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Strongly nonnorming subspaces and prequojections

by

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Abstract. We give an alternative proof of the fact, already established in [8], that strongly nonnorming subspaces exist in the dual of every non-quasi-reflexive Banach space. Via a "lifting lemma", the result is obtained from the explicit construction of a strongly nonnorming subspace in the dual of a suitable subspace of the given Banach space. The emphasis here is on the constructive method, which is then applied to produce a Fréchet space with remarkable properties.

Introduction. Let X be a Banach space. A closed subspace M of the dual X' is called *total* if it is w^*-dense in X' and *norming* if its unit ball is w^*-dense in some multiple of the unit ball of X'. Define the *derived set* M^1 of M as the collection of all limits of w^*-convergent and bounded nets in M and, inductively, the *derived set of order* n, M^n, of M, by $M^n = (M^{n-1})^1$ for $n > 1$. Then M is nonnorming if (and only if) $M^1 \neq X'$, while we shall say that M is *strongly nonnorming* if it is total and $M^n \neq X'$ (i.e., $M^n \neq M^{n+1}$) for all n.

In his book (cf. [1, p. 213]) Banach had already shown that there are subspaces of $l^1 = (c_0)'$ whose successive derived sets are all different (note that for separable X w^*-convergent sequences suffice and our definition reduces to Banach's). These subspaces are not necessarily total, but obviously they are so in their w^*-closures, and are, therefore, examples of strongly nonnorming subspaces in duals of suitable quotients of c_0.

More recently, in [6] S. Dierolf and the author related the existence of strongly nonnorming subspaces to a certain problem in Fréchet space theory raised in [3] and this renewed the interest in strongly nonnorming subspaces. Thus, in [2] a weaker form of Banach's result was rediscovered via Banach's "Théorème 1" of [1, Annexe], while in [11, Problem 17] we conjectured that a strongly nonnorming subspace exists in every non-quasi-reflexive Banach space. This conjecture was proved in [9] for separable spaces and in [8] for general Banach spaces, directly.

The fact that strongly nonnorming subspaces give rise to a very interesting (and unsuspected before) class of Fréchet spaces, which we name *prequojections*, is amply discussed in [7], to which we refer for details (but see also § 2). However, all the proofs in [1], [2], [8] and [9] are "existential" and no

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concrete subspace is produced, while for the purpose of applications, especially to Fréchet space theory, it is necessary to have examples of manageable subspaces. In this spirit, we present here the following proof, alternative to the one in [8], whose main thrust is the actual construction of explicit strongly nonnorming subspaces in the duals of suitable spaces. This is done in § 1. The power of this constructive method wil then be amply displayed in § 2 where, by way of application, we use it to exhibit a prequojection with some very remarkable properties.

1. Strongly nonnorming subspaces... In order to obtain the general result from a special case, we need the following "lifting lemma" whose proof is a matter of routine checking.

LEMMA. *Let X be a Banach space, let Y be a subspace of X and let $Q\colon X'\to Y'$ be the canonical quotient map. If M is a subspace of Y' then, for any n, $Q^{-1}(M^n)=(Q^{-1}(M))^n$. Therefore, if M is strongly nonnorming in Y', so is $Q^{-1}(M)$ in X'.*

THEOREM 1. *Let X be a Banach space with $\dim X''/X=\infty$. Then X' contains a strongly nonnorming subspace.*

Proof. We shall index all sequences from 0 to ∞.

By [4, Theorem 2], X contains a basic sequence (x_n) such that

$$\sup\Big(\Big\|\sum_{i=j}^{k} x_{n_i+j}\Big\|\colon 0\leqslant j\leqslant k<\infty\Big)=C<\infty, \tag{1}$$

with $n_i=i(i+1)/2$ and, by the lemma, it will suffice to prove the assertion for $X=[x_n]$.

Let (N_m) be a partition of the nonnegative integers into disjoint infinite subsets. For every $m\geqslant 0$, $(x_{n_i+j}\colon j\in N_m,\ i\geqslant j)$ is a subsequence of (x_n) satisfying (1) and it may be written as (x_n^m) by relabelling the integers in N_m. In this way (x_n) is partitioned into disjoint infinite subsequences (x_n^m) each of which satisfies (1). If (f_n) is the sequence of biorthogonal functionals associated to (x_n), then (f_n) is also partitioned into corresponding subsequences (f_n^m) and we may assume that $\|f_n^m\|\leqslant 1$ for all $m,n\geqslant 0$. Next, we observe that $(n_i+j\colon i\geqslant j\geqslant 0)$ is an enumeration of the nonnegative integers, so that the map $s\colon n_i+j\to j$ is a mapping of the nonnegative integers onto themselves for which $s^{-1}(j)$ is infinite for every j. Finally, we choose a sequence (ε_n) such that $0<\varepsilon_n\leqslant 2^{-n-1}$ and form the subspace M of X' as

$$M=[f_n^0+\varepsilon_{s(n)}f^1_{s(n)}+\varepsilon_{s(n)}\varepsilon_{s^2(n)}f^2_{s^2(n)}+\ldots\colon n\geqslant 0] \tag{2}$$

$$=\Big[\sum_{m=0}^{\infty}(\varepsilon_{s(n)}\ldots\varepsilon_{s^m(n)})f^m_{s^m(n)}\colon n\geqslant 0\Big].$$

We assert that M is strongly nonnorming in X'. In fact, M is closed and to see that it is total we argue as follows.

Suppose that $x\in X$ and that $f(x)=0$ for all $f\in M$. For $s(n)=j$ put

$$g_j=f_j^1+\varepsilon_{s(j)}f^2_{s(j)}+\ldots; \tag{3}$$

then $f_n^0(x)+\varepsilon_j g_j(x)=0$ and, letting $n\to\infty$ through $s^{-1}(j)$, we have $g_j(x)=0$. Since j is arbitrary, $f_n^0(x)=0$ for all n. Now, starting with $g_j(x)=0$, repeat the argument using (3) to obtain $f_j^1(x)=0$ for all j and, inductively, $f_n^m(x)=0$ for all m, n. Therefore, $x=0$ and M is total.

Now we show that $M^n\neq X'$ for all n. To start with, by (3) we may write

$$M=[f_n^0+\varepsilon_j g_j\colon j\geqslant 0,\ s(n)=j].$$

Since $f_n^0\xrightarrow[w^*]{}0$, $g_j\in M^1$ for all j. Supposing $M^1=X'$, there must be a $\delta>0$ for which $\bar{B}_M^{w^*}\supset 2\delta B_{X'}$, where B_M and $B_{X'}$ are the unit balls of M and X' respectively. Now $\|g_j\|\leqslant 2$ by (3) and hence $\delta g_j\in\bar{B}_M^{w^*}$ for all j. Thus, since X is separable, for any fixed m there is a sequence (u_l), which we may take of the form

$$u_l=\sum_{n=0}^{k_l}a_{ln}(f_n^0+\varepsilon_{s(n)}g_{s(n)}),\qquad \|u_l\|\leqslant 1, \tag{4}$$

which w^*-converges to δg_m. Given arbitrary j and l, choose a $k\geqslant j$ such that $n_k+j\geqslant k_l$. Then, remembering that $s(n_i+j)=j$ for all $i\geqslant j$, it follows from (1) that

$$\Big|\sum_{\substack{n=0\\ s(n)=j}}^{k_l}a_{ln}\Big|=\Big|\sum_{n=0}^{k_l}a_{ln}\sum_{i=j}^{k}f_n^0(x^0_{n_i+j})\Big| \tag{5}$$

$$=\Big|u_l\Big(\sum_{i=j}^{k}x^0_{n_i+j}\Big)\Big|\leqslant\|u_l\|\,\Big\|\sum_{i=j}^{k}x^0_{n_i+j}\Big\|\leqslant C$$

and, in particular,

$$\Big|\sum_{\substack{n=0\\ s(n)=m}}^{k_l}a_{ln}\Big|\leqslant C. \tag{6}$$

Since (5) holds for all l we have, with a suitable j_l,

$$\Big\|\sum_{n=0}^{k_l}a_{ln}\varepsilon_{s(n)}g_{s(n)}\Big\|=\Big\|\sum_{j=0}^{j_l}\Big(\sum_{\substack{n=0\\ s(n)=j}}^{k_l}a_{ln}\Big)\varepsilon_j g_j\Big\|\leqslant\sum_{j=0}^{j_l}\Big|\sum_{\substack{n=0\\ s(n)=j}}^{k_l}a_{ln}\Big|\varepsilon_j\|g_j\|\leqslant 2C.$$

Thus, a subsequence of $\big(\sum_{n=0}^{k_l}a_{ln}\varepsilon_{s(n)}g_{s(n)}\big)$, which we denote the same way, is w^*-convergent, hence so is also $\big(\sum_{n=0}^{k_l}a_{ln}f_n^0\big)$ and necessarily

$$\sum_{n=0}^{k_l}a_{ln}f_n^0\xrightarrow[w^*]{}0,\qquad \sum_{n=0}^{k_l}a_{ln}\varepsilon_{s(n)}g_{s(n)}\xrightarrow[w^*]{}\delta g_m.$$

But by (6) we may also assume that for this subsequence we have

$$\sum_{\substack{n=0\\ s(n)=m}}^{k_l} a_{ln} \to a,$$

so that

$$\sum_{\substack{n=0\\ s(n)\neq m}}^{k_l} a_{ln}\varepsilon_{s(n)} g_{s(n)} \xrightarrow[w^*]{} (\delta - a\varepsilon_m) g_m. \tag{7}$$

However, recalling (3) we see that

$$g_m(x_m^1) = f_m^1(x_m^1) = 1, \quad g_{s(n)}(x_m^1) = 0 \quad \text{for } s(n) \neq m,$$

which, together with (7), implies $a = \delta/\varepsilon_m$, and we get a contradiction with (6) for m sufficiently large. Thus $\bar{B}_M^{w^*}$ contains no multiple of $B_{X'}$ and hence $M^1 \neq X'$.

Similarly, putting

$$h_k = f_k^2 + \varepsilon_{s(k)} f_{s(k)}^3 + \ldots,$$

we see from (3) that $g_j = f_j^1 + \varepsilon_{s(j)} h_{s(j)}$, so that, as before, $h_k \in M^2$ for all k, while the same argument as above shows that $M^2 \neq X'$.

Proceeding this way, we see that $M^n \neq X'$ for all n, i.e. that M is strongly nonnorming as asserted.

We conclude this section by noting that the above method of proof yields also the following result, which solves Problem 18 in [11].

THEOREM 2. *Let X be a Banach space with $\dim X''/X = \infty$. Then, for any $k \geq 1$, X' contains a closed subspace M_k such that $(M_k)^k \neq X'$ and $(M_k)^{k+1} = X'$.*

Proof. By the lemma we may assume, as in the proof of Theorem 1, that $X = [x_n]$ with (x_n) satisfying (1). Partition (x_n) into $k+1$ disjoint subsequences (x_n^i) $(i = 0, \ldots, k)$ satisfying (1) and let (f_n^i) be the corresponding partition of (f_n). Then (notation as in the proof of Theorem 1)

$$M_k = [f_n^0 + \varepsilon_{s(n)} f_{s(n)}^1 + \ldots + (\varepsilon_{s(n)} \ldots \varepsilon_{s^k(n)}) f_{s^k(n)}^k \colon n \geq 0]$$

is the required subspace.

2. ... and prequojections. Here we apply the method of § 1 to the case when $X = c_0$. Thus, we write $X' = l^1$ as $l^1 = \left(\bigoplus_{m=0}^{\infty} l_m^1\right)_1$, with $l_m^1 \equiv l^1$ for all m, we let (f_n^m) be the standard basis for l_m^1 and we define M as in (2). The situation now is simpler and it is enough to take $0 < \varepsilon_n \leq \varepsilon < 1$ for all n. Also, s may be any mapping of the set of nonnegative integers onto itself such that $s^{-1}(n)$ is infinite for every n.

We know from Theorem 1 that M is total and strongly nonnorming in l^1. Now let g_j be defined as in (3), so that $M = [f_n^0 + \varepsilon_{s(n)} g_{s(n)}]$. For all choices of scalars (a_n) we have

$$\sum_{n=0}^{k} |a_n| = \Big\| \sum_{n=0}^{k} a_n f_n^0 \Big\| \leq \Big\| \sum_{n=0}^{k} a_n (f_n^0 + \varepsilon_{s(n)} g_{s(n)}) \Big\|, \tag{8}$$

hence $(f_n^0 + \varepsilon_{s(n)} g_{s(n)})$ is equivalent to the usual basis of l^1 and $M \simeq l^1$ (actually M is $1/(1-\varepsilon)$-isomorphic to l^1).

Now, as in the proof of Theorem 1, $(g_j) \subset M^1$, hence also $(f_n^0) \subset M^1$ and, therefore,

$$\overline{M^1} \supset [f_n^0] + [g_j] = l_0^1 \oplus [g_j].$$

We show that equality holds. Suppose $f \in M^1$; then there exists a sequence $(u_l) \subset M$ such that $\|u_l\| \leq C$ and $u_l \xrightarrow[w^*]{} f$. We may assume that (u_l) is as in (4). Since the natural projection of l^1 onto l_0^1 is w^*-continuous, the sequences

$$\Big(\sum_{n=0}^{k_l} a_{ln} f_n^0 \Big) \quad \text{and} \quad \Big(\sum_{n=0}^{k_l} a_{ln} \varepsilon_{s(n)} g_{s(n)} \Big)$$

must w^*-converge to some $f_0 \in l_0^1$ and $g \in \left(\bigoplus_{m=1}^{\infty} l_m^1\right)_1$ respectively. By (8), $\sum_{n=0}^{k_l} |a_{ln}| \leq \|u_l\| \leq C$ and hence we may write

$$f_0 = \sum_{n=0}^{\infty} a_n f_n^0 \quad \text{and} \quad g = \sum_{j=0}^{\infty} \Big(\sum_{\substack{n=0\\ s(n)=j}}^{\infty} a_n \Big) \varepsilon_j g_j$$

for a suitable sequence (a_n) with $\sum_{n=0}^{\infty} |a_n| \leq C$. Therefore, $f = f_0 + g \in l_0^1 \oplus [g_j]$ and

$$\overline{M^1} = l_0^1 \oplus [g_j] \tag{9}$$

as asserted. Also, as for M we recognize that $\overline{M^1} \simeq l^1$.

Now, let $f \in l^1$ and write $f = \sum_{n=0}^{\infty} a_n f_n^0 + g$, with $g \in \left(\bigoplus_{m=1}^{\infty} l_m^1\right)_1$. We have

$$\Big\| \sum_{n=0}^{\infty} a_n (f_n^0 + \varepsilon_{s(n)} g_{s(n)}) \Big\| \leq \frac{1}{1-\varepsilon} \sum_{n=0}^{\infty} |a_n| \leq \frac{1}{1-\varepsilon} \|f\|,$$

which shows that the map defined by

$$f = \sum_{n=0}^{\infty} a_n f_n^0 + g \to \sum_{n=0}^{\infty} a_n (f_n^0 + \varepsilon_{s(n)} g_{s(n)})$$

is a continuous projection of all of l^1 onto M.

Finally, recalling (9), it is clear that the whole argument above may be applied again to the subspace $[g_j]$ of $\overline{M^1}$, since $[g_j]$ is of the same form as M, and by induction we obtain

THEOREM 3. *$l^1 = (c_0)'$ contains a strongly nonnorming subspace M such that, for all n:*

(a) *$\overline{M^n} \simeq l^1$;*

(b) *$\overline{M^n}$ is complemented in l^1 (hence in $\overline{M^{n+1}}$).*

Now we define a *prequojection* as a Fréchet space whose strong bidual is a quojection or, equivalently, whose strong dual has a representation as a strict (LB)-space (cf. [6]). A prequojection is *proper* (or *nontrivial*) if it is not a quojection. By [6, § 4] every strongly nonnorming subspace M in a dual Banach space X' gives rise to a prequojection F with a continuous norm (obviously proper). F is just the projective limit of the sequence (F_n), where F_n is the completion of X for the norm generated by the polar of the unit ball of M^n, so that $F'_n = M^n$. Hence Theorem 3 enables us to assert

THEOREM 4. *There is a prequojection F such that:*

(a) *F is separable and has a continuous norm (is even countably normed by* [6, § 4]);

(b) $F'_\beta \simeq \bigoplus_n l^1$.

Therefore:

(c) *F and the quojection $\prod_n c_0$ have isomorphic duals;*

(d) *F'_β has an unconditional (even absolute) basis;*

(e) *F has the approximation property but not the bounded approximation property.*

((e) follows from [5, Proposition 4.1(b)] and [10, Remark 4].)

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Unconditional bases and the Radon–Nikodým property

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Abstract. It is known that if X has an unconditionally basic finite-dimensional decomposition (UBFDD), then each of RNP, KMP, and PCP is equivalent to X not having a subspace isomorphic with c_0. If X is a subspace of a space with an unconditional basis, then RNP and KMP are equivalent. It is shown that there is a Banach space X which is a subspace of a space with an unconditional basis, but X does not have RNP or KMP, X has PCP, and no subspace of X is isomorphic with c_0.

A Banach space X has the *Radon–Nikodým property* (RNP) if the Radon–Nikodým theorem is valid for Bochner integration and bounded-variation measures with values in X; X has the *Krein–Milman property* (KMP) if each closed convex subset of X is the closure of the convex span of its extreme points; and X has the *point-of-continuity property* (PCP) if, for each bounded closed nonempty subset C of X, there is a point x of C such that the weak and norm topologies (restricted to C) coincide at x.

Rather than using the definition of RNP, we will use the fact that X has RNP if and only if X does not contain a bush (for an easy proof of this, see [5, Theorem 7, p. 354]). A *bush* in a Banach space X is a bounded partially ordered subset B of X for which each member has at least two (but finitely many) successors and is a convex combination of its successors, there is a positive *separation constant* δ such that $\|v-u\| \geqslant \delta$ if v is a successor of u, and B has a first member to which each member of B can be joined by a linearly ordered chain of successive members of B. If the chain that joins a member b of B to the first member has n members, then b is said to be of *order n*. An *approximate bush* is a set B^n that satisfies all the hypotheses for a bush except that instead of requiring that each member is a convex combination of its successors, it is assumed that there is a sequence of positive numbers $\{\delta_n\}$ for which $\sum_1^\infty \delta_n < \infty$ and each member of B^n of order n differs from a convex combination of its successors by less than δ_n.

It has long been known that a Banach space X has RNP if each separable subspace is isomorphic to a subspace of a separable dual (e.g., if X is reflexive or if X is isomorphic to a subspace of a space that has a boundedly complete basis). The converse is false (see [1] or [8]). Also, RNP$\Rightarrow$KMP [9, Theorem 2]. If X has PCP, then RNP$\Leftrightarrow$KMP [10, Theorem 2.1]. If X is a subspace of a space with an unconditional basis, then RNP$\Leftrightarrow$KMP and KMP$\Rightarrow$PCP.