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Metric characterization of first Baire class linear forms and octahedral norms

by

GILLES GODEFROY (Paris and Columbia, Mo.)

Abstract. We characterize those elements z of the bidual X^{**} of a Banach space X which are of the first Baire class by means of the set $\Omega(z)$ of points of X at which the norm of X^{**} is smooth in the direction z. New characterizations of Banach spaces not containing $l^1(N)$ are given. When X contains $l^1(N)$, we construct an octahedral norm on X; this norm enjoys optimal properties of roughness.

0. Introduction. Since H. P. Rosenthal proved his fundamental result ([26]), the first Baire class functions, their topological properties, and the pointwise compact spaces they generate have attracted a renewed attention (see e.g. [4], [10], [18]). Indeed, these functions—leaving apart their intrinsic interest — turn out to be a basic tool for studying the Banach spaces not containing an isomorphic copy of $l^1(N)$.

This class of Banach spaces strictly contains (by [19]) the class of Asplund spaces, and it is well known (see [5]) that this latter class can be characterized by differentiability properties of norms; it is therefore natural to try to characterize the spaces not containing $l^1(N)$ in terms of "smoothness" of norms; several results along these lines can be found in [24] and [14].

Our goal in the present work is to tighten the link between these two aspects of the theory, by characterizing the elements z of the bidual which are of the first Baire class by the smoothness properties of the bidual norm in the direction z; this will allow us to obtain some information about the geometry of the Banach spaces containing or not $l^1(N)$.

Let me briefly describe the content of this article. The main result of the first section (Theorem I.2) is a characterization of the elements $z \in X^{**}$ which are of the first Baire class by means of the set $\Omega(z)$ of points of X at which the bidual norm is differentiable in the direction z; this permits us to show that the set of points of continuity of a first Baire class function on a metrizable compact convex set K meets the set $*$-$\mathrm{Exp}(K)$ of exposed points of K in a dense set (Corollary I.5) and to give a new characterization of spaces not containing $l^1(N)$ by a smoothness property of their norms (Corollary I.7).

The results of Section I suggest that "$X \not\supset l^1(N)$" is the natural pointwise analogue of the property "X Asplund"; the first two results of Section II show indeed that one switches from one property to another by interchanging the order of the quantifiers in the expressions of the smoothness of the norms (Proposition II.1 and Theorem II.2). We construct (Theorem II.4) an "octahedral" norm on every Banach space containing $l^1(N)$; such a norm turns out to be optimal for several conditions of roughness (see Remarks II.5); it also allows us to characterize (Theorem II.6) the spaces not containing $l^1(N)$ by the behavior of the norming subspaces of the dual space.

Acknowledgements. I am glad to thank B. Maurey for his interest in this matter. The present work relies heavily on a previous unpublished joint work [15] of B. Maurey and the author. My gratitude also to N. J. Kalton who suggested condition (4) of Theorem I.2; and to Michèle Godefroy who was kind enough to type this article.

Notation. The weak-star topology on the dual X^* of a Banach space X is denoted by w^*. The points of X — resp. of X^*, X^{**}, X^{***} — are denoted by $x, x', \ldots$ — resp. $y, y', \ldots, z, z', \ldots, t, t', \ldots$ The same notation is used for a norm on X and its dual norm on X^*; the initial norm is denoted by $\|\ \|$. When there is no ambiguity on the norm, the closed unit ball of a Banach space Y is denoted by Y_1, and the unit sphere by $S_1(Y)$. If K is a w^*-compact convex subset of a dual X^*, the set of extreme points is $\operatorname{Ext}(K)$ and $*$-$\operatorname{Exp}(K)$ is the set of points which are exposed in K by an element of X. If $z \in X^{**}$, $\mathscr{C}_*(z; K)$ is the set of points of continuity of z on (K, w^*). $\operatorname{Osc}(z; S) = \sup_S z - \inf_S z$ is the oscillation of a function z on a set S, while $\operatorname{Osc}(z)(\cdot)$ denotes the pointwise oscillation.

The subspaces we consider are always assumed to be closed in the norm topology, and the different norms we construct are always equivalent, hence these words will often be omitted.

All the Banach spaces we consider are real.

I. Metric characterizations of first Baire class functions. Let us start this section by defining the set of smoothness $\Omega(\|\ \|; z)$ of an element z of X^{**}; this notion will play a crucial role in our study.

DEFINITION I.1. Let X be a Banach space with the norm $\|\ \|$, and let z be in X^{**}. The *set of smoothness* $\Omega(\|\ \|; z)$ of z is the set of x's in $X\setminus\{0\}$ such that

$$\lim_{\lambda\to 0} \lambda^{-1}[\|x+\lambda z\| + \|x-\lambda z\| - 2\|x\|] = 0.$$

In other words, $\Omega(\|\ \|; z)$ consists of the points of X at which the bidual norm is differentiable in the direction z.

It turns out that the elements z of X^{**} which are of the first Baire class can be characterized in terms of $\Omega(z)$, as follows:

THEOREM I.2. *Let X be a Banach space, and $z \in X^{**}$. The following are equivalent*:

(1) *For every w^*-compact subset K of X^*, the restriction of z to K has a point of w^*-continuity.*

(2) *For every equivalent norm $\|\ \|$ on X, the set $\Omega(\|\ \|; z)$ is a dense G_δ of X.*

(3) *For every equivalent norm $\|\ \|$ on X, the set $\Omega(\|\ \|; z)$ is nonempty.*

If X is separable, then (1)–(3) *are equivalent to*:

(4) *There exists an equivalent norm $|||\ |||$ on X such that $\Omega(|||\ |||; z) = X\setminus\{0\}$.*

Proof. The following technical lemma will be useful.

LEMMA I.3. *With the above notation, $x \in \Omega(\|\ \|; z)$ if and only if*

$$\inf_{\alpha>0} \operatorname{Osc}(z; S(x, \alpha)) = 0$$

where $S(x, \alpha) = \{y \in X_1^* \mid y(x) > \|x\| - \alpha\}$.

Proof. Consider the following subset of X_1^{***}:

$$G = \{t \in X_1^{***} \mid t(x) = \|x\|\}.$$

It is easily checked that $x \in \Omega(\|\ \|; z)$ if and only if $\operatorname{Osc}(z; G) = 0$. On the other hand, we have

$$G = \bigcap_{\alpha>0} S(x, \alpha)^{\sim}$$

where $\sim$ denotes the closure in (X^{***}, w^*). By w^*-compactness, this implies that

$$\operatorname{Osc}(z; G) = \inf_{\alpha>0} \operatorname{Osc}(z; S(x, \alpha))$$

and this shows the lemma. ∎

Let us now proceed to the proof of the theorem.

(1)⇒(2). If x and x' in X are such that $\|x-x'\| < \varepsilon$, one has $S(x', \alpha-\varepsilon) \subseteq S(x, \alpha)$. This clearly implies that for every $\eta > 0$, the set

$$O_\eta = \{x \in X\setminus\{0\} \mid \inf_{\alpha>0} \operatorname{Osc}(z; S(x, \alpha)) < \eta\}$$

is open in X. We have to show that O_η is dense. Pick $x_0 \in X$ and $\varepsilon > 0$. By a lemma of Bishop–Phelps [2] if $x \in X$ is such that

$$(*) \qquad \forall\, y \in \operatorname{Ker} x_0,\ \|y\| \leqslant (2\varepsilon)^{-1}, \quad |x(y)| \leqslant 1,$$

then $\|x - x_0\| < \varepsilon\|x_0\|$ or $\|x + x_0\| \leqslant \varepsilon\|x_0\|$.

We let now

$$K_\varepsilon = \operatorname{conv}(\{\|y\| \leqslant 1\} \cup \{y \in \operatorname{Ker} x_0 \mid \|y\| \leqslant (2\varepsilon)^{-1}\}).$$

The set K_ε is the dual unit ball of an equivalent norm $|||\ |||_\varepsilon$ on X. By assumption (1) and a standard Baire category argument, the set $\mathscr{C}_*(z; K_\varepsilon)$ of points of continuity of $z\colon (K_\varepsilon, w^*)\to \boldsymbol{R}$ is dense in K_ε. Hence by ([27], Proposition 8) the set $\mathscr{C}_*(z; K_\varepsilon)$ meets $\mathrm{Ext}(K_\varepsilon)$ in a w^*-dense subset of $\mathrm{Ext}(K_\varepsilon)$. Since we have $\mathrm{Ext}(K_\varepsilon)\subseteq\{\|y\|\leqslant 1\}\cup \mathrm{Ker}\,x_0$, there exists $y_0\in\mathscr{C}_*(z; K_\varepsilon)\cap \mathrm{Ext}(K_\varepsilon)$ with $\|y_0\|=1$.

Since y_0 belongs to this intersection, there exists $x\in X$ with $|||x|||_\varepsilon=1$ and $\alpha<1$ such that $x(y_0)>\alpha$ and

$$\mathrm{Osc}(z;\ \{y\in K_\varepsilon \mid x(y)>\alpha\})<\eta.$$

Since $|||x|||_\varepsilon=1$, the condition $(*)$ is satisfied and thus we may assume that $\|x-x_0\|<\varepsilon\|x_0\|$; on the other hand, the oscillation of z on the nonempty set $\{y\in X^* \mid \|y\|\leqslant 1,\ x(y)>\alpha\}$ is less than η and thus $x\in O_\eta$; this shows that O_η is dense. Since by Lemma I.3 we have

$$\Omega(z)=\bigcap_{k=1}^{\infty} O_{k^{-1}}$$

Baire's theorem concludes the proof.

(2)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(1). Assume (1) is not satisfied, and let K be a w^*-compact subset of X such that $\mathscr{C}_*(z; K_\varepsilon)=\emptyset$; by Baire's theorem, this implies that there exists $\varepsilon_0>0$ such that

$$K'=\{y\in K \mid \mathrm{Osc}(z|_K)(y)\geqslant\varepsilon_0\}$$

has a nonempty interior in (K, w^*); we let K_0 be the w^*-closure of the w^*-interior of K' in K; we clearly have

$$\forall\ y\in K_0,\quad \mathrm{Osc}(z|_{K_0})(y)\geqslant\varepsilon_0.$$

Let $C=\overline{\mathrm{cv}}^*(K_0\cup(-K_0))$. The function $\mathrm{Osc}(z|_C)$ is easily seen to be concave and w^*-u.s.c.; since K_0 is w^*-compact and $K_0\cup(-K_0)\supseteq \mathrm{Ext}(C)$, there exists for every $y\in C$ a probability measure μ on $K_0\cup(-K_0)$ such that $y=r(\mu)$. Since $\mathrm{Osc}(z|_C)$ is the infimum of the w^*-continuous affine functions which maximize it, Fatou's lemma shows that

$$\mathrm{Osc}(z|_C)(y)\geqslant\int \mathrm{Osc}(z|_C)(e)\,d\mu(e)$$

and since $\mathrm{Osc}(z|_C)\geqslant \mathrm{Osc}(z|_{K_0})$ on K_0, this shows that $\mathrm{Osc}(z|_C)\geqslant\varepsilon_0$ on C.

We now define the set C' by

$$C'=\{\lambda y_1+(1-\lambda)y_2 \mid y_1\in C,\ \|y_2\|\leqslant 1,\ \tfrac{1}{2}\leqslant\lambda\leqslant 1\}.$$

C' is clearly the dual unit ball of an equivalent norm $|||\ |||$ on X. The relation

$$\forall\ y\in C,\quad \mathrm{Osc}(z|_{C'})(y)\geqslant \mathrm{Osc}(z|_C)(y)\geqslant\varepsilon_0,$$

together with the concavity of $\mathrm{Osc}(z|_{C'})$ and the definition of C', shows that

$$\forall\ y\in C',\quad \mathrm{Osc}(z|_{C'})(y)\geqslant\varepsilon_0/2$$

and thus the oscillation of z is at least $\varepsilon_0/2$ on every w^*-open slice of C'; by Lemma I.3, this shows that $\Omega(|||\ |||; z)=\emptyset$, and thus (3)$\Rightarrow$(1).

We will now connect the property (4) with (1)–(3); hence we are now assuming that X is separable.

(1)$\Rightarrow$(4). Since X is separable, (X_1^*, w^*) is a metrizable compact space. Then (1) is equivalent to the existence of a sequence $(x_n)_{n\geqslant 1}$ in X such that

$$z=w^*\text{-}\lim_{n\to\infty} x_n \quad \text{in } (X^{**}, w^*).$$

Let $(x_k')_{k\geqslant 1}$ be a norm-dense sequence in the unit ball of X. The norm

$$\|y\|_1=\|y\|+\Big(\sum_{k=1}^{\infty}2^{-k}(x_k'(y))^2\Big)^{1/2}$$

is an equivalent strictly convex dual norm on X^*. We now define an equivalent dual norm $|||\ |||$ on X^* by the formula

$$|||y|||=\|y\|_1+\sum_{n=1}^{\infty}2^{-n}\sup_{k,l\geqslant n}|x_k(y)-x_l(y)|.$$

$|||\ |||$ is strictly convex since $\|\ \|_1$ is; by Lemma I.3, for the proof of $\Omega(|||\ |||; z)=X\backslash\{0\}$ it is enough to show that

$$(**)\qquad |||y_i|||\leqslant 1,\quad |||y|||=1,\quad y_i\xrightarrow{w^*}y$$

implies that $\lim_i z(y_i)=z(y)$. Since the functionals which define $|||\ |||$ are w^*-l.s.c. the conditions $(**)$ imply that

$$\forall\ n\geqslant 1,\quad \lim_i\sup_{k,l\geqslant n}|x_k(y_i)-x_l(y_i)|=\sup_{k,l\geqslant n}|x_k(y)-x_l(y)|.$$

Take $\varepsilon>0$, and let N be such that $\sup_{k,l\geqslant N}|x_k(y)-x_l(y)|<\varepsilon/4$. There exists I_0 such that

$$\forall\ i\geqslant I_0,\quad \sup_{k,l\geqslant N}|x_k(y_i)-x_l(y_i)|<\varepsilon/3.$$

By letting k tend to infinity, we get

$$\forall\ i\geqslant I_0,\quad |z(y_i)-x_N(y_i)|\leqslant\varepsilon/3.$$

Also $|z(y)-x_N(y)|\leqslant\varepsilon/4$. Now since x_N is w^*-continuous, there exists $I_1\geqslant I_0$ such that

$$\forall\ i\geqslant I_1,\quad |x_N(y)-x_N(y_i)|<\varepsilon/3$$

and thus

$$\forall\ i\geqslant I_1,\quad |z(y)-z(y_i)|\leqslant\varepsilon/3+\varepsilon/3+\varepsilon/4<\varepsilon.$$

(4)⇒(1). Let $|||\ |||$ be a norm such that $\Omega(|||\ |||; z) = X\backslash\{0\}$ and let $X_1^* = \{|||y||| \leqslant 1\}$. Condition (1) is equivalent to the fact that for every α, β in $\boldsymbol{R}$, $X_1^* \cap z^{-1}((\alpha, \beta))$ is w^*-K_σ in X_1^* [1], and this is what we will show. Take $\alpha < \beta$ and $k \in N$ such that $k^{-1} < (\beta-\alpha)/2$. Let

$$S_k = \{y \in X^* \mid |||y||| = 1,\ \alpha+k^{-1} < z(y) < \beta-k^{-1}\}$$

and let O_k be the union of the w^*-slices S of X_1^* such that $S \cap \{|||y||| = 1\} \subseteq S_k$. Since we restricted ourselves to slices, we have

$$O_k \subseteq z^{-1}\big((\alpha+k^{-1},\ \beta-k^{-1})\big) \cap X_1^*.$$

On the other hand, Lemma I.3 shows that O_k contains all the points of S_k which attain their norm on $X_1 = \{|||x||| \leqslant 1\}$; by the Bishop–Phelps theorem, this set is norm-dense in S_k, and thus we have

$$S_k \subseteq O_k + (2k)^{-1}|||z|||^{-1} X_1^* = F_k.$$

Since O_k is an open subset of the metrizable compact space (X_1^*, w^*), it is w^*-K_σ and thus $F_k \cap X_1^*$ is also w^*-K_σ. Observe now that $F_k \subseteq z^{-1}((\alpha, \beta))$ for every k; this implies that

$$\bigcup_{k\geqslant 1} S_k \subseteq \Big(\bigcup_{k\geqslant 1} F_k\Big) \cap X_1^* \subseteq X_1^* \cap z^{-1}((\alpha, \beta)).$$

But we clearly have

$$\operatorname{conv}\Big(\bigcup_{k\geqslant 1} S_k\Big) = X_1^* \cap z^{-1}((\alpha, \beta))$$

and thus

$$\operatorname{conv}\Big(\Big(\bigcup_{k\geqslant 1} F_k\Big) \cap X_1^*\Big) = X_1^* \cap z^{-1}((\alpha, \beta)).$$

Since $(\bigcup_{k\geqslant 1} F_k) \cap X_1^*$ is w^*-K_σ, this shows that the set $X_1^* \cap z^{-1}((\alpha, \beta))$ is also w^*-K_σ, and this concludes the proof. ■

Remarks I.4. 1) It is easy to establish a quantitative version of Lemma I.3. This permits one to show, by the proof of (3)⇒(1), that if $z \in X^{**}$ is not of the first Baire class, there exists an equivalent norm $\|\ \|$ on X such that the bidual norm is "uniformly rough" in the direction z; namely, there exists $\varepsilon_1 = \varepsilon_0/2 > 0$ such that

$$\|x+\lambda z\| + \|x-\lambda z\| \geqslant 2\|x\| + \varepsilon_1|\lambda|$$

for every $x \in X$ and every $\lambda \in \boldsymbol{R}$.

2) Theorem I.2 furnishes us with a condition of roughness of the norm which implies the weak sequential completeness (w.s.c.): namely, if a Banach space X is such that $\tilde{\Omega}(\|\ \|; z) \neq X$ for every $z \in X^{**}\backslash X$, then X is w.s.c. The only natural examples of this situation seem to be the spaces which are L^1-complemented in their bidual (see [13]). It is not known if every w.s.c. Banach space has an equivalent norm such that $\tilde{\Omega}(\|\ \|; z) \neq X$ for every $z \in X^{**}\backslash X$.

3) Theorem I.2 applies of course to the case where $z \in X$; then (1)⇒(2) reads: if X is any Banach space and $z \in X$, the norm of X is differentiable in the direction z at each point of a dense G_δ, $\Omega(z)$, of X.

4) Let X be separable and let $z \in X^{**}$ be of the second affine Baire class, i.e., $z = w^*\text{-}\lim z_n$ where the z_n's are of the first Baire class. It is not difficult to adapt the proof of (1)⇒(4) to deduce that there is an equivalent norm on X so that every z_n is w^*-continuous on the unit sphere $S_1(X^*)$ of X^*; in particular, the restriction of z to $(S_1(X^*), w^*)$ is of the first Baire class, and $\mathscr{C}_*(z; X_1^*)$ is a w^*-dense G_δ of (X_1^*, w^*). In view of (4)⇒(1), this appears to be the optimal result for such a z.

Let us now mention another consequence of Theorem I.2.

Corollary I.5. *Let X be a separable Banach space, and K a w^*-compact convex subset of X^*. Let $*$-Exp(K) be the set of $*$-exposed points of K. If $z \in X^{**}$ is of the first Baire class, the set $\mathscr{C}_*(z; K)$ of points of w^*-continuity of z on K meets $*$-Exp(K) in a w^*-dense subset of $*$-Exp(K).*

Proof. Standard techniques show that we can assume without loss of generality that $K = X_1^*$ is a dual unit ball; we will make the proof only in this case.

By Theorem I.2 the set $\Omega(z)$ is a dense G_δ of X; on the other hand, Mazur's theorem shows that the set

$$\Omega = \{x \in X \mid \exists\ !\ y \in X_1^* \text{ s.t. } y(x) = \|x\|\}$$

is a dense G_δ of X; hence $\Omega(z) \cap \Omega = \Omega'$ is dense in Ω.

Let $x_0 \in \Omega'$ and $y_0 \in X_1^*$ be such that $y_0(x_0) = \|x_0\|$. Since $x_0 \in \Omega$, the family $S(x_0, \alpha) = \{y \in X_1^* \mid y(x_0) > \|x_0\| - \alpha\}$, where $\alpha > 0$, is a basis of neighborhoods of y_0 in (X_1^*, w^*); since $x_0 \in \Omega(z)$, Lemma I.3 shows that $y_0 \in \mathscr{C}_*(z; X_1^*)$ and thus y_0 belongs to $\mathscr{C}_*(z; X_1^*) \cap *\text{-Exp}(X_1^*)$.

If now $y \in *\text{-Exp}(X_1^*)$ is exposed by $x \in \Omega$, consider a sequence (x_n) in Ω' with $\lim \|x-x_n\| = 0$. By w^*-compactness, the corresponding y_n's in $*\text{-Exp}(X_1^*) \cap \mathscr{C}_*(z; X_1^*)$ are w^*-converging to y, and this concludes the proof. ■

Remarks I.6. 1) One has very little information on the topology of $*$-Exp(K) — in contrast with Ext(K) — and this makes the study of this set quite delicate. In the case where $*$-Exp(K) is replaced by Ext(K), Corollary I.5 is a special case of ([27], Proposition 8). The present proof of Corollary I.5 and that of ([8], Th. 1) are similar.

2) If X^* has the C*PCP (see [8], [11], [28]) and if $\mathscr{C}_*(w^*; \|\ \|)$ denotes the set of points of continuity of Id: $(X_1^*, w^*) \to (X_1^*, \|\ \|)$, it is unknown ([16], p. 31) whether $\mathscr{C}(w^*; \|\ \|)$ always meets the set of norm-attaining

elements of X^*. Corollary I.5 gives an affirmative answer for the "pointwise" analog, namely for the set $\mathscr{C}_*(z; X_1^*)$; however, I tend to believe that the answer should be negative in general. The main difference here is that the function $\operatorname{Osc}(z)$ is w^*-u.s.c. and concave on X_1^*, but the corresponding function

$$w(z) = \inf\{\| \ \|\text{-diam}(W) \mid W \text{ neigh. of } z \text{ in } (X_1^*, w^*)\}$$

is clearly w^*-u.s.c. but *not* concave in general.

Our next corollary expresses the fact that the class of spaces not containing $l^1(N)$ can be characterized by a weak smoothness property of their norms.

Corollary I.7. *Let X be a Banach space. The following are equivalent:*

(1) *X does not contain $l^1(N)$.*

(2) *For every equivalent norm on X and every $z \in X^{**}$, the set $\Omega(z)$ is a dense G_δ of X, in particular is nonempty.*

Proof. This is clear by Theorem I.2 and the Odell–Rosenthal theorem [25], or Haydon's theorem [17] in the nonseparable case. ■

Let us observe that the above implication (2)⇒(1) relies on (3)⇒(1) in Theorem I.2, i.e., on rough norms on spaces containing $l^1(N)$. A much stronger result is actually available and will be shown below (Theorem II.4).

II. Construction of rough and octahedral norms. We start this section with a couple of results which show that replacing the assumption "X^* separable" by "X does not contain $l^1(N)$" leads to an interchange of the order of the quantifiers in the expression of the smoothness properties of norms.

Let us first show the easy

Proposition II.1. *Let X be a separable Banach space. Then:*

(a) *X^* is separable ⇔ there exists an equivalent norm on X such that*

$$\forall\, z \in X^{**}, \quad \mathscr{C}_*(z; X_1^*) \supseteq S_1(X^*).$$

(b) *X does not contain $l^1(N)$ ⇔ for every $z \in X^{**}$, there exists an equivalent norm such that $\mathscr{C}_*(z; X_1^*) \supseteq S_1(X^*)$.*

Proof. (a) If X^* is separable, the Kadec–Klee renorming technique (see [21]) gives a norm on X such that the w^*- and norm topologies coincide on $S_1(X^*)$ and thus ⇒ holds.

Conversely, if $\mathscr{C}_*(z; X_1^*)$ contains $S_1(X^*)$ for every z in X^{**} then the w^*- and weak topologies coincide on $S_1(X^*)$, and thus $S_1(X^*)$ is separable for the weak topology; it follows that X^* is separable.

(b) X does not contain $l^1(N)$ if and only if every $z \in X^{**}$ is of the first Baire class ([25], [17]) and thus the proof of (1)⇔(4) in Theorem I.2 shows (b). ■

The next result is, up to the terminology, Theorem II.1 of [15]. We reproduce here for completeness the proof given in [15].

Theorem II.2. *Let X be a Banach space. The following are equivalent:*

(1) *X is an Asplund space.*

(2) *For every equivalent norm $\| \ \|$ on X, the intersection over $z \in X^{**}$ of the sets $\Omega(\| \ \|; z)$ is a dense G_δ of X, in particular is nonempty.*

Proof. (1)⇒(2). X is Asplund if and only if every equivalent norm on X is Fréchet smooth at every point of a dense G_δ. Now if $x \in X$ is a point of Fréchet smoothness of the norm of X then, by Shmul'yan's lemma (see [9]) or by the local reflexivity principle (see [23], p. 33), x is also a point of Fréchet smoothness of the bidual norm and thus belongs to $\Omega(\| \ \|; z)$ for every z in X^{**}.

(2)⇒(1). We denote by $\tilde{C}$ the closure of a subset C of a Banach space Z in (Z^{**}, w^*).

If a Banach space Z does not have the Radon–Nikodym property, then by [30] there exists a closed convex balanced and bounded subset C of Z and $\varepsilon_0 > 0$ such that

$$(***) \qquad \operatorname{dist}(z, Z) \geqslant \varepsilon_0 \quad \text{for every } z \in \operatorname{Ext}(\tilde{C}).$$

If X is not an Asplund space, we may apply this result to $Z = X^*$.

We define $K_0 = (\varepsilon_0/2)X_1^* + C$ and let $K = \overline{K_0}^*$ be the closure of K_0 in (X^*, w^*). The set K is the unit ball of a dual norm $|||\ |||$ on X^*. We have $\tilde{K}_0 = (\varepsilon_0/2)X_1^{***} + \tilde{C}$ and thus

$$\operatorname{Ext}(\tilde{K}_0) \subseteq (\varepsilon_0/2)\operatorname{Ext}(X_1^{***}) + \operatorname{Ext}(\tilde{C}).$$

By (***) this implies $\operatorname{Ext}(\tilde{K}_0) \cap X^* = \varnothing$.

Let us now show that the predual norm $|||\ |||$ on X satisfies

$$\bigcap_{z \in X^{**}} \Omega(|||\ |||; z) = \varnothing.$$

Take $x \in X \setminus \{0\}$, and let $y \in \operatorname{Ext}(K)$ be such that $y(x) = |||x|||$. If $\pi\colon X^{***} \to X^*$ denotes the canonical projection of kernel $X^\perp$, let

$$F = \{t \in \tilde{K}_0 \mid \pi(t) = y\}.$$

Since $y \in \operatorname{Ext}(K)$, the set F is a nonempty w^*-closed face of $\tilde{K}_0$ and thus $F \cap \operatorname{Ext}(\tilde{K}_0)$ is nonempty. Pick t in this set; we have $t \notin X^*$ and thus $t \neq y$; we choose $z \in X^{**}$ such that $z(t-y) > 0$. If now

$$S(x, \alpha) = \{y' \in K \mid y'(x) > |||x||| - \alpha\}$$

we have $y \in S(x, \alpha)$ and $t \in S(x, \alpha)^{\sim}$ for every $\alpha > 0$ and thus

$$\inf_{\alpha>0} \operatorname{Osc}(z; S(x, \alpha)) \geqslant z(t-y) > 0$$

and Lemma I.3 shows that $x \notin \Omega(|||\ |||; z)$. ■

Remarks II.3. 1) The above proof can be adapted to give a "uniform" version of the result—like in Remark I.4.1—which is, at least formally, an improvement of [22] and ([20], Prop. 3). This is actually done in ([15], Theorem II.3). For recent improvements (in particular from the quantitative point of view) of this result, the reader is referred to [29] and [31].

2) It is interesting to compare these statements with some results of [28] and [11]. If X^* does not have the Radon–Nikodym property, there exists by the above proof an equivalent norm on X and $\varepsilon_1 > 0$ such that every w^*-open slice of X^* has a diameter at least ε_1, although this cannot be done in general for w^*-open sets (example: $X = JT$ [11]). Also, the norm constructed in Theorem II.2 is such that no point of X_1^{***} which is exposed by an element of X belongs to X^*; this cannot be done in general for the $*$-exposed points of X_1^{***}; the example is again $X = JT$ (by [28]).

3) If X^* has the Radon–Nikodym property, then clearly

$$\exists\ \Omega\ w^*\text{-dense } G_\delta \text{ of } X_1^* \text{ s.t. } \Omega \subseteq \mathscr{C}_*(X_1^*; z)$$

for every $z \in X^{**}$.

On the other hand, if $X \not\supset l^1(N)$, then

$$\forall\ z \in X^{**}, \quad \mathscr{C}_*(X_1^*; z) = \Omega(z) \text{ is a } w^*\text{-dense } G_\delta \text{ in } X_1^*.$$

There is no example, up to now, where this weaker statement is optimal; in other words, the following question is open: does there exist $X \not\supset l^1(N)$ such that the map Id: $(X_1^*, w^*) \to (X_1^*, w)$ has no point of continuity?

We will now show that one can construct on any space which contains $l^1(N)$ an equivalent norm which enjoys optimal properties of roughness. Theorem II.4 is proven in ([15], Theorem III.1) in the case where X is separable. The proof given in [15] is very different, and much more difficult. This theorem also gives an independent proof and an improvement of (2)⇒(1) in Corollary I.7.

THEOREM II.4. *Let X be a Banach space. The following are equivalent:*

(1) *X contains $l^1(N)$.*

(2) *There exists an equivalent norm $|||\ |||$ on X and $z \in X^{**}\setminus\{0\}$ such that*

$$\forall\ x \in X, \quad |||x+z||| = |||x|||+|||z|||.$$

Proof. (2)⇒(1). We have $\Omega(|||\ |||; z) = \emptyset$ and thus Corollary I.7 applies.

Let us mention that the local reflexivity principle and an easy induction argument allow one to deduce from (2) the existence of a $(1+\varepsilon)$-copy of $l^1(N)$ in X (see [15]), thus providing us with an alternative proof.

(1)⇒(2). It is standard to deduce from the Hahn–Banach theorem that if a Banach space X contains a subspace Y isomorphic to a Banach space Z, there exists an equivalent norm $\|\ \|$ on X such that Y equipped with the norm induced by $\|\ \|$ is isometric to Z.

We may therefore assume that X contains a subspace Y which is isometric to $l^1(N)$. Let r_Y: $X^* \to X^*/Y^{\perp} = Y^*$ be the canonical quotient map. Let Ψ: $l^1 \to Y$ be a surjective isometry; the set $K = \operatorname{Ext}(Y_1^*)$ is w^*-homeomorphic to $\{-1, 1\}^N$ and thus it is w^*-compact. By Zorn's lemma, there exists a minimal w^*-compact subset of X_1^*, say K_0, such that $r_Y(K_0) = K$.

We pick a nontrivial ultrafilter $\mathscr{U}_0$ on N, and let $z_0 = \Psi^{**}(\mathscr{U}_0)$ be the corresponding element of Y^{**}. Clearly, $\|z_0\| = |z_0(y)|$ for every $y \in K$. Moreover, the sets

$$A^+ = \{y \in K \mid z_0(y) = 1\}, \quad A^- = \{y \in K \mid z_0(y) = -1\}$$

are both w^*-dense in K; by minimality, the sets $r_Y^{-1}(A^+) \cap K_0$ and $r_Y^{-1}(A^-) \cap K_0$ are also w^*-dense in K_0.

We let now $K_1 = K_0 \cup (-K_0)$ and $z = r_Y^*(z_0)$; clearly $\|z\| = 1 = |z(y)|$ for every $y \in K_1$ and the sets $z^{-1}(\pm 1) \cap K_1$ are both w^*-dense in K_1. We finally define

$$B = \overline{\operatorname{conv}}^*(K_1 + \{y \in Y^{\perp} \mid \|y\| \leqslant 2\}).$$

We have $r_Y(B) = Y_1^* = \overline{\operatorname{conv}}^*(K)$; hence for every $y \in X^*$ with $\|y\| \leqslant 1$, there exists $y_0 \in \overline{\operatorname{conv}}^*(K_1)$ such that $r_Y(y) = r_Y(y_0)$, and thus $y_0 - y \in Y^{\perp}$ and $\|y_0 - y\| \leqslant 2$, which shows that $y \in B$.

It follows that B is the unit ball of an equivalent dual norm on X^*; we will show that the predual norm $|||\ |||$ and $z = r_Y^*(z_0)$ work.

Indeed, since $r_Y(B) = Y_1^*$ we have $|||z||| = 1$; moreover, the set $K_2 = K_1 + \{y \in Y^{\perp} \mid \|y\| \leqslant 2\}$ is w^*-compact and thus contains $\operatorname{Ext}(B)$, and the sets $z^{-1}(\pm 1) \cap K_2$ are both w^*-dense in K_2; hence the Krein–Milman theorem shows that the sets $z^{-1}(\pm 1) \cap B$ are also w^*-dense in B.

Pick $x \in X$ and $\varepsilon > 0$, and let V be a w^*-open subset of B such that

$$\forall\ y \in V, \quad x(y) > |||x||| - \varepsilon.$$

Since there exists $y_0 \in V$ such that $z(y_0) = 1$, we have $|||z+x||| \geqslant y_0(z+x) > |||x|||+1-\varepsilon$ and thus $|||z+x||| \geqslant 1+|||x|||$; since $|||z||| = 1$, the result follows. ■

Remarks II.5. 1) The above proof shows that if X contains $l^1(N)$, there are a dual unit ball B in X^* and $z \in X^{**}$ such that $|z| \leqslant 1$ on B, $|z| = 1$ on $\operatorname{Ext}(B)$, and $z^{-1}(\pm 1) \cap \operatorname{Ext}(B)$ are w^*-dense in $\operatorname{Ext}(B)$. This seems to be the optimal "oscillation result" one could expect (see [25], [17]).

2) The local reflexivity principle implies that every norm $|||\ |||$ which satisfies condition (2) of Theorem II.4 satisfies:

(0) For every $F \subseteq X$ with $\dim F < \infty$ and every $\varepsilon > 0$, there exists $x_{F,\varepsilon} \in X$ with $|||x_{F,\varepsilon}||| = 1$ and

$$|||x + \lambda x_{F,\varepsilon}||| \geqslant (1-\varepsilon)(|||x||| + |\lambda|) \quad \text{for every } x \in F \text{ and } \lambda \in \mathbf{R}.$$

The converse is true if X is separable [14]; it is therefore natural to call such a norm an *octahedral norm*. This condition (0) is also equivalent [14] to the following: if $B_1, \ldots, B_n$ are a finite number of balls such that $X_1 \subseteq B_1 \cup \ldots \cup B_n$, there exists i, $1 \leqslant i \leqslant n$, such that $X_1 \subseteq B_i$. Moreover, (0) is also equivalent to the fact that every finite convex combination of w^*-open slices of X^* has diameter 2 (see [7]); i.e., all the slices of the family are big "in the same direction"; this implies that every w^*-open subset of X^* has diameter 2. For the related notion of "strongly regular" set, see e.g. [3], [10], [16].

3) This leads to the question of whether a norm can be constructed on $X \supset l^1(N)$ such that every convex combination of weakly open slices of X^* has diameter 2. For solving affirmatively this (open) question, it would be sufficient to answer positively the following

QUESTION. If X contains $l^1(N)$, does there exist an equivalent norm on X such that the bidual norm is octahedral on X^{**}?

Let us mention that the canonical norm of l^{1**} is clearly octahedral while the canonical norm of $\mathscr{C}([0, 1])^{**}$ is not, although $\mathscr{C}([0, 1])$ is octahedral.

4) It is easy to deduce from the proof of Theorem II.4 that if $z \in X^{**}\setminus X$ is such that there exists $Y \subseteq X$ isomorphic to $l^1(N)$ such that $z \in Y^{\perp\perp}$, then there is a norm $|||\ |||$ such that z satisfies condition (2) of II.4. However, the converse is not true, as shown by the example of

$$z = \mathbf{1}_{Q\cap[0,1]} - \mathbf{1}_{[0,1]\setminus Q}$$

in $\mathscr{C}([0, 1])^{**}$ which is of the second Baire class and thus cannot belong to $Y^{\perp\perp}$ with Y isomorphic to $l^1(N)$ (by [6]). The question for which $z \in X^{**}$ such a norm exists appears to be delicate. An obvious—e.g. by Theorem I.2—necessary condition is that z cannot be of the first Baire class.

5) The element z of $\mathscr{C}([0, 1])^{**}$ which is defined above permits one to show that the function "w^*-Osc(z)" depends heavily upon the norm. Indeed, the w^*-oscillation of z on the canonical dual unit ball of $\mathscr{M}(0, 1)$ is identically 2; on the other hand, there exists by Remark I.4.4 a dual unit ball K such that $z|_K$ is w^*-continuous at every point of a w^*-dense G_δ of K.

Let us conclude this work by a characterization of Banach spaces not containing $l^1(N)$ in terms of norming subspaces of the dual space. This result was proved in [15] (Theorem IV.4) in the case of X separable.

Recall that a subspace Y of the dual X^* of X is said to be *norming* if

$$\forall\, x \in X, \quad \|x\| = \sup\{y(x) \mid y \in Y,\ \|y\| \leqslant 1\}.$$

With this terminology, the following holds:

THEOREM II.6. *Let X be a Banach space. Then either*

(i) *for every equivalent norm on X, the dual X^* contains a smallest norming subspace, or*

(ii) *there exists a norm on X such that every w^*-closed hyperplane of X^* contains the intersection of two norming hyperplanes.*

Moreover, (i) *is equivalent to: X does not contain $l^1(N)$.*

Proof. Y is norming if and only if the canonical quotient map $r_Y\colon X^{**} \to X^{**}/Y^\perp$ induces an isometric embedding on X, which means that

$$\forall\, x \in X, \quad \|x\| = \inf\{\|x+z\| \mid z \in Y^\perp\}, \quad \text{i.e.,}$$

$$\forall\, x \in X,\ \forall\, z \in Y^\perp, \quad \|x+z\| \geqslant \|x\|.$$

Consider the cone

$$\mathscr{C} = \{z \in X^{**} \mid \|x+z\| \geqslant \|x\| \ \ \forall\, x \in X\}.$$

By the above, Y is norming if and only if $Y^\perp$ is contained in $\mathscr{C}$.

If $X \not\supset l^1(N)$, then by ([14], Corollary V.5) the cone $\mathscr{C}$ is a w^*-closed vector subspace (see [12] for a simpler proof when X is separable). If $N = \mathscr{C}^\top$ is the orthogonal of $\mathscr{C}$ in X^* then

$$Y \text{ norming} \Leftrightarrow Y^\perp \subseteq \mathscr{C} = N^\perp \Leftrightarrow Y \supseteq N$$

and this shows that $X \not\supset l^1(N)$ implies (i).

Assume now that $X \supset l^1(N)$, and let $|||\ |||$ and $z \in X^{**}\setminus\{0\}$ be such that condition (2) of Theorem II.4 holds. Pick $x \in X\setminus\{0\}$ and let

$$z_1 = x + \frac{|||x|||}{|||z|||} z, \quad z_2 = x - \frac{|||x|||}{|||z|||} z.$$

For every $x' \in X$ and $i = 1, 2$, one has

$$|||z_i + x'||| = |||x + x'||| + |||x||| \geqslant |||x'|||$$

and thus z_1 and z_2 belong to $\mathscr{C}$; hence, the spaces $\operatorname{Ker} z_i$ $(i = 1, 2)$ are norming hyperplanes and since $x = \frac{1}{2}(z_1 + z_2)$, we have

$$\operatorname{Ker} z_1 \cap \operatorname{Ker} z_2 \subseteq \operatorname{Ker} x.$$

Hence the norm $|||\ |||$ satisfies (ii). ∎

Remark II.7. When X^* contains a smallest norming subspace N, then X has at most one isometric predual—namely the space N. This method was used in [12]; for much more along these lines, see [14]. Theorem II.6 expresses the fact that the class of Banach spaces not containing $l^1(N)$ is exactly the isomorphic class to which this technique applies.

References

[1] R. Baire, *Sur les fonctions de variables réelles*, Annali di Mat. (3) 3 (1899), 1–123.

[2] E. Bishop and R. R. Phelps, *The support functionals of a convex set*, in: Proc. Sympos. Pure Math. 7, Amer. Math. Soc., Providence, R.I., 27–35.

[3] J. Bourgain, *La propriété de Radon–Nikodym*, Publ. Math. Univ. Pierre et Marie Curie 36 (1979).

[4] J. Bourgain, D. H. Fremlin and M. Talagrand, *Pointwise compact sets of Baire-measurable functions*, Amer. J. Math. 100 (4) (1978), 845–886.

[5] R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon–Nikodym Property*, Lecture Notes in Math. 993, Springer, 1983.

[6] J. P. R. Christensen, *Topology and Borel Structure*, North-Holland Math. Stud. 10, 1974.

[7] R. Deville, *A dual characterization of the existence of small combinations of slices*, Bull. Austral. Math. Soc. 37 (1988), 113–120.

[8] R. Deville, G. Godefroy, D. Hare and V. Zizler, *Differentiability of convex functions and the convex point of continuity property in Banach spaces*, Israel J. Math. 59 (1987), 245–255.

[9] J. Diestel, *Geometry of Banach Spaces. Selected Topics*, Lecture Notes in Math. 485, Springer, 1975.

[10] N. Ghoussoub, G. Godefroy, B. Maurey and W. Schachermayer, *Some topological and geometrical structures in Banach spaces*, Mem. Amer. Math. Soc. 378 (1987).

[11] N. Ghoussoub, B. Maurey and W. Schachermayer, *Geometrical implications of certain infinite dimensional decompositions*, Trans. Amer. Math. Soc., to appear.

[12] G. Godefroy, *Points de Namioka, espaces normants, applications à la théorie isométrique de la dualité*, Israel J. Math. 38 (3) (1981), 209–220.

[13] —, *Sous-espaces bien disposés de L^1—applications*, Trans. Amer. Math. Soc. 286 (1) (1984), 227–249.

[14] G. Godefroy and N. Kalton, *The ball topology and its applications*, Contemp. Math. 85 (1989), 131–190.

[15] G. Godefroy et B. Maurey, *Normes lisses et normes anguleuses sur les espaces de Banach séparables*, unpublished preprint.

[16] D. Hare, *A duality theory for Banach spaces with the convex-point-of-continuity property*, Ph.D. Dissertation, Univ. of British Columbia, 1987.

[17] R. Haydon, *Some more characterizations of Banach spaces containing l_1*, Math. Proc. Cambridge Philos. Soc. 80 (1976), 269–276.

[18] R. Haydon, E. Odell and H. P. Rosenthal, *On certain classes of Baire-1 functions with applications to Banach space theory*, in: Longhorn Notes, Texas Funct. Anal. Seminar, to appear.

[19] R. C. James, *A separable somewhat reflexive Banach space with nonseparable dual*, Bull. Amer. Math. Soc. 80 (1974), 738–743.

[20] K. John and V. Zizler, *On rough norms on Banach spaces*, Comm. Math. Univ. Carolinae 19 (2) (1978), 335–349.

[21] V. L. Klee, *Convex bodies and periodic homeomorphisms in Hilbert space*, Trans. Amer. Math. Soc. 74 (1953), 10–43.

[22] E. B. Leach and J. H. M. Witfield, *Differentiable functions and rough norms on Banach spaces*, Proc. Amer. Math. Soc. 33 (1) (1972), 120–126.

[23] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Vol. I, *Sequence Spaces*, Springer, 1977.

[24] B. Maurey, *Types and l_1-subspaces*, in: Longhorn Notes, Texas Funct. Anal. Seminar, Austin, Texas, 1982/1983, 123–137.

[25] E. Odell and H. P. Rosenthal, *A double-dual characterization of separable Banach spaces containing l^1*, Israel J. Math. 20 (1975), 375–384.

[26] H. P. Rosenthal, *A characterization of Banach spaces containing l^1*, Proc. Nat. Acad. Sci. U.S.A. 71 (1974), 2411–2413.

[27] E. Saab and P. Saab, *A dual geometric characterization of Banach spaces not containing l_1*, Pacific J. Math. 105 (2) (1983), 415–425.

[28] W. Schachermayer, *Some new properties of the James-tree space JT*, to appear.

[29] W. Schachermayer, A. Sersouri, and E. Werner, *Some geometrical results concerning the Radon–Nikodym property of Banach spaces*, Israel J. Math., to appear.

[30] C. Stegall, *The Radon–Nikodym property in Banach spaces*, part I, based on lectures given in Essen, 1982.

[31] E. Werner, *Non-dentable solid subsets in Banach lattices failing R.N.P. Applications to renorming*, Proc. Amer. Math. Soc., to appear.

ÉQUIPE D'ANALYSE
UNIVERSITÉ PARIS VI
Tour 46-0, 4ème étage
4, Pl. Jussieu, 75252 Paris Cedex 05, France

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MISSOURI
Columbia, Missouri 65211, U.S.A.

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