Now we define a *prequojection* as a Fréchet space whose strong bidual is a quojection or, equivalently, whose strong dual has a representation as a strict (LB)-space (cf. [6]). A prequojection is *proper* (or *nontrivial*) if it is not a quojection. By [6, § 4] every strongly nonnorming subspace M in a dual Banach space X' gives rise to a prequojection F with a continuous norm (obviously proper). F is just the projective limit of the sequence (F_n), where F_n is the completion of X for the norm generated by the polar of the unit ball of M^n, so that $F'_n = M^n$. Hence Theorem 3 enables us to assert

THEOREM 4. *There is a prequojection F such that:*

(a) *F is separable and has a continuous norm (is even countably normed by* [6, § 4]);

(b) $F'_\beta \simeq \bigoplus_n l^1$.

Therefore:

(c) *F and the quojection $\prod_n c_0$ have isomorphic duals*;

(d) *F'_β has an unconditional (even absolute) basis*;

(e) *F has the approximation property but not the bounded approximation property.*

((e) follows from [5, Proposition 4.1(b)] and [10, Remark 4].)

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Unconditional bases and the Radon–Nikodým property

by

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Abstract. It is known that if X has an unconditionally basic finite-dimensional decomposition (UBFDD), then each of RNP, KMP, and PCP is equivalent to X not having a subspace isomorphic with c_0. If X is a subspace of a space with an unconditional basis, then RNP and KMP are equivalent. It is shown that there is a Banach space X which is a subspace of a space with an unconditional basis, but X does not have RNP or KMP, X has PCP, and no subspace of X is isomorphic with c_0.

A Banach space X has the *Radon–Nikodým property* (RNP) if the Radon–Nikodým theorem is valid for Bochner integration and bounded-variation measures with values in X; X has the *Krein–Milman property* (KMP) if each closed convex subset of X is the closure of the convex span of its extreme points; and X has the *point-of-continuity property* (PCP) if, for each bounded closed nonempty subset C of X, there is a point x of C such that the weak and norm topologies (restricted to C) coincide at x.

Rather than using the definition of RNP, we will use the fact that X has RNP if and only if X does not contain a bush (for an easy proof of this, see [5, Theorem 7, p. 354]). A *bush* in a Banach space X is a bounded partially ordered subset B of X for which each member has at least two (but finitely many) successors and is a convex combination of its successors, there is a positive *separation constant* δ such that $\|v-u\| \geqslant \delta$ if v is a successor of u, and B has a first member to which each member of B can be joined by a linearly ordered chain of successive members of B. If the chain that joins a member b of B to the first member has n members, then b is said to be of *order* n. An *approximate bush* is a set B^a that satisfies all the hypotheses for a bush except that instead of requiring that each member is a convex combination of its successors, it is assumed that there is a sequence of positive numbers $\{\delta_n\}$ for which $\sum_1^\infty \delta_n < \infty$ and each member of B^a of order n differs from a convex combination of its successors by less than δ_n.

It has long been known that a Banach space X has RNP if each separable subspace is isomorphic to a subspace of a separable dual (e.g., if X is reflexive or if X is isomorphic to a subspace of a space that has a boundedly complete basis). The converse is false (see [1] or [8]). Also, RNP $\Rightarrow$ KMP [9, Theorem 2]. If X has PCP, then RNP $\Leftrightarrow$ KMP [10, Theorem 2.1]. If X is a subspace of a space with an unconditional basis, then RNP $\Leftrightarrow$ KMP and KMP $\Rightarrow$ PCP.

Also, if the space with an unconditional basis has no subspace isomorphic with c_0, then RNP, KMP, and PCP are equivalent for X [6, Theorem 4.7]. If X itself has an UBFDD, then each of RNP, KMP, and PCP is equivalent to X not having a subspace isomorphic with c_0 [6, Theorem 4.8]. The purpose of this paper is to show that this conclusion is false if it is assumed only that X is a subspace of a space with an unconditional basis.

The proof of the following theorem, as well as attempts to prove there do not exist such X and Z that have properties (a), (b), and (c) of this theorem, was motivated heavily by the fact that, if $X \subset Z$, X fails RNP, and Z has a FDD $\{\varphi_n\}$, then there is a positive number δ such that, for any sequence of positive numbers $\{\beta_n\}$, there is a bush B in the unit ball of X with separation constant δ and a sequence $\{\Phi_n : n \geqslant 0\}$ of consecutive blocks of $\{\varphi_n\}$ such that $\Phi(0)$ is empty and, if Δ is a difference of order n for B, then

$$\Delta \in \operatorname{lin}\{\Phi_{n-1}, \Phi_n\} + N(Z, \beta_n),$$

where $N(Z, \beta_n)$ is the β_n-neighborhood of 0 in the space Z (see Lemma 1.2 of [6]). Actually, there may be some connection between the need for $N(Z, \beta_n)$ and the apparent need to use an approximate bush in the proof of the theorem.

Note that X having properties (a)–(c) of the theorem gives a negative answer to the question raised in [2, Remark 3.4]: Is X a subspace of a space with a boundedly complete unconditional basis if X is a subspace of a space with an unconditional basis and X has no subspace isomorphic with c_0?

THEOREM. *There is a Banach space X which is a subspace of a Banach space Z for which X and Z have the properties:*

(a) *Z has an UBFDD (and therefore is contained in a space with an unconditional basis* [7, p. 51]).

(b) *X does not have RNP.*

(c) *X has no subspace isomorphic with c_0.*

(d) *X has PCP.*

(e) *X does not have KMP.*

(f) *X does not have an UBFDD.*

Proof. To define X and Z, we first let $\{N_i\}$ be a sequence of sets for which $\sum |N_i|^{-1/2} < \infty$ and $|N_i| \geqslant 2$ for each i. Then we introduce, for each $k \geqslant 1$, the set of symbols

$$D_k = \{\Delta_k^\alpha : \alpha \in \prod_1^k N_i\} \cup \{\delta_k^\alpha : \alpha \in \prod_1^{k+1} N_i\}.$$

For notational convenience later, we let $\delta_0^\alpha = 0$. Let V be the natural vector space of all formal linear combinations with real coefficients of members of $\bigcup_{k=1}^\infty D_k$. After a norm has been introduced on V, the resulting normed linear space will be denoted by Z_0 and the completion of Z_0 will be denoted by Z. We let Φ_k denote $\operatorname{lin}(D_k)$ for each k. It will be seen that $\{\Phi_k\}$ is an UBFDD for Z.

Let V^+ be the subspace of V for which

$$V^+ = \operatorname{lin}\{\delta_{k-1}^\alpha + \Delta_k^\alpha : \alpha \in \prod_1^k N_i,\ k \geqslant 1\}.$$

After the norm has been introduced on V^+, the resulting normed linear space will be denoted by X_0 and the completion of X_0 will be denoted by X.

If $n_i \in N_i$ for each i, we say that the sequence $\{n_i\}$ determines the *branch of* V that is the linear span of

$$\{\Delta_1^{\alpha(1)};\ \delta_1^{\alpha(2)}, \Delta_2^{\alpha(2)};\ \delta_2^{\alpha(3)}, \Delta_3^{\alpha(3)};\ \ldots;\ \delta_{k-1}^{\alpha(k)}, \Delta_k^{\alpha(k)};\ \ldots\},$$

where $\alpha(k) = (n_1, n_2, \ldots, n_k)$ for each k. Also, $\{n_i\}$ determines the *branch of* V^+ that is the linear span of

$$\{\Delta_1^{\alpha(1)},\ \delta_1^{\alpha(2)} + \Delta_2^{\alpha(2)},\ \delta_2^{\alpha(3)} + \Delta_3^{\alpha(3)},\ \ldots,\ \delta_{k-1}^{\alpha(k)} + \Delta_k^{\alpha(k)},\ \ldots\}.$$

If β and β^+ are the branches of V and V^+ determined by $\{n_i\}$ and $\alpha(k) = (n_1, n_2, \ldots, n_k)$ for each k, then a *segment of* β is a subset s of V for which there are integers m and n such that

$$s = \operatorname{lin}\Big[\bigcup_{m \leqslant k \leqslant n} \{\delta_{k-1}^{\alpha(k)}, \Delta_k^{\alpha(k)}\}\Big],$$

and a *segment of* β^+ is a subset s^+ of β^+ for which there are integers m and n such that

$$s^+ = \operatorname{lin}\{\delta_{k-1}^{\alpha(k)} + \Delta_k^{\alpha(k)} : m \leqslant k \leqslant n\}.$$

The empty set is a segment of each branch in V and also a segment of each branch in V^+. The vectors $\delta_{k-1}^{\alpha(k)}$, $\Delta_k^{\alpha(k)}$, and $\delta_{k-1}^{\alpha(k)} + \Delta_k^{\alpha(k)}$ are said to be of *order* k. A *block* of $\{\Phi_k\}$ is the linear combination of a finite set of consecutive members of $\{\Phi_k\}$. If s is a segment of a branch β in V and $z \in V$, we let $s(z)$ denote the truncation of z to s. If β is determined by $\{n_i\}$, then $s(z)$ has a unique representation as

$$s(z) = \sum_{k=1}^{p} a_k(\delta_{k-1}^{\alpha(k)} + \Delta_k^{\alpha(k)}) + \sum_{k=1}^{q} c_k(\Delta_k^{\alpha(k)} + \delta_k^{\alpha(k+1)}), \tag{1}$$

where $\alpha(k) = (n_1, n_2, \ldots, n_k)$ for each k. We let

$$\left[\!\left[s(z) \right]\!\right] = \sum_{k=1}^{p} |a_k - a_{k+1}| + \sup\{|c_k| : 1 \leqslant k \leqslant q\}, \tag{2}$$

where $a_{p+1} = 0$. Note that $s(\cdot)$ is a linear map of V onto s and $\left[\!\left[s(z) \right]\!\right]$ is a semi-norm on V and a norm on s. Let $\{\Gamma(\lambda)\}$ and $\{\Psi(\lambda)\}$ be sequences of positive numbers for which $\Psi(1) = 1$, each $\Psi(\lambda)$ is an integer,

$$\sum_{\lambda=1}^{\infty} \Gamma(\lambda) = 1 \quad \text{and} \quad \lim_{\lambda \to \infty} \Gamma(\lambda)[\Psi(\lambda)]^{1/2} = \infty. \tag{3}$$

For members z of V, we let

$$(4)\qquad \|z\| = \sup\{\sum\Gamma(\lambda)[\sum_{i=1}^{\Psi(\lambda)} [\![s_i^\lambda(z)]\!]^2]^{1/2}\},$$

where the sup is for sums over sets of λ's such that each λ has an associated set $S^\lambda = \{s_i^\lambda: 1 \leqslant i \leqslant \Psi(\lambda)\}$ of segments of branches in V for which no two segments in S^λ contain nonzero vectors in the same branch and any two sets S^μ and S^ν have the property that there are at least two Φ_i's that separate these sets in the sense that all segments of one set precede these Φ_i's and all segments of the other set follow all these Φ_i's.

(a) *Z has an UBFDD.* Suppose z and z^* in Z_0 are identical except that all vectors in some Φ_k have been deleted from z to obtain z^*. Let

$$s(z) = \sum a_i(\delta_{i-1}^{\alpha(i)} + \Delta_i^{\alpha(i)}) + \sum c_i(\Delta_i^{\alpha(i)} + \delta_i^{\alpha(i+1)})$$

be the representation of $s(z)$ as in (1). Then

$$s(z^*) = s(z) - (a_k\Delta_k^{\alpha(k)} + a_{k+1}\delta_k^{\alpha(k+1)}) - c_k(\Delta_k^{\alpha(k)} + \delta_k^{\alpha(k+1)})$$
$$= s(z) + (a_k - a_{k+1})\delta_k^{\alpha(k+1)} - a_k(\Delta_k^{\alpha(k)} + \delta_k^{\alpha(k+1)}) - c_k(\Delta_k^{\alpha(k)} + \delta_k^{\alpha(k+1)}).$$

Therefore, if z^* is obtained from z by removing terms in some finite collection of sets $\{\Phi_k: k\in K\}$, then

$$[\![s(z^*)]\!] \leqslant [\![s(z)]\!] + [\![s[\sum_{k\in K}(a_k - a_{k+1})\delta_k^{\alpha(k+1)}]]\!]$$
$$(5)\qquad + [\![s[\sum_{k\in K} a_k(\Delta_k^{\alpha(k)} + \delta_k^{\alpha(k+1)})]]\!]$$
$$+ [\![s[\sum_{k\in K} c_k(\Delta_k^{\alpha(k)} + \delta_k^{\alpha(k+1)})]]\!].$$

Since

$$\delta_k^{\alpha(k+1)} = [(\Delta_1^{\alpha(1)} + \delta_1^{\alpha(2)}) + (\Delta_2^{\alpha(2)} + \delta_2^{\alpha(3)}) + \ldots + (\Delta_k^{\alpha(k)} + \delta_k^{\alpha(k+1)})]$$
$$- [\Delta_1^{\alpha(1)} + (\delta_1^{\alpha(2)} + \Delta_2^{\alpha(2)}) + \ldots + (\delta_{k-1}^{\alpha(k)} + \Delta_k^{\alpha(k)})],$$

we have $[\![s(\delta_k^{\alpha(k+1)})]\!]$ equal to 0 or 2 (0 if $\delta_k^{\alpha(k+1)} \notin s$). Similarly, $[\![s(\Delta_k^{\alpha(k)})]\!]$ is 0, 1, or 2 (0 if $\Delta_k^{\alpha(k)} \notin s$ and 1 if $k=1$ and $\Delta_1^{\alpha(1)} \in s$). Thus in (5),

$$[\![s[\sum_{k\in K}(a_k - a_{k+1})\delta_k^{\alpha(k+1)}]]\!] \leqslant 2\sum_{k\in K}|a_k - a_{k+1}| \leqslant 2[\![s(z)]\!].$$

The sum $\sum_{k\in K} a_k(\Delta_k^{\alpha(k)} + \delta_k^{\alpha(k+1)})$ in (5) may consist only of terms of type $a_k(\Delta_k^{\alpha(k)} + \delta_k^{\alpha(k+1)})$ that are in s, but it may also contain one or two terms of type $a_p\delta_p^{\alpha(p+1)}$ or $a_q\Delta_q^{\alpha(q)}$. Thus

$$[\![s[\sum_{k\in K} a_k(\Delta_k^{\alpha(k)} + \delta_k^{\alpha(k+1)})]]\!] \leqslant \sup\{|a_k|: k\in K\} + 4\sup\{|a_k|: k\in K\}$$
$$\leqslant 5\sup\{|a_i|: i\geqslant 1\} \leqslant 5\sum_{i\geqslant 1}|a_i - a_{i+1}| \leqslant 5[\![s(z)]\!].$$

Similarly,

$$[\![s[\sum_{k\in K} c_k(\Delta_k^{\alpha(k)} + \delta_k^{\alpha(k+1)})]]\!] \leqslant \sup\{|c_k|: k\in K\} + 4\sup\{|c_k|: k\in K\} \leqslant 5[\![s(z)]\!].$$

Now it follows from (5) that $[\![s(z^*)]\!] \leqslant 13[\![s(z)]\!]$. For an arbitrary positive ε, choose $\{s_i^\lambda\}$ as in (4) for suitable values of λ so that

$$\|z^*\| - \varepsilon < \sum\Gamma(\lambda)[\sum_1^{\Psi(\lambda)} [\![s_i^\lambda(z^*)]\!]^2]^{1/2}.$$

Then

$$\|z^*\| - \varepsilon < 13\sum\Gamma(\lambda)[\sum_1^{\Psi(\lambda)} [\![s_i^\lambda(z)]\!]^2]^{1/2} \leqslant 13\|z\|.$$

Thus $\{\Phi_k\}$ is an UBFDD wih unconditional constant not greater than 13.

(b) *X does not have RNP.* We will show first that X contains an approximate bush B^a. Let 0 be the first member of B^a. The other members are to be sums of type

$$w = \Delta_1^{\pi(1)} + (\delta_1^{\pi(2)} + \Delta_2^{\pi(2)}) + \ldots + (\delta_{k-1}^{\pi(k)} + \Delta_k^{\pi(k)}),$$

for which there is a set $\{n_i\}$ with $n_i\in N_i$ for each i and $\pi(j) = (n_1, n_2, \ldots, n_j)$ for each j. The successors of this w are the $|N_{k+1}|$ members of the set

$$\{w + (\delta_k^\pi + \Delta_{k+1}^\pi): \pi = (n_1, n_2, \ldots, n_k, n) \text{ with } n\in N_{k+1}\}.$$

The difference Δ^π between w and the πth successor of w is $\delta_k^\pi + \Delta_{k+1}^\pi$. It follows from (2), (4), and $\Psi(1) = 1$ that $\|\Delta^\pi\| \geqslant 2\Gamma(1)$ if $k\geqslant 1$, and $\|\Delta^\pi\| \geqslant \Gamma(1)$ if $k=0$ (so $\Delta^\pi = \Delta_1^{(n)}$, $n\in N_1$). Thus B^a has separation constant $\Gamma(1)$. Also, B^a is bounded, since it follows from (2), (4), and $\sum_{\lambda=1}^\infty\Gamma(\lambda) = 1$ that $\|w\| \leqslant 1$. Similarly, $\|\Delta^\pi\| \leqslant 2$. Since all Δ^π have the same order, only one S^λ can be used in (4) to determine the norm of the average of the differences between w and successors of w. Also, each s_i^λ for this S^λ can contain at most one $\Delta^\pi|N_{k+1}|^{-1}$. Therefore, the average has norm not greater than

$$\Gamma(\lambda)[\sum_{i=1}^{\Psi(\lambda)} \|s_i^\lambda(z)\|^2]^{1/2} \leqslant 2\Gamma(\lambda)[\sum_{i=1}^{|N_{k+1}|} |N_{k+1}|^{-2}]^{1/2} \leqslant 2|N_{k+1}|^{-1/2}.$$

Since $\sum|N_k|^{-1/2} < \infty$, B^a is an approximate bush. It follows that X contains a bush [3] and therefore fails RNP.

(c) *X has no subspace isomorphic with c_0.* It is important to keep in mind that when $x\in X^0$ and x is in a branch, then each $c_i = 0$ for any representation of $s(x)$ as in (1). Suppose X has a subspace isomorphic with c_0. Then X has a subspace that is almost isometric with c_0 [4, p. 548]. Therefore there are members $\{e_n\}$ of X_0 for which e_k is in a block Ω_k of $\{\Phi_k\}$ for each k, the block Ω_k

precedes and is separated from Ω_{k+1} by at least two Φ's, $\|e_k\| > 1$ for each k, and $[\![\sum_{i\in A} e_i]\!] < 3/2$ for each finite set A. For each e_k, choose a set $A(k)$ such that, for each $\lambda \in A(k)$, there is a set $S_k^\lambda = \{s_{ki}^\lambda\colon 1 \leqslant i \leqslant \Psi(\lambda)\}$ as in (4) so that

$$\sum_{\lambda\in A(k)} \Gamma(\lambda)\Big[\sum_{i=1}^{\Psi(\lambda)} [\![s_{ki}^\lambda(e_k)]\!]^2\Big]^{1/2} > 1.$$

There is no loss of generality if we assume that, for any k and $\varkappa$ with $k < \varkappa$, there exist at least two Φ's which separate $\operatorname{lin}\{S_k^\lambda\colon \lambda \in A(k)\}$ from $\operatorname{lin}\{S_\varkappa^\lambda\colon \lambda \in A(\varkappa)\}$. Suppose there is an $\varepsilon > 0$ for which there are infinitely many e_k's for each of which there is an s_{ki}^λ with $[\![s_{ki}^\lambda(e_k)]\!] > \varepsilon$, where λ and i are functions of k. Associate with each such e_k exactly one such $s_{ki}^\lambda = s_k$. Then one of the following is true.

(i) There is a branch β that contains infinitely many of these s_k's.
(ii) Each branch contains only finitely many of these s_k's.

Suppose (i) is true. Choose $\{e_k\colon k \in A\}$ so that $\varepsilon\Gamma(1)|A| > 3/2$ and the s_k corresponding to e_k is in β for each $k \in A$. Let s be a segment that contains each of these e_k's. Then

$$\Big\|\sum_{k\in A} e_k\Big\| \geqslant \Gamma(1)\Big[\!\Big[s\Big(\sum_{k\in A} e_k\Big)\Big]\!\Big] \geqslant \varepsilon\Gamma(1)|A| > 3/2.$$

Therefore, (i) is false.

Suppose (ii) is true. Then there is a set S of infinitely many s_k's for which no two members of S contain nonzero vectors in the same branch. To see this, observe first that there is some n_1 for which more than one vector $v = \delta_{n_1-1}^\alpha + \Delta_{n_1}^\alpha$ in V^+ of order n_1 has the property that there is an s_k, spanned by vectors of order greater than n_1, that is in a branch that contains v. Moreover, some $v = v^*$ has infinitely many such s_k's. Choose one such s_k for each of the other such v's. Now apply this procedure again, using an n_2 for which more than one vector of V^+ of order n_2 on a branch containing v^* has the property ..., etc., etc. Now choose λ for which $\varepsilon\Gamma(\lambda)[\Psi(\lambda)]^{1/2} > 3/2$ and let $\{e_{k(i)}\colon 1 \leqslant i \leqslant \Psi(\lambda)\}$ be a set of $\Psi(\lambda)$ vectors whose corresponding $s_{k(i)}$'s are in S. Then it follows from (4) that, if $w = \sum_{i=1}^{\Psi(\lambda)} e_{k(i)}$, then

$$\|w\| \geqslant \Gamma(\lambda)\Big[\sum_{i=1}^{\Psi(\lambda)} [\![s_{k(i)}(w)]\!]^2\Big]^{1/2} > 3/2.$$

Since both (i) and (ii) are false, we know that, for any positive ε, there is an N such that, if $k > N$, then $[\![s_{ki}^\lambda(e_k)]\!] < \varepsilon$ for all λ and i. For e_1, the set $A(1)$ and $\{s_{1i}^\lambda\}$ for $\lambda \in A(1)$ satisfy

$$\sum_{\lambda\in A(1)} \Gamma(\lambda)\Big[\sum_{i=1}^{\Psi(\lambda)} [\![s_{1i}^\lambda(e_1)]\!]^2\Big]^{1/2} > 1.$$

Choose $\varepsilon > 0$ so that $\sum_{\lambda\in A(1)}\Gamma(\lambda)[\Psi(\lambda)\varepsilon^2]^{1/2} < \frac{1}{2}$. For this ε, choose N as above. Let k be any integer greater than N. Recall that

$$\sum_{\lambda\in A(k)} \Gamma(\lambda)\Big[\sum_{i=1}^{\Psi(\lambda)} [\![\sigma_{ki}^\lambda(e_k)]\!]^2\Big]^{1/2} > 1.$$

We have

$$\|e_1 + e_k\| \geqslant \sum_{\lambda\in A(1)} \Gamma(\lambda)\Big[\sum_{i=1}^{\Psi(\lambda)} [\![s_{1i}^\lambda(e_1)]\!]^2\Big]^{1/2} + \sum_{\lambda\in[A(k)-A(1)]} \Gamma(\lambda)\Big[\sum_{i=1}^{\Psi(\lambda)} [\![\sigma_{ki}^\lambda(e_k)]\!]^2\Big]^{1/2}.$$

Since

$$\sum_{\lambda\in A(1)} \Gamma(\lambda)\Big[\sum_{i=1}^{\Psi(\lambda)} [\![\sigma_{ki}^\lambda(e_k)]\!]^2\Big]^{1/2} < \tfrac{1}{2},$$

this implies $\|e_1 + e_k\| > 1 + \frac{1}{2} = 3/2$ and completes the proof that X has no subspace isomorphic with c_0.

(d) *X has PCP*. It is known that if X fails PCP and is contained in a space with an unconditional basis, then X has a subspace isomorphic with c_0 [6, Theorem 4.5]. Since X has no subspace isomorphic with c_0, X must have PCP.

(e) *X does not have KMP*. This follows from the fact that RNP$\Leftrightarrow$KMP in any space with PCP [10].

(f) *X does not have an UBFDD*. If X had an UBFDD, then RNP, KMP, and PCP would have been equivalent [6, Theorem 4.8].

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Extending holomorphic maps from compact sets in infinite dimensions

by

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Abstract. The aim of this paper is to study the extension of holomorphic maps from compact sets in metric vector spaces with values in some locally convex spaces and in complete C-manifolds. Moreover, the theorem of Siciak–Zakharyuta for continuous separately holomorphic functions with values in Banach–Lie groups is established.

Introduction. The extension of separately holomorphic functions defined on special subsets of C^n has been investigated by many authors, for example Siciak [8], Zakharyuta [11]. In [9] and [10] Siciak and Waelbroeck have considered this problem for compact sets in C^n. Moreover, Waelbroeck has also considered this problem for unique compact sets in a Banach space. Here a compact set K in a topological vector space is called *unique* if for every holomorphic function f on a neighbourhood of K such that $f|K = 0$ there exists a neigbourhood U of K such that $f|U = 0$. This paper is devoted to the study of the extension of continuous functions on compact sets K in topological vector spaces with values in locally convex complex manifolds to holomorphic functions on a neighbourhood of K.

In Section 1 we investigate the interrelation between the holomorphic extendability and weakly holomorphic extendability of continuous functions on a compact set K in a metric vector space E with values in a locally convex space F such that F^* is a Baire space. We prove that if either E or F is nuclear, then holomorphic extendability and weakly holomorphic extendability are equivalent. This has been established by Siciak in [9] and Waelbroeck in [10] in the case where $\dim E < \infty$. Our method in the case where E is a nuclear metric vector space is based on an idea of Waelbroeck [10]. We first prove the nuclearity of the DF-space $\operatorname{inj\,lim}\{H^\infty(U)/A_U\colon U \supset K\}$ where $H^\infty(U)$ denotes the Banach space of bounded holomorphic functions on U equipped with the sup norm and $A_U := \{f \in H^\infty(U)\colon f|K = 0\}$. (In the case where $K \subset C^n$ this proof is not difficult.) Next following Waelbroeck, using the closed graph theorem for maps of barrelled locally convex spaces into B-complete spaces, we obtain the above result.

In the case where E and F are Banach spaces we prove that there exists a Banach space $\tilde{F}$ containing F as a closed subspace such that every continuous function on a compact set in E with values in F having the weakly holomorphic extension property can be extended to a holomorphic function on a neighbourhood of K but with values in $\tilde{F}$.