nonincreasing in x, or nondecreasing in y, are considered separately. In the last section we deal with the necessity of our conditions.

The Hardy operator $Tf(x) = \int_0^x f$, $x > 0$, and the modified Hardy operators $T_\eta f(x) = x^{-\eta} \int_0^x f$, with real η, are examples of the above operators. Several authors have obtained inequalities for weighted Lebesgue norms for these operators (cf. [2]–[4], [7], [9] and [10]). Our results compare with others in the literature as follows. If we restrict ourselves to the Hardy operator, the sufficient condition (1.3) of Theorem 1 is known to be also a necessary condition ([8]). The same is true for condition (1.5) of Theorem 2 when restricted to the modified Hardy operators. If we consider only Lebesgue norms our results are related to those of Andersen and Heinig [1] as follows. Our monotonicity conditions on $K(x, y)$ are more general than those in [1], while the weights considered by Andersen and Heinig are in a class larger than ours.

1980 *Mathematics Subject Classification* (1985 *Revision*): Primary 26D15.

* Research partially supported by CONICOR under Grant 981/87, Córdoba, Argentina.