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On the rank of a class of bijective substitutions

by

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Abstract. We consider the problem raised by M. K. Mentzen of whether it is possible, for any pair (k, n), $k \leqslant n$, to find an ergodic automorphism T with rank equal to n and maximal spectral multiplicity k. We show that a general class of bijective substitutions over r symbols have rank r. This result is used to solve Mentzen's problem for the case $(2, n)$ (previously solved by Mentzen in the case $(1, n)$). The maximal spectral type of the main examples is explicitly constructed.

§ 1. Introduction. For each natural number $r \geqslant 2$, Mentzen [10] constructed an ergodic automorphism with rank r and simple spectrum. It appears to be quite difficult to give examples of ergodic automorphisms with rank r, $r \geqslant 2$, and few such examples are known. Requiring these examples to have nonsimple spectrum adds to the difficulty. Mentzen suggested that it should be possible to construct, for any pair of natural numbers (k, n), $k \leqslant n$, an ergodic automorphism with rank n and maximal spectral multiplicity k. (It was shown by Chacon that the rank is an upper bound for the maximal spectral multiplicity.)

Our main theorem is a general result concerning the rank of a class of bijective substitutions of length r over r symbols. This result is applied to some examples first studied in Goodson [3], where the Morse sequence $x = 010 \times 010 \times \ldots$ over $\mathbf{Z}_3$ was shown to have maximal spectral multiplicity equal to two. We generalize this transformation by constructing for each $r \geqslant 3$, a bijective substitution over r symbols with maximal spectral multiplicity equal to two. The case $r = 2$ turns out to be the well known Thue–Morse sequence, shown by del Junco [4] to have rank 2 and simple spectrum. These examples are particularly interesting because it is possible to give an explicit formula for their maximal spectral type, again the case $r = 2$ being well known. An application of our main theorem now shows that these transformations have rank r.

The proof of our main theorem depends on a general result and methods of M. K. Mentzen for estimating the rank of a substitution. We use results of Coquet, Kamae and Mendès France [2] and Queffélec [11] to determine the maximal spectral multiplicity of our examples. See also the recent book by M. Queffélec [12].

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§ 2. Definitions. Let T be an ergodic automorphism of the Lebesgue space $(X, \mathscr{B}, \mu)$. A sequence of partitions ξ_n, $n \geq 1$, is said to *converge to* $\mathscr{B}$ if for every $\varepsilon > 0$ there exists n_0 such that for each $n \geq n_0$ there is a set B_n, a union of atoms of ξ_n, for which $\mu(B \triangle B_n) < \varepsilon$.

DEFINITION 1. (i) We say that T has *rank at most* r if there exist sets F_k^j, $1 \leq j \leq r$, $k = 1, 2, \ldots$, and integers n_k^j, $1 \leq j \leq r$, $k = 1, 2, \ldots$, such that for fixed k the sets $\{T^i F_k^j\}_{i=0}^{n_k^j - 1}$, $1 \leq j \leq r$, are pairwise disjoint and the partitions

$$\xi_k = \{G_k, T^i F_k^j \colon 0 \leq i \leq n_k^j, \ 1 \leq j \leq r\}$$

converge to $\mathscr{B}$, where

$$G_k = X - \bigcup_{j=1}^{r} \bigcup_{i=0}^{n_k^j - 1} T^i F_k^j.$$

(ii) We say T has *rank* r if it has rank at most r, but not rank at most $r-1$. If T is of rank r for no $r \geq 1$, we say T has *infinite rank.*

We now outline the definitions and properties of substitutions of constant length (see [8] or [12] for more details).

Let $r \geq 2$ be an integer and write $N_r = \{0, 1, \ldots, r-1\}$, $N_r^* = \bigcup_{n \geq 1} N_r^n$. The members of N_r^* are called *blocks*. If $B \in N_r^*$, $B = (b_0, b_1, \ldots, b_{n-1})$ then $B[s, t] = (b_s, \ldots, b_t)$ and $B[s] = B[s, s]$ for $0 \leq s \leq t$. n is called the *length* of B and is denoted by $|B|$.

DEFINITION 2. Let $\lambda \geq 2$ be an integer and $\theta \colon N_r \to N_r^\lambda$ a function. For any $n > 0$ there is a natural extension of θ, $\theta \colon N_r^n \to N_r^{n\lambda}$, and also to a map from $N_r^{\mathbf{Z}}$ to itself, given by

$$\theta(B) = \theta(b_0)\theta(b_1)\ldots\theta(b_{n-1}) \quad \text{if } B = (b_0, b_1, \ldots, b_{n-1})$$

and

$$\theta(x) = \ldots\theta(b_{-1})\theta(b_0)\theta(b_1)\ldots \quad \text{if } x = \ldots b_{-1} b_0 b_1 \ldots \in N_r^{\mathbf{Z}},$$

where in the latter case, by convention the 0th symbol of $\theta(x)$ coincides with the initial symbol of $\theta(b_0)$. θ is assumed to be a one-one map; the n-fold composition of θ is denoted by θ^n.

If there exists $n \geq 1$ such that for any $i, j \in N_r$, $\theta^n(i)[k] = j$ for some k, $0 \leq k \leq \lambda^n - 1$, then θ is called a *substitution of constant length* λ on r symbols.

For a substitution of constant length there is a fixed point $x_0 \in N_r^{\mathbf{Z}}$ such that if T_θ is the shift on $N_r^{\mathbf{Z}}$ then the restriction of T_θ to

$$X(\theta) = \overline{\{T_\theta^n(x_0) \colon n \in \mathbf{Z}\}},$$

the orbit closure of x_0, is a uniquely ergodic dynamical system with unique T-invariant measure μ_θ satisfying

$$\mu_\theta(B) = \lim_{k \to \infty} \frac{\mathrm{fr}(B, \theta^k(0))}{\lambda^k}$$

for any block B, where we may assume $x_0[0] = 0$, and

$$\mathrm{fr}(B, C) = \mathrm{card}\{t \colon C[t, t+|B|-1] = B\}.$$

A metric d can be defined on N_r^n, $n = 1, 2, \ldots$, by

$$d(B, C) = \mathrm{card}\{t \colon B[t] \neq C[t]\}/n$$

for $B, C \in N_r^n$.

DEFINITION 3. With a substitution θ of constant length λ we associate a *λ-automaton* $\{\varphi_j \colon j = 0, \ldots, \lambda-1\}$ whose jth instruction φ_j is the map $\varphi_j \colon N_r \to N_r$ defined by $\varphi_j(i) = \theta(i)[j]$, the jth letter in the word $\theta(i)$.

(i) If the instructions φ_j are bijections then θ is said to be a *bijective* substitution.

(ii) If the instructions φ_j commute then the substitution is said to be *commutative.*

Substitutions which are both bijective and commutative arise as abelian Morse sequences.

We now state the result of Mentzen [10] which gives an estimate of the rank of a substitution. This result applies to substitutions which satisfy for some constant $c > 0$

$$(*) \qquad d(\theta^n(i), \theta^n(j)) \geq c > 0$$

for each $i \neq j$, $n \geq 1$. Note that it is well known that a substitution on r symbols has rank at most r.

THEOREM 1 ([10]). (i) *If θ is a substitution of constant length there is a constant M_θ such that for every block $B \in N_r^n$*

$$|B|\mu_\theta(B) \leq M_\theta.$$

(ii) *If the substitution of constant length θ satisfies condition* $(*)$ *and if $M_\theta < 1/m$ then* rank $T_\theta \geq m+1$.

Remark. Note that bijective substitutions necessarily satisfy $(*)$.

§ 3. Bijective substitutions of length r. We now state our main result concerning the rank of bijective substitutions over r symbols of length r.

THEOREM 2. *Let θ be a bijective substitution on r symbols of length r.*

(1) *If $\theta(i)[k, k+1] = \theta(i')[k', k'+1]$ implies $i = i'$ and $k = k'$, then T_θ has rank r.*

Proof. Let B be any finite block and write $M_\theta = 1/r$. We show that

$$|B|\mu_\theta(B) \leqslant M_\theta < \frac{1}{r-1}$$

and hence by Theorem 1, rank $T_\theta \geqslant r$, and since rank $T_\theta \leqslant r =$ number of symbols, we have rank $T_\theta = r$.

We split the proof into a number of lemmas.

LEMMA 1. *If θ is a bijective substitution on r symbols of length λ then* $\mu_\theta(i) = 1/r$, $i \in N_r$.

Proof. Clearly $\sum_{k\in N_r} \mathrm{fr}(i, \theta^n(k)) = \lambda^n$. Dividing by λ^n, letting $n \to \infty$ and using the unique ergodicity of T_θ we obtain $r\mu_\theta(i) = 1$.

For the rest of the proof we assume that θ is a substitution of length r satisfying (1).

LEMMA 2. $\mu_\theta(ij) \leqslant 1/(r(r-1))$.

Proof. If $\mathscr{B}_2$ is the set of blocks of length 2 occurring in x_0 (i.e. $\mathscr{B}_2 = \{ij\colon \mu_\theta(ij) > 0\}$) we define a *partial function* $\varphi\colon \mathscr{B}_2 \to \mathscr{B}_2$ by

$$\varphi(ij) = i'j' \quad \text{iff} \quad \theta(i')[r-1] = i,\ \theta(j')[0] = j.$$

Since θ is bijective, φ is a partial function, for suppose $i'j' = \varphi(ij) = i''j''$; then

$$\theta(i')[r-1] = i = \theta(i'')[r-1], \qquad \theta(j')[0] = j = \theta(j'')[0]$$

and since θ is bijective, $i' = i''$ and $j' = j''$.

We split the proof of the lemma into a number of cases.

Case A: φ is not defined on ij. This means that $\mu_\theta(ij) > 0$ but $\mu_\theta(i'j') = 0$. It follows from (1) that there is a unique $k_{ij} \in N_r$ such that ij appears in $\theta(k_{ij})$ in a unique position. Therefore $\mathrm{fr}(ij, \theta^n(0)) = \mathrm{fr}(k_{ij}, \theta^{n-1}(0))$ and hence $\mu_\theta(ij) = (1/r)\,\mu(k_{ij}) = 1/r^2$.

Case B: ij appears in $\theta(k_{ij})$ and $\varphi(ij), \varphi^2(ij), \ldots, \varphi^s(ij)$ are well defined with $\varphi^s(ij) = ij$. Then

$$\begin{aligned}
\mathrm{fr}(ij, \theta^n(0)) &= \mathrm{fr}(k_{ij}, \theta^{n-1}(0)) + \mathrm{fr}(\varphi(ij), \theta^{n-1}(0))\\
&= \mathrm{fr}(k_{ij}, \theta^{n-1}(0)) + \mathrm{fr}(k_{\varphi(ij)}, \theta^{n-2}(0))\\
&\quad + \mathrm{fr}(\varphi^2(ij), \theta^{n-2}(0))\\
&= \ldots\\
&= \mathrm{fr}(k_{ij}, \theta^{n-1}(0)) + \ldots + \mathrm{fr}(k_{\varphi^{s-1}(ij)}, \theta^{n-s}(0))\\
&\quad + \mathrm{fr}(ij, \theta^{n-s}(0)).
\end{aligned}$$

Hence

$$\mu_\theta(ij) = (1/r)(1/r) + (1/r^2)(1/r) + \ldots + (1/r^s)(1/r) + (1/r^s)\mu_\theta(ij)$$

and it follows that $\mu_\theta(ij) = 1/(r(r-1))$.

Remark. If $k_{\varphi^l(ij)}$ were not defined for some $1 < l \leqslant s$ it would mean that $\mu_\theta(ij) < 1/(r(r-1))$.

Case C: ij appears in $\theta(k_{ij})$ with $\varphi(ij), \ldots, \varphi^{s-1}(ij)$ well defined, but φ is not defined on $\varphi^{s-1}(ij)$. Then

$$\mathrm{fr}(ij, \theta^n(0)) = \mathrm{fr}(k_{ij}, \theta^{n-1}(0)) + \ldots + \mathrm{fr}(k_{\varphi^{s-1}(ij)}, \theta^{n-s}(0)),$$

so that $\mu_\theta(ij) = (r^s-1)/(r^{s+1}(r-1)) < 1/(r(r-1))$, with the same remark applying as in Case B.

Case D: ij appears in $\theta(k_{ij})$ with $\varphi(ij), \ldots, \varphi^s(ij)$ well defined and $\varphi^l(ij) = \varphi^{l+\varrho}(ij)$ where $l+\varrho = s$. By Case B, $\mu_\theta(\varphi^l(ij)) = 1/(r(r-1))$ (or the inequality by the remark after Case B) and

$$\begin{aligned}
\mathrm{fr}(ij, \theta^n(0)) = \mathrm{fr}(k_{ij}, \theta^{n-1}(0)) + \mathrm{fr}(k_{\varphi(ij)}, \theta^{n-2}(0)) + \ldots\\
+ \mathrm{fr}(k_{\varphi^{l-1}(ij)}, \theta^{n-1}(0)) + \mathrm{fr}(\varphi^l(ij), \theta^{n-1}(0)).
\end{aligned}$$

Hence

$$\begin{aligned}
\mu_\theta(ij) &= (1/r)(1/r) + (1/r^2)(1/r) + \ldots + (1/r^l)(1/r) + (1/r^l)(1/(r(r-1)))\\
&\leqslant 1/(r(r-1)).
\end{aligned}$$

Case E: ij does not appear in any $\theta(k)$, $k \in N_r$. Then there exists a unique (i', j') such that $\theta(i')[r-1] = i$ and $\theta(j')[0] = j$. This implies that $\mu_\theta(ij) \leqslant \mu_\theta(\theta(i')[0, 1]) \leqslant 1/(r(r-1))$.

LEMMA 3. $\mu_\theta(ijk) \leqslant 1/r^2$ *for any 3-block* ijk.

Proof. *Case* A: There is a unique $s_{ijk} \in N_r$ such that ijk appears in $\theta(s_{ijk})$; then $\mu_\theta(ijk) = (1/r)\,\mu_\theta(s_{ijk}) = 1/r^2$.

Case B: There exist i', j', i'', j'' such that

$$ijk = \theta(i')[r-2, r-1]\,\theta(j')[0] \quad \text{and/or}$$

$$ijk = \theta(i'')[r-1]\,\theta(j'')[0, 1].$$

Then $\mathrm{fr}(ijk, \theta^n(0)) \leqslant \mathrm{fr}(i'j', \theta^{n-1}(0)) + \mathrm{fr}(i''j'', \theta^{n-1}(0))$ and so

$$\begin{aligned}
\mu_\theta(ijk) &\leqslant (1/r)\,\mu_\theta(i'j') + (1/r)\,\mu_\theta(i''j'')\\
&\leqslant 2/(r^2(r-1)) \leqslant 1/r^2 \quad \text{for } r \geqslant 3.
\end{aligned}$$

LEMMA 4. *There is a constant $M < 1/(r-1)$ such that for each block B with $|B| \leqslant r$ and $\mu_\theta(B) > 0$ we have*

$$|B|\,\mu_\theta(B) \leqslant M.$$

LEMMA 5. *If $\theta^m(i) = \theta^m(j)\,\theta^m(j')[u, u+r^m-1]$ then either $u = 0$ or $u = r^m$.*

Proof. For $m = 1$ the result follows from (1). The general result now follows by induction in a straightforward manner.

LEMMA 6. *For any block B with $|B| = n$ and $r^m+1 \leqslant n \leqslant r^{m+1}$, $m \geqslant 1$, there is a unique representation of B in the form*

$$B = B_1\,\theta^m(i_1)\,\theta^m(i_2)\ldots\theta^m(i_t)\,B_2,$$

$0 \leqslant t \leqslant r-2$, $0 < |B_i| \leqslant r^m$, $B_1 = \theta^m(j_1)[u, r^m-1]$, $B_2 = \theta^m(j_2)[0, v]$, *where* $j_1, j_2, i_1, \ldots, i_t$ *are unique.*

Proof. B is a block appearing in x_0 which is a concatenation of θ^m-blocks, so by Lemma 5 we can write B in the form

$$B = B_1\,\theta^m(i_1)\,\theta^m(i_2)\ldots\theta^m(i_t)\,B_2$$

for unique $i_1, \ldots, i_t$, $0 \leqslant t \leqslant r-2$, where $0 < |B_i| \leqslant r^m$. It follows that it suffices to consider the case when $B = B_1 B_2$, $0 < |B_i| < r^m$. In this case B must be of the form

$$B = \ \ldots k\theta^{m-1}(i)\,\theta^{m-1}(j)\,l\ldots \qquad (r \geqslant 3).$$

The result now follows from Lemma 5 and (1).

LEMMA 7. *For any block B with $\mu_\theta(B) > 0$*

$$|B|\,\mu_\theta(B) \leqslant M_\theta \quad \textit{where} \quad M_\theta = 1/r.$$

Proof. We have shown this if $|B| \leqslant r$ so suppose $|B| \geqslant r+1$, $|B| = n$ say. Choose $m \in \mathbf{N}$ satisfying $r^m+1 \leqslant n \leqslant r^{m+1}$; then by Lemma 6, B can be uniquely represented in the form

$$B = B_1\,\theta^m(i_1)\ldots\theta^m(i_t)\,B_2$$

where $0 \leqslant t \leqslant r-2$, $0 < |B_i| \leqslant r^m$ and $B_1 = \theta^m(j_1)[u, r^m-1]$, $B_2 = \theta^m(j_2)[0, v]$, so

$$B = \theta^m(j_1)\,\theta^m(i_1)\ldots\theta^m(i_t)\,\theta^m(j_2)[u, v].$$

It follows that

$$n\mu_\theta(B) = \frac{n\mu_\theta(j_1 i_1 \ldots i_t j_2)}{r^m} = \frac{n(t+2)\,\mu_\theta(j_1 i_1 \ldots i_t j_2)}{(t+2)\,r^m} \leqslant \frac{nM_\theta}{(t+2)\,r^m} \leqslant M_\theta$$

since $n = tr^m + |B_1| + |B_2| \leqslant tr^m + 2r^m$.

§ 4. **The rank and maximal spectral multiplicity of examples.** Let $G = \{e, g_1, \ldots, g_{r-1}\}$ be a finite (possibly nonabelian) group with r distinct elements and identity e. Define θ_G on G by

$$\theta(e) = e, g_{i_1}, \ldots, g_{i_{r-1}},$$
$$\theta(g) = g, gg_{i_1}, \ldots, gg_{i_{r-1}}, \qquad g \in G,$$

where $i_1, \ldots, i_{r-1} \in \{0, \ldots, r-1\}$ and $g_0 = e$.

Subject to suitable restrictions, θ_G satisfies the conditions of Definition 2 and is thus a substitution on G of constant length r. Under these conditions we have:

THEOREM 3. (i) *θ_G is a bijective substitution.*

(ii) *If G is an abelian group, θ_G is commutative.*

(iii) *The uniquely ergodic transformation arising from θ_G has rank r if*

$$(g_{i_j}, g_{i_{j+1}}) \neq (gg_{i_s}, gg_{i_{s+1}})$$

for each $g \in G$ whenever $j \neq s$.

Proof. (i) and (ii). The instructions $\varphi_k(g) = \theta_G(g)[k] = gg_{i_k}$ are clearly bijective and commute if G is abelian.

(iii) Suppose $\theta(g)[j, j+1] = \theta(h)[s, s+1]$. Then

$$(gg_{i_j}, gg_{i_{j+1}}) = (hg_{i_s}, hg_{i_{s+1}}) \quad \text{or}$$
$$(g_{i_j}, g_{i_{j+1}}) = (g^{-1}hg_{i_s}, g^{-1}hg_{i_{s+1}}),$$

hence by the hypothesis we must have $g = h$ and $j = s$, so the conditions of Theorem 2 hold.

The following example shows that the above construction is nonvacuous.

EXAMPLE 1. Let $G = \{e, g_1, \ldots, g_{r-1}\}$ be a finite group. The substitution θ defined by

$$\theta(e) = e, g_1, g_2 g_1, g_3 g_2 g_1, \ldots, g_{r-1} g_{r-2} \ldots g_2 g_1, \qquad \theta(g) = g\theta(e),$$

satisfies the conditions of Theorem 3(iii).

Proof. Suppose that $g \in G$ and j, s satisfy $(g_{i_j}, g_{i_{j+1}}) = (gg_{i_s}, gg_{i_{s+1}})$. Then $g_j g_{j-1} \ldots g_2 g_1 = gg_s g_{s-1} \ldots g_2 g_1$ and $g_{j+1} g_j \ldots g_2 g_1 = gg_{s+1} g_s \ldots g_2 g_1$ so $g_{j+1} = g_{s+1}$, $j = s$ and $g = e$.

It remains to prove that θ is a substitution. It is enough to show that there exists $n \in \mathbf{N}$ such that all g_i's appear in $\theta^n(e)$. First of all we see that if h appears in $\theta^k(e)$ then h appears in $\theta^{k+m}(e)$, $m \geqslant 1$, as $\theta(g) = g\ldots$ for each $g \in G$. Now, we observe that g_1 appears in $\theta(e)$. Since $g_2 g_1$ appears in $\theta(e)$, the element $(g_2 g_1) g_1 = g_2 g_1^2$ appears in $\theta^2(e)$, the element $(g_2 g_1^2) g_1 = g_2 g_1^3$ appears in $\theta^3(e)$ and in general $g_2 g_1^k$ appears in $\theta^k(e)$. Hence there must exist n_2 such that g_2 appears in $\theta^{n_2}(e)$. The element $g_3 g_2 g_1$ appears in $\theta(e)$. Therefore as before $g_3(g_2 g_1)^2$ appears in $\theta^2(e)$ and in general $g_3(g_2 g_1)^k$ appears in $\theta^k(e)$. Thus there exists n_3 such that g_3 appears in $\theta^{n_3}(e)$. The same arguments show that for each g_i there exists n_i, $i = 1, \ldots, r-1$, such that g_i appears in $\theta^{n_i}(e)$.

Remarks. 1. The transformation T_θ arising from the substitution θ_G can be represented as a (possibly nonabelian) Morse sequence of the form $x_0 = b \times b \times \ldots$ and hence as a G-extension of a discrete spectrum transformation (see Robinson [14]). It follows from Robinson [13] (and also Queffélec

[12]) that the maximal spectral multiplicity of such a transformation is bounded from below by the maximal dimension of the irreducible representations of G. It follows that for the transformation T_θ of Example 1, rank $T_\theta = r$ and $\mathscr{D}_G \leqslant m(T_\theta) \leqslant r$ where $m(T) =$ maximal spectral multiplicity of T and $\mathscr{D}_G =$ maximal dimension of the irreducible representations of G.

2. Suppose r is prime. Then $G = \mathbf{Z}_r$ and θ_G is an abelian Morse sequence of the form $x_0 = b \times b \times \ldots$ over $\mathbf{Z}_r$. In this situation, Kwiatkowski and Sikorski [7] have shown that $m(T_{\theta_G}) = 1$ or 2 and is 2 precisely when the block $b = \theta(0)$ is symmetric.

Example 2. We specialize Example 1 to the case $G = \mathbf{Z}_r$, $r \geqslant 2$, to obtain for each $r \geqslant 3$ a substitution θ_r with rank $T_{\theta_r} = r$ and $m(T_{\theta_r}) = 2$, thus solving Mentzen's problem for the case $(2, n)$.

We have

$$\theta_r(0) = 0136 \ldots \frac{k(k-1)}{2} \ldots \frac{r(r-1)}{2} \pmod r,$$

$$\theta_r(i) = \theta_r(0) + i \pmod r.$$

For example, when $r = 2$ we get

$$\theta_2(0) = 01, \quad \theta_2(1) = 10,$$

the Thue–Morse sequence, shown by del Junco [4] to have rank 2 and simple spectrum. For $r = 3$

$$\theta_3(0) = 010, \quad \theta_3(1) = 121, \quad \theta_3(2) = 202$$

gives rise to the Morse sequence $x_0 = 010 \times 010 \times \ldots$ over $\mathbf{Z}_3$, shown in Goodson [3] to have maximal spectral multiplicity equal to two. We generalize this to prove

Theorem 4. *If θ_r is the bijective substitution*

$$\theta_r(0) = 0136 \ldots r(r-1)/2 \pmod r,$$

$$\theta_r(i) = \theta_r(0) + i \pmod r,$$

then rank $T_{\theta_r} = r$ and, for $r \geqslant 3$, $m(T_{\theta_r}) = 2$.

Proof. Denote by U_T the unitary operator $U_T\colon L^2(X(\theta_r), \mu_\theta) \to L^2(X(\theta_r), \mu_\theta)$ induced by T, $U_T f(x) = f(T^{-1}x)$, where μ_θ is the unique invariant measure. For each $p \in \{0, 1, \ldots, r-1\}$ and for $w = e^{2\pi i/r}$, we define subspaces H_p of $L^2(X(\theta_r))$ by

$$H_p = \{f \in L^2(X(\theta_r))\colon f \circ \sigma(x) = w^p f(x)\}.$$

Then $L^2(X(\theta_r)) = \bigoplus_{p=0}^{r-1} H_p$ and each H_p is invariant under U_T (where $\sigma\colon X(\theta_r) \to X(\theta_r)$ is the homeomorphism defined by adding 1 to each component of $X(\theta_r)$). It follows from Goodson [3] that $U_T|H_p$, $p \neq 0$, has simple continuous spectrum and $U_T|H_0$ has discrete spectrum. (This result is true more generally, see Kwiatkowski and Sikorski [67] and Queffélec [12] for the simplicity of spectrum and Martin [9] for the continuity of spectrum.)

Denote by λ_p the maximal spectral type of U_T restricted to the invariant subspace H_p corresponding to $p \in \mathbf{Z}_r$. If $\hat{\lambda}_p$ is the Fourier transform of λ_p then $\hat{\lambda}_p(0) = 1$ and $\hat{\lambda}_p(-n) = \overline{\hat{\lambda}_p(n)}$, furthermore the recurrence formula of Coquet, Kamae and Mendès France [2; Theorem 3] implies that

$$\hat{\lambda}_p(rn+a) = A_p(a)\hat{\lambda}_p(n) + B_p(a)\hat{\lambda}_p(n+1)$$

for $n = 0, 1, \ldots$; $a = 0, 1, \ldots, r-1$ and $p \in \mathbf{Z}_r$, where

$$A_p(a) = \frac{1}{r}\sum_{k=0}^{r-a-1} \zeta_p(a+k)\overline{\zeta_p(k)}, \qquad B_p(a) = \frac{1}{r}\sum_{k=r-a}^{r-1} \zeta_p(a+k-r)\overline{\zeta_p(k)},$$

and $\zeta_p(k) = w^{pb[k]}$ where $w = e^{2\pi i/r}$ and $b[k]$ is the kth member of the block $b = \theta_r(0)$. Furthermore, it is known that the measures λ_p, $p \in \mathbf{Z}_r$, are either equal or mutually singular (see Keane [6], Queffélec [12] or Kwiatkowski and Sikorski [7]).

Lemma 8. *For the substitution θ_r and for $p = 1, \ldots, r-1$; $n = 0, 1, \ldots$; $a = 0, 1, \ldots, r-1$, we have the recurrence relation*

$$\hat{\lambda}_p(rn+a) = \frac{\sin\frac{\pi a^2 p}{r}}{r\sin\frac{\pi ap}{r}}\{(-1)^{(r+1)p}\hat{\lambda}_p(n+1) - \hat{\lambda}_p(n)\} \quad \textit{if } ap \neq 0 \pmod r$$

$$= \frac{(-1)^{k(a+1)}}{r}\{(-1)^{(r+1)p} a\hat{\lambda}_p(n+1) + (r-a)\hat{\lambda}_p(n)\} \quad \textit{if } ap = kr.$$

Proof. $A_p(a) = r^{-1}\sum_{k=0}^{r-a-1} w^{p(b[a+k]-b[k])}$ where

$$b[a+k] - b[k] = \frac{(a+k)(a+k+1)}{2} - \frac{k(k+1)}{2} = \frac{a(a+1)}{2} + ak.$$

Therefore

$$A_p(a) = \frac{1}{r} w^{ap(a+1)/2} \sum_{k=0}^{r-a-1} w^{apk}$$

$$= \frac{1}{r} w^{ap(a+1)/2} \frac{1 - w^{(r-a)ap}}{1 - w^{ap}} \quad \text{if } w^{ap} \neq 1$$

$$= -\frac{1}{r}\frac{\sin\frac{\pi a^{2}p}{r}}{\sin\frac{\pi ap}{r}} \quad \text{if } ap \neq 0 \pmod r.$$

In the case that $ap = 0 \pmod r$

$$A_p(a) = \frac{1}{r} w^{ap(a+1)/2}(r-a) = \frac{r-a}{r} e^{\pi i a p(a+1)/r}$$

$$= \frac{r-a}{r}(-1)^{k(a+1)} \quad \text{where } ap = kr.$$

Similarly

$$B_p(a) = \frac{(-1)^{(r+1)p}}{r} \frac{\sin\dfrac{\pi a^2 p}{r}}{\sin\dfrac{\pi a p}{r}} \quad \text{if } ap \neq 0 \pmod r$$

$$= \frac{a}{r}(-1)^{k(a+1)}(-1)^{(r+1)p} \quad \text{if } ap = kr$$

and the result follows.

Remark. The lemma holds even when $r = 2$, giving the recurrence relation for the Fourier coefficients of the Thue–Morse sequence $x = 01 \times 01 \times \ldots$ over $\mathbf{Z}_2$, given by Kakutani in [5].

Lemma 9. *For $r \geqslant 3$, T_{θ_r} has maximal spectral multiplicity equal to two.*

Proof. Let $p, q \in \mathbf{Z}_r - \{0\}$. Then since λ_p and λ_q are either equal or mutually singular it suffices to show that $\lambda_p = \lambda_q$ if and only if $p = q$ or $p = r - q$.

Suppose $ap \neq 0 \pmod r$. Then $a(r-p) \neq 0 \pmod r$ and

$$\frac{\sin\dfrac{\pi a^2(r-p)}{r}}{\sin\dfrac{\pi a(r-p)}{r}} = \frac{\sin\dfrac{\pi a^2 p}{r}}{\sin\dfrac{\pi a p}{r}};$$

it easily follows that $\hat\lambda_p(rn+a) = \hat\lambda_{r-p}(rn+a)$ for all $n = 0, 1, \ldots$

If $ap = 0 \pmod r$, say $ap = kr$, we have $a(r-p) = (a-k)r$ and we again see that $\hat\lambda_p(rn+a) = \hat\lambda_{r-p}(rn+a)$ for $n = 0, 1, \ldots$ Note that $\hat\lambda_p(0) = 1$ for all p, thus $\lambda_p = \lambda_{r-p}$, $p = 0, 1, \ldots, r-1$.

Conversely, suppose $\lambda_p = \lambda_q$ where $p, q \in \mathbf{Z}_r - \{0\}$. Then $\hat\lambda_p(rn+a) = \hat\lambda_q(rn+a)$ for all $n = 0, 1, \ldots$; $a = 0, 1, \ldots, r-1$. We split the proof into a number of cases.

Case 1: Suppose r is odd. If $a = 2$ then $pa \neq 0 \pmod r$ and $qa \neq 0 \pmod r$ so

$$\hat\lambda_p(rn+2) = \frac{\sin(4\pi p/r)}{r\sin(2\pi p/r)}\{\hat\lambda_p(n+1) - \hat\lambda_p(n)\}$$

and similarly for $\hat\lambda_q$, thus

$$\frac{\sin(4\pi p/r)}{\sin(2\pi p/r)} = \frac{\sin(4\pi q/r)}{\sin(2\pi q/r)} \quad \text{or} \quad \cos(2\pi p/r) = \cos(2\pi q/r),$$

so $p = q$ or $p = r - q$.

Case 2: Suppose r is even and $\lambda_p = \lambda_q$ where p is odd and q is even. Then if we take $a = 1$,

$$\hat\lambda_p(rn+1) = \frac{\sin(\pi p/r)}{r\sin(\pi p/r)}\{(-1)^p \hat\lambda_p(n+1) - \hat\lambda_p(n)\}$$

$$= -\hat\lambda_p(n+1) - \hat\lambda_p(n),$$

and $\hat\lambda_q(rn+1) = \hat\lambda_q(n+1) - \hat\lambda_q(n)$, so we must have $\hat\lambda_q(n+1) = 0$ for all $n \geqslant 0$, which is impossible.

Case 3: If r is even with $2p \neq 0 \pmod r$ and $2q \neq 0 \pmod r$ with p and q both even or both odd we can argue as in Case 1 to see that $p = q$ or $p = r - q$. On the other hand, if $2p = 0 \pmod r$ and $2q = 0 \pmod r$ we must have $p = q = r/2$.

Case 4: If r is even with $2p \neq 0 \pmod r$ and $2q = 0 \pmod r$ with p and q both odd or both even, then

$$\hat\lambda_p(rn+2) = \frac{2}{r}\cos\frac{2\pi p}{r}\{(-1)^p \hat\lambda_p(n+1) - \hat\lambda_p(n)\},$$

$$\hat\lambda_q(rn+2) = -\frac{1}{r}\{(-1)^q 2\hat\lambda_q(n+1) - (r-2)\hat\lambda_q(n)\}.$$

Put $n = 0$ and equate the above using $\hat\lambda_p(1) = \hat\lambda_q(1) = -1/(r-(-1)^p)$, $\hat\lambda_p(0) = \hat\lambda_q(0) = 1$; we see that Case 4 cannot arise.

We are now able to give an explicit formula for the maximal spectral type of T_{θ_r}.

Corollary. *For $r \geqslant 2$ prime, λ_p is the Riesz product measure*

$$\lambda_p = \prod_{n \geqslant 0}\left\{1 - \frac{2}{r}\sum_{a=1}^{r-1} \frac{\sin\dfrac{\pi a^2 p}{r}}{\sin\dfrac{\pi a p}{r}} \cos(2\pi a r^n x)\right\}.$$

A similar formula holds when r is not prime.

Proof. Use Lemma 8 and the method of Queffélec [11].

Remarks. 1. Theorem 4 gives a new proof of the result due to Kwiatkowski and Sikorski [7] that for each $r > 2$ there are continuous Morse sequences over $\mathbf{Z}_r$ which have nonsimple spectra.

2. The proof of Theorem 4 implies that the spectrum of T_{θ_r} restricted to $\bigoplus_{p=1}^{r-1} H_p$ is homogeneous for r odd and nonhomogeneous if r is even ($r > 2$).

3. It is still an open question whether a generalized Morse sequence over a finite abelian group can have maximal spectral multiplicity greater than two.

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A smooth subadditive homogeneous norm on a homogeneous group

by

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Abstract. We prove that on every homogeneous group there exists a smooth, subadditive and homogeneous norm.

Introduction. Around 1970 E. M. Stein introduced the notion of a homogeneous group. Such a group G admits a homogeneous norm $\|\cdot\|$, which for a $\gamma \geqslant 1$ satisfies

$$\|xy\| \leqslant \gamma(\|x\| + \|y\|) \quad \text{for all } x, y \in G.$$

The group equipped with $\|\cdot\|$ and the Haar (Lebesgue) measure is a space of homogeneous type in the sense of [1]. A number of estimates become easier if $\gamma = 1$, i.e. if the homogeneous norm is subadditive, so that it gives rise to a left-invariant metric. It is known that for some homogeneous groups such a norm exists, e.g. for Heisenberg groups and the like [2]. Also for stratified groups the optimal control metric is homogeneous.

The aim of this note is to show that a homogeneous and subadditive norm exists for every homogeneous group and in fact the construction is quite simple. More information about such norms is supplied by Theorem 2.

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A smooth subadditive homogeneous norm on a homogeneous group. A family of *dilations* on a nilpotent Lie algebra G is a one-parameter group $\{\delta_t\}_{t>0}$ ($\delta_t \circ \delta_s = \delta_{ts}$) of automorphisms of G determined by

$$\delta_t e_j = t^{d_j} e_j,$$

where $e_1, \ldots, e_n$ is a linear basis for G, the d_j are real numbers and $d_n \geqslant \ldots \geqslant d_1 \geqslant 1$. If we put $(x_1, \ldots, x_n) = \sum x_i e_i$, then

$$\delta_t(x_1, \ldots, x_n) = (t^{d_1} x_1, \ldots, t^{d_n} x_n).$$

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