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Some properties of endomorphisms of Lipschitz algebras

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Abstract. In this note we consider endomorphisms of Lipschitz algebras $\mathrm{Lip}_\alpha(K, d)$ and $\mathrm{lip}_\alpha(K, d)$ where (K, d) is a compact metric space. We determine necessary and sufficient conditions for such endomorphisms to be compact and further show that the spectrum of a nonzero compact endomorphism consists only of the points 0 and 1.

Let (K, d) be a compact metric space with metric d. Following [5] we denote by $\mathrm{Lip}(K, d)$ the Banach algebra of complex-valued functions f on K for which

$$\|f\|_{\mathrm{Lip}(K,d)} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x)-f(y)|}{d(x, y)} < \infty.$$

These are classical algebras when $K = [0, 1]$ or $\mathbf{T}$, the unit circle, with the usual metrics [4]. General Lipschitz algebras $\mathrm{Lip}(K, d)$ have been studied by, among others, Sherbert [5], [6] and Bade, Curtis and Dales [1]. It was shown in [5] that $\mathrm{Lip}(K, d)$ is a regular commutative semisimple Banach algebra with maximal ideal space K. It also follows from [5] that a linear map T from $\mathrm{Lip}(K, d)$ to $\mathrm{Lip}(K, d)$ is a nonzero endomorphism if and only if there exists a map φ: $K \to K$ such that $Tf = f\circ\varphi$ for all $f \in \mathrm{Lip}(K, d)$ and $d(\varphi(x), \varphi(y)) \leqslant Md(x, y)$ for some $M > 0$, all $x, y \in K$. Although the thrust of [1] related to Lipschitz algebras concerns the question of amenability and weak amenability of these algebras, that paper contains other nice properties of Lipschitz algebras $\mathrm{Lip}(K, d)$ and related algebras $\mathrm{Lip}_\alpha(K, d)$ and $\mathrm{lip}_\alpha(K, d)$. The objective of this note is to study compact endomorphisms of $\mathrm{Lip}(K, d)$. We will show that an endomorphism T: $f \to f\circ\varphi$ of $\mathrm{Lip}(K, d)$ is compact if and only if

$$\lim_{d(x,y)\to 0} d(\varphi(x), \varphi(y))/d(x, y) = 0.$$

Further we will determine the spectra of these operators.

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DEFINITION. If (K, d) is a metric space, a map $\varphi\colon K\to K$ will be called *a supercontraction* if

$$\lim_{d(x,y)\to 0} d(\varphi(x), \varphi(y))/d(x, y) = 0.$$

We observe that constant functions are clearly the only supercontractions of $K = [0, 1]$ with the usual metric. This seemed to confirm an earlier conjecture that every nonzero compact endomorphism T of a regular commutative semisimple Banach algebra with connected maximal ideal space X has the form $Tf = \hat{f}(x_0)1$ for some $x_0\in X$. However, in [3] we constructed an example of a nontrivial compact endomorphism of a regular commutative semisimple Banach algebra with connected maximal ideal space. Since, as we now show, there exists a compact connected metric space (K, d) and a nonconstant supercontraction $\varphi\colon K\to K$ the resulting Banach algebra $\mathrm{Lip}(K, d)$ is then another example of a regular commutative semisimple Banach algebra with a nontrivial compact endomorphism.

EXAMPLE (of a connected metric space with a nonconstant supercontraction). Let $K = [\frac{1}{4}, 1]$ and define $d\colon K\times K\to \mathbf{R}_+$ by

$$d(x, y) = \sqrt{|x-y|} \quad \text{if } \tfrac{1}{4}\leqslant x, y\leqslant \tfrac{1}{2},$$

$$d(x, y) = d(y, x) = \sqrt{\tfrac{1}{2}-x}+(y-\tfrac{1}{2}) \quad \text{if } \tfrac{1}{4}\leqslant x\leqslant \tfrac{1}{2}\leqslant y\leqslant 1,$$

$$d(x, y) = |x-y| \quad \text{if } \tfrac{1}{2}\leqslant x, y\leqslant 1.$$

It is easily checked that (K, d) is a compact connected metric space and that the function $\varphi\colon K\to K$ defined by $\varphi(x) = 2x$ if $\frac{1}{4}\leqslant x\leqslant\frac{1}{2}$ and $\varphi(x) = 1$ for $\frac{1}{2}\leqslant x\leqslant 1$ is a supercontraction.

As Theorem 1 will show, this φ induces a nontrivial compact endomorphism of $\mathrm{Lip}(K, d)$.

THEOREM 1. *Let (K, d) be a compact metric space and let T be an endomorphism of $\mathrm{Lip}(K, d)$ induced by a map $\varphi\colon K\to K$. Then T is compact if and only if φ is a supercontraction.*

Proof. First assume that $T\neq 0$ is a compact endomorphism of $\mathrm{Lip}(K, d)$ with $Tf = f\circ\varphi$. Suppose φ is not a supercontraction, i.e.

$$\lim_{d(x,y)\to 0} d(\varphi(x), \varphi(y))/d(x, y) \neq 0.$$

It then follows that there exist $\varepsilon > 0$ and $x_n, y_n\in K$ satisfying $d(x_n, y_n) < 1/n^2$ and $d(\varphi(x_n), \varphi(y_n))/d(x_n, y_n)\geqslant\varepsilon > 0$. Let

$$F_n(x) = \frac{1-e^{-nd(x,\varphi(y_n))}}{n}.$$

Then $\|F_n\|_\infty < 1/n$ and for each $x, y\in K$, $x\neq y$,

$$\frac{|F_n(x)-F_n(y)|}{d(x, y)} = \frac{|e^{-nd(y,\varphi(y_n))}-e^{-nd(x,\varphi(y_n))}|}{nd(x, y)} = \frac{e^{-n\xi}\,|d(y, \varphi(y_n))-d(x, \varphi(y_n))|}{d(x, y)}$$

where $\xi > 0$ is between $d(x,\varphi(y_n))$ and $d(y, \varphi(y_n))$. (Mean Value Theorem for $g(t) = e^{-nt}$ on the interval between $d(x, \varphi(y_n))$ and $d(y, \varphi(y_n))$.) Since $e^{-n\xi} < 1$ and $|d(y, \varphi(y_n))-d(x, \varphi(y_n))|\leqslant d(x, y)$ we have

$$\sup_{x\neq y}|F_n(x)-F_n(y)|/d(x, y) < 1;$$

hence $\|F_n\|_{\mathrm{Lip}(K,d)} < 1/n+1$. As we are assuming that the map T is compact, there exist F_{n_k} and G in $\mathrm{Lip}(K, d)$ with $F_{n_k}\circ\varphi\to G$ in $\mathrm{Lip}(K, d)$. However, since $F_{n_k}\to 0$ uniformly, $G = 0$ and so $F_{n_k}\circ\varphi\to 0$ in $\mathrm{Lip}(K, d)$ norm. Thus

$$\sup_{x\neq y}\left|F_{n_k}(\varphi(x))-F_{n_k}(\varphi(y))\right|/d(x, y)\to 0.$$

With $\varepsilon > 0$ as before, it follows that $|F_{n_k}(\varphi(x))-F_{n_k}(\varphi(y))|/d(x, y) < \varepsilon/2$ for all $x, y\in K$, $x\neq y$, k large. Therefore

$$\frac{|e^{-n_kd(\varphi(y),\varphi(y_{n_k}))}-e^{-n_kd(\varphi(x),\varphi(y_{n_k}))}|}{n_k\, d(x, y)} < \frac{\varepsilon}{2}$$

for all $x, y\in K$, $x\neq y$, k large. In particular, the last inequality holds if $x = x_{n_k}$ and $y = y_{n_k}$. Thus

$$\frac{\varepsilon}{2} > \frac{1-e^{-n_kd(\varphi(x_{n_k}),\varphi(y_{n_k}))}}{n_k\, d(x_{n_k}, y_{n_k})} = \frac{e^{-n_k\xi}\, d(\varphi(x_{n_k}), \varphi(y_{n_k}))}{d(x_{n_k}, y_{n_k})}$$

for some ξ, depending on k, where $0 < \xi < d(\varphi(x_{n_k}), \varphi(y_{n_k}))$. ($d(\varphi(x_{n_k}), \varphi(y_{n_k}))$ is positive since $d(\varphi(x_{n_k}), \varphi(y_{n_k}))/d(x_{n_k}, y_{n_k})\geqslant\varepsilon > 0$.) Also since φ induces an endomorphism, $d(\varphi(x), \varphi(y))\leqslant Md(x, y)$ for some $M > 0$, all $x, y\in K$. This implies that $d(\varphi(x_{n_k}), \varphi(y_{n_k}))\leqslant M/n_k^2$. Thus $0 < \xi\leqslant M/n_k^2$ and so for large k we have

$$0 < \varepsilon\leqslant\frac{d(\varphi(x_{n_k}), \varphi(y_{n_k}))}{d(x_{n_k}, y_{n_k})} < \frac{\varepsilon}{2}e^{n_k\xi} < \frac{\varepsilon}{2}e^{M/n_k}.$$

Letting $k\to\infty$ gives the contradiction $0 < \varepsilon\leqslant\varepsilon/2$. Thus if φ induces a compact endomorphism, then

$$\lim_{d(x,y)\to 0} d(\varphi(x), \varphi(y))/d(x, y) = 0.$$

Conversely, assume that T is an endomorphism of $\mathrm{Lip}(K, d)$ induced by a supercontraction φ. Let $\{f_n\}$ be a sequence in $\mathrm{Lip}(K, d)$ with $\|f_n\|\leqslant 1$. Then $\{f_n\}$ is uniformly bounded, and since there exists $M > 0$ such that $|f_n(x)-f_n(y)|\leqslant Md(x, y)$, the sequence $\{f_n\}$ is equicontinuous on K. By the Ascoli–Arzelà Theorem there exists a subsequence $\{f_{n_k}\}$ and $g\in C(K)$ with

$f_n \to g$ uniformly. We claim that $\{f_{n_k} \circ \varphi\}$ is a Cauchy sequence in $\mathrm{Lip}(K, d)$. Indeed, fix $\varepsilon > 0$ and let $\delta > 0$ be such that $0 < d(x, y) < \delta$ implies that

$$d(\varphi(x), \varphi(y))/d(x, y) < \varepsilon.$$

When $0 < d(x, y) < \delta$ and $\varphi(x) \neq \varphi(y)$, we have

$$\begin{aligned}
&\frac{|[f_{n_k}(\varphi(x)) - f_{n_{k'}}(\varphi(x))] - [f_{n_k}(\varphi(y)) - f_{n_{k'}}(\varphi(y))]|}{d(x, y)} \\
&\qquad \leqslant \frac{|f_{n_k}(\varphi(x)) - f_{n_k}(\varphi(y))|}{d(\varphi(x), \varphi(y))} \frac{d(\varphi(x), \varphi(y))}{d(x, y)} \\
&\qquad\quad + \frac{|f_{n_{k'}}(\varphi(x)) - f_{n_{k'}}(\varphi(y))|}{d(\varphi(x), \varphi(y))} \frac{d(\varphi(x), \varphi(y))}{d(x, y)} \\
&\qquad \leqslant [\|f_{n_k}\| + \|f_{n_{k'}}\|] \frac{d(\varphi(x), \varphi(y))}{d(x, y)} < 2\varepsilon
\end{aligned}$$

for all k, k' where $x, y \in K$ and $0 < d(x, y) < \delta$. (The inequality is certainly true if $\varphi(x) = \varphi(y)$.) On the other hand, for $d(x, y) \geqslant \delta$,

$$\frac{|[f_{n_k}(\varphi(x)) - f_{n_{k'}}(\varphi(x))] - [f_{n_k}(\varphi(y)) - f_{n_{k'}}(\varphi(y))]|}{d(x, y)} \leqslant \frac{2\|f_{n_k} - f_{n_{k'}}\|_\infty}{d(x, y)} \leqslant \frac{2\|f_{n_k} - f_{n_{k'}}\|_\infty}{\delta}.$$

Since $\{f_{n_k}\}$ is a Cauchy sequence in the uniform norm on $C(K)$, we have $(2/\delta)\|f_{n_k} - f_{n_{k'}}\|_\infty < \varepsilon$ for large k, k'.

Thus for all $x, y \in K$, $x \neq y$,

$$\frac{|(f_{n_k} - f_{n_{k'}})(\varphi(x)) - (f_{n_k} - f_{n_{k'}})(\varphi(y))|}{d(x, y)} < 2\varepsilon$$

for large k, k'. Since $\varepsilon > 0$ is arbitrary we conclude that $\{f_{n_k} \circ \varphi\}$ is a Cauchy sequence in $\mathrm{Lip}(K, d)$. Hence there exists $G \in \mathrm{Lip}(K, d)$ with $f_{n_k} \circ \varphi \to G$, showing that T is a compact endomorphism. Clearly $G = g \circ \varphi$.

We next show that the spectrum $\sigma(T)$ of a compact endomorphism $T \neq 0$ of $\mathrm{Lip}(K, d)$ consists of two points, 0 and 1. We recall [2] that if $T \neq 0$ is a compact endomorphism of a commutative semisimple Banach algebra B and if $(Tf)^\wedge = \hat{f} \circ \varphi$, then $\bigcap \varphi_n(X)$ is finite, where X is the maximal ideal space of B, $\varphi\colon X \to X$ and φ_n denotes the nth iterate of φ. Moreover, if X is compact and connected, then $1 \in B$ and $\bigcap \varphi_n(X)$ is a singleton.

THEOREM 2. *Let (K, d) be a compact metric space. If $T \neq 0$ is a compact endomorphism of* $\mathrm{Lip}(K, d)$, *then* $\sigma(T) = \{0, 1\}$.

Proof. Let T be a compact endomorphism of $\mathrm{Lip}(K, d)$ with $Tf = f \circ \varphi$. Clearly $0 \in \sigma(T)$ and also $1 \in \sigma(T)$ since $T1 = 1$. Assume that K is connected and $\{x_0\} = \bigcap \varphi_n(K)$. Next suppose there exists $\lambda \in \sigma(T) \setminus \{0, 1\}$. Since T is compact, λ is an eigenvalue of T. Let $Tf = \lambda f$. We will show that $f = 0$. To this end, we observe first that since x_0 is a fixed point of φ and $\lambda \neq 0, 1$, then $Tf = \lambda f$ implies that $f(x_0) = f(\varphi(x_0)) = \lambda f(x_0)$, whence $f(x_0) = 0$. Since by Theorem 1,

$$\lim_{d(x, x_0) \to 0} \frac{d(\varphi(x), \varphi(x_0))}{d(x, x_0)} = \lim_{d(x, x_0) \to 0} \frac{d(\varphi(x), x_0)}{d(x, x_0)} = 0,$$

we can choose $\delta > 0$ such that $0 < d(x, x_0) < \delta$ implies that

$$d(\varphi(x), x_0)/d(x, x_0) < |\lambda|/2.$$

Assume that m is a positive integer such that $\varphi_m(K) \subset \{t\colon d(t, x_0) < \delta\}$ and let x be a given element of $\varphi_m(K)$. Then $\varphi_j(x) \in \varphi_{m+j}(K) \subset \varphi_m(K)$ for $j \geqslant 0$ so that $d(\varphi_j(x), x_0) < \delta$ for such x and j. Clearly if $\varphi_k(x) = x_0$ for some k, then $f(x) = \lambda^{-k} f(x_0) = 0$. Suppose $\varphi_k(x) \neq x_0$ for all k. Then $f(\varphi_n(x)) = T^n f(x) = \lambda^n f(x)$ for all positive integers n, so that

$$\begin{aligned}
\frac{|\lambda|^n |f(x)|}{d(x, x_0)} &= \frac{|f(\varphi_n(x)) - f(x_0)|}{d(x, x_0)} \\
&= \frac{|f(\varphi_n(x)) - f(x_0)|}{d(\varphi_n(x), x_0)} \frac{d(\varphi(\varphi_{n-1}(x)), x_0)}{d(\varphi_{n-1}(x), x_0)} \cdots \frac{d(\varphi(x), x_0)}{d(x, x_0)}.
\end{aligned}$$

Since $d(\varphi_k(x), x_0) < \delta$ for all $k \geqslant 0$ it follows that

$$d(\varphi(\varphi_{k-1}(x)), x_0)/d(\varphi_{k-1}(x), x_0) < |\lambda|/2$$

for all $k \geqslant 1$. This implies that

$$\frac{|\lambda|^n |f(x)|}{d(x, x_0)} \leqslant \frac{|f(\varphi_n(x)) - f(x_0)|}{d(\varphi_n(x), x_0)} \left|\frac{\lambda}{2}\right|^n$$

for all positive integers n. Moreover, $|f(\varphi_n(x)) - f(x_0)| \leqslant M d(\varphi_n(x), x_0)$ for some $M > 0$. Thus $f(x) = 0$ for all $x \in \varphi_m(K)$. But for each $t \in K$, $\varphi_m(t) \in \varphi_m(K)$, so $\lambda^m f(t) = f(\varphi_m(t)) = 0$. Since $\lambda \neq 0$, $f(t) = 0$ for all $t \in K$. Hence if $\lambda \neq 0, 1$, then λ is not an eigenvalue of T and therefore $\sigma(T) = \{0, 1\}$ as claimed. Finally, if K is not connected, then by considering powers of T and the action of φ on components of K it is not hard to show that again $\sigma(T) = \{0, 1\}$.

We also remark that in the case that K is connected, the proof shows that 1 is an eigenvalue of multiplicity 1; that is, if $Tf = f$, then f is a constant function.

Remark 1. Let (K, d) be a compact metric space. Then for each α, $0 < \alpha \leqslant 1$, $d^\alpha\colon (x, y) \to d(x, y)^\alpha$ is a metric on K. Let $\mathrm{Lip}_\alpha(K, d) = \mathrm{Lip}(K, d^\alpha)$

and define $\operatorname{lip}_\alpha(K, d)$, $0 < \alpha < 1$, by

$$\operatorname{lip}_\alpha(K, d) = \left\{ f \in \operatorname{Lip}_\alpha(K, d):\ \lim_{d(x,y) \to 0} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} = 0 \right\}.$$

(See [1]). Then $\operatorname{lip}_\alpha(K, d)$ is a closed subalgebra of $\operatorname{Lip}_\alpha(K, d)$ with maximal ideal space K. Further, it was shown in [1] that for $0 < \alpha < 1$, $\operatorname{lip}_\alpha(K, d)^{**} = \operatorname{Lip}_\alpha(K, d)$ and that $\operatorname{Lip}(K, d)$ is dense in $\operatorname{lip}_\alpha(K, d)$ for all α, $0 < \alpha < 1$.

We claim that every nonzero compact endomorphism of $\operatorname{lip}_\alpha(K, d)$, $0 < \alpha < 1$, is induced by a supercontraction, and conversely, every supercontraction induces such an endomorphism. To see this we first assume that T is a nonzero compact endomorphism of $\operatorname{lip}_\alpha(K, d)$. Then $Tf = f \circ \varphi$ for some $\varphi\colon K \to K$. Since $\operatorname{lip}_\alpha(K, d)^{**} = \operatorname{Lip}_\alpha(K, d)$ the map T^{**} is a compact endomorphism of $\operatorname{Lip}_\alpha(K, d) = \operatorname{Lip}(K, d^\alpha)$, whence φ is a supercontraction. To show the converse, let φ be a supercontraction of K and $Tf = f \circ \varphi$ on $\operatorname{lip}_\alpha(K, d)$. Suppose $\{f_n\}$ is a bounded sequence in $\operatorname{lip}_\alpha(K, d)$. Then $\{f_n\}$ is bounded in $\operatorname{Lip}_\alpha(K, d)$ so that there exists $\{f_{n_k}\}$ and $g \in \operatorname{Lip}_\alpha(K, d)$ with $f_{n_k} \circ \varphi \to g$ in $\operatorname{Lip}_\alpha(K, d)$. But $\operatorname{lip}_\alpha(K, d)$ is closed in $\operatorname{Lip}_\alpha(K, d)$. Therefore $g \in \operatorname{lip}_\alpha(K, d)$, which shows that $T\colon f \to f \circ \varphi$ is compact on $\operatorname{lip}_\alpha(K, d)$, $0 < \alpha < 1$, if φ is a supercontraction. Clearly $\sigma(T) = \{0, 1\}$ as before.

Remark II. It was shown in [6] that if (K_1, d_1) and (K_2, d_2) are compact metric spaces then $0 \neq T\colon \operatorname{Lip}(K_1, d_1) \to \operatorname{Lip}(K_2, d_2)$ is a homomorphism if and only if $Tf = f \circ \varphi$, $\varphi\colon K_2 \to K_1$ with $d_1(\varphi(x), \varphi(y)) \leqslant M d_2(x, y)$ for some $M > 0$. With appropriate modifications of the terminology and proofs it can be shown that T is a compact homomorphism of $\operatorname{Lip}(K_1, d_1)$ into $\operatorname{Lip}(K_2, d_2)$ if and only if

$$\lim_{d_2(x,y) \to 0} d_1(\varphi(x), \varphi(y))/d_2(x, y) = 0.$$

Specifically, it follows easily that if $\beta > \alpha$, then the identity map $\iota\colon \operatorname{Lip}_\beta(K, d) \to \operatorname{Lip}_\alpha(K, d)$ is compact.

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