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Received February 29, 1988 (2412)
Revised version October 17, 1988

STUDIA MATHEMATICA, T. XCV (1989)

A new look on Hankel forms over Fock space

by

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Abstract. By a generalized Hankel form over Fock space $F^{\alpha,2}(C)$ we mean a bilinear form of the type

$$\Gamma_{b,h}(f_1, f_2) = \iint_{C^2} \bar{h}\left(\frac{z_1-z_2}{2}\right)\bar{b}\left(\frac{z_1+z_2}{2}\right) f_1(z_1) f_2(z_2)\, d\lambda_\alpha(z_1) d\lambda_\alpha(z_2).$$

Here $d\lambda_\alpha$ is the Gaussian measure in C with density $(\alpha/\pi)e^{-\alpha|z|^2}$ $(\alpha > 0)$ and an entire function $f = f(z)$ belongs to $F^{\alpha,2}(C)$ iff it is square integrable with respect to it. A trace ideal criterion is proved for such forms which generalizes the corresponding results for ordinary Hankel forms, the case $h \equiv 1$ ([JPR], [W]).

0. Introduction. In [JP] the following new point of view in the theory of Hankel forms (operators) was advocated. Let V be a Hilbert space of analytic functions over a "homogeneous" domain Ω in C^d on which the corresponding symmetry group G acts via unitary operators. Identifying Hilbert—Schmidt forms over V with elements of $V \otimes V$, one has also an action of G on bilinear forms. Defining a Hankel form as a bilinear form Γ such that the value $\Gamma(f, g)$ for $f, g \in V$ depends only on the combination $f \cdot g$, we may identify the space of all Hankel forms as a special irreducible component of $V \otimes V$ under the above action.

Consider, for instance, the weighted Bergman case ($d = 1$, $\Omega = \Delta =$ unit disk in C, $G = \mathrm{SU}(1, 1)/\pm 1$, $V = A^{\alpha,2}(\Delta)$). Then $V \otimes V$ comes as a discrete sum of irreducible G-modules, one of which then consists of Hankel forms, and the elements of the other components are termed Hankel forms of higher weight. This case was studied at length in [JP].

The situation of the Fock space ($d = 1$, $\Omega = C$, $G = \mathscr{H} =$ Heisenberg group, $V = F^{\alpha,2}(C) =$ Fock space), actually a limiting case of the previous and briefly alluded to in [JP], is somewhat special, in the respect that now all

1980 *Mathematics Subject Classification*: 47B35, 47B10.
Key words and phrases: Hankel form, Fock space, Heisenberg group, Hilbert–Schmidt class, trace class.

irreducible "Heisenberg modules" in $F^{\alpha,2}(C)$ are "isotypic" (isomorphic to $F^{2\alpha,2}(C)$).

This forces a changed point of view. As above we identify bilinear Hilbert–Schmidt forms over $F^{\alpha,2}(C)$ with functions in $F^{\alpha,2}(C^2)$. If $\beta \in F^{\alpha,2}(C^2)$ then the corresponding form L_β is given by

$$L_\beta(f, g) = \iint_{C^2} \overline{\beta(z_1, z_2)}\, f(z_1) f(z_2)\, d\lambda_\alpha(z_1)\, d\lambda_\alpha(z_2)$$

with $d\lambda_\alpha(z) = (\alpha/\pi) e^{-\alpha|z|^2} dE(z)$ (E = Euclidean area measure). Recall also that

$$f \in F^{\alpha,2}(C) \Leftrightarrow \int_C |f(z)|^2\, d\lambda_\alpha(z) < \infty.$$

(More generally, we define for $0 < p \leqslant \infty$

$$f \in F^{\alpha,p}(C) \Leftrightarrow \int_C |f(z) e^{-\alpha|z|^2/2}|^p\, dE(z) < \infty,$$

with the usual interpretation if $p = \infty$.)

Definition. By a *generalized Hankel form* is meant a form that generates an irreducible Heisenberg module (an $\mathscr{H}$-module).

By general principles (see Sec. 1 for the motivation) then β is expected to be of the form

$$\beta(z_1, z_2) = h\left(\frac{z_1 - z_2}{2}\right) b\left(\frac{z_1 + z_2}{2}\right),$$

with $b, h \in F^{2\alpha,2}(C)$. The 2 factors are chosen mostly out of convenience. We will denote the corresponding form by $\Gamma_{b,h}$.

In Sections 2–3 of this paper we investigate the smoothness and boundedness properties of such forms. Our objective is to carry the theory to the same level as for "ordinary" Hankel forms over Fock space ([JPR], [W]). Section 1 is devoted to merely formal considerations.

All our considerations generalize *mutatis mutandis* to the case of higher dimension ($d > 1$) but for simplicity we have confined our attention to the case $d = 1$.

The Fock space corresponds to "bosons". Probably there is also a corresponding theory with "fermions". Then the rôle of the Heisenberg group $\mathscr{H}$ is taken by the Clifford algebra. We plan to return to the fermionic case in a subsequent publication.

1. Formal considerations on generalized Hankel forms. We begin with some general nonsense.

Quite generally, consider a finite-dimensional G-module $V = V_1 \oplus \ldots \oplus V_r$ which is the sum of *isotypic* irreducible G-modules V_i; G is now any group, say, finite. Let $V = V_1' \oplus \ldots \oplus V_r'$ be any other such decomposition. Then there exists an endomorphism of G-modules $T: V \to V$ such that $V_i' = TV_i$. By

Schur's lemma T must be of the form

$$x' = Tx \Leftrightarrow x_i' = \sum a_{ij} x_j,$$

where x_i and x_i' are the components of x and x' in the decompositions $V = V_1 \oplus \ldots \oplus V_r$ and $V = V_1' \oplus \ldots \oplus V_r'$ and the a_{ij}'s are scalars. It follows that each V_i' consists of vectors of the form $(a_{1i}x, \ldots, a_{ri}x)$ with $x \in V_i$. This gives the general form of an irreducible submodule of V. ∎

The statement about the form of β made in the introduction, apparently, is just a "continuous" analogue of the above fact.

There arises now the question how to pick up h in a natural way. Let M_h denote the $\mathscr{H}$-module corresponding to a given function h:

$$M_h = \left\{\beta:\ \exists b\quad \beta(z_1, z_2) = h\left(\frac{z_1 - z_2}{2}\right) b\left(\frac{z_1 + z_2}{2}\right)\right\}.$$

We observe that the metaplectic group $\mathrm{Mp}(2, \boldsymbol{R})$ (a double cover of the symplectic group $\mathrm{Sp}(2, \boldsymbol{R}) = \mathrm{SL}(2, \boldsymbol{R})$) permutes the spaces M_h. In the notation of [Pee1]

$$T_{\tilde g} C(a) T_{\tilde g^{-1}} = C(b) \qquad (b = ga),$$

where $T_{\tilde g}$ denotes the Shale–Weil representation of $\mathrm{Mp}(2, \boldsymbol{R})$ and C is the Bargmann–Segal representation of the Heisenberg group $\mathscr{H}$, $\tilde g$ denoting an element of $\mathrm{Mp}(2, \boldsymbol{R})$ which projects onto the matrix g in $\mathrm{Sp}(2, \boldsymbol{R})$. (Let M be an arbitrary $\mathscr{H}$-module and $y \in T_{\tilde g} M$, i.e. $y = T_{\tilde g} x$, $x \in M$. Then

$$C(b) y = C(b) T_{\tilde g} x = T_{\tilde g} C(a) x,$$

i.e. (as $C(a)x \in M$) $C(b) y \in T_{\tilde g} M$ and $T_{\tilde g} M$ too is an $\mathscr{H}$-module.) More explicitly, (for $\alpha = 1$) let

$$T_{\tilde g} f(z) = \frac{B^{1/2}}{\pi} \int_C \exp\{\tfrac{1}{2}(Az^2 + 2Bz\bar w + C\bar w^2)\} f(w)\, d\lambda_1(w).$$

Then we find

$$\begin{aligned} T_{\tilde g} \beta(z_1, z_2) = \frac{B}{\pi^2} \iint_{C^2} & \exp\{\tfrac{1}{2}(Az_1^2 + 2Bz_1 \bar w_1 + C\bar w_1^2) \\ & + \tfrac{1}{2}(Az_2^2 + 2Bz_2 \bar w_2 + C\bar w_2^2)\}\, \beta(w_1, w_2)\, d\lambda_1(w_1)\, d\lambda_1(w_2) \end{aligned}$$

or, using the identities $\left(\frac{z_1 - z_2}{2}\right)^2 + \left(\frac{z_1 + z_2}{2}\right)^2 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2$ etc.,

$$\begin{aligned} &= \frac{B}{\pi^2} \int_C \exp\left\{A\left(\frac{z_1 + z_2}{2}\right)^2 + 2B\frac{z_1 + z_2}{2}\bar w + C\bar w^2\right\} h(w)\, d\lambda_2(w) \\ &\quad \times \int_C \exp\left\{A\left(\frac{z_1 - z_2}{2}\right)^2 + 2B\frac{z_1 - z_2}{2}\bar w + C\bar w^2\right\} b(w)\, d\lambda_2(w) \\ &= T_{\tilde g}^2 h \cdot T_{\tilde g}^2 b. \end{aligned}$$

A natural choice seems therefore to be ("Gauss–Weierstrass functions")

$$h = e_{ac} = e^{az^2/2+cz}$$

because this family is permuted by $\mathrm{Mp}(2, \boldsymbol{R})$ [Pee1]. One can take $a = 0$, i.e. one is led to consider generalized Hankel forms corresponding to

$$\beta(z_1, z_2) = e^{c(z_1-z_2)/2}\, b\left(\frac{z_1+z_2}{2}\right).$$

The case $c = 0$, apparently, is treated in [JPR], i.e.

$$\beta(z_1, z_2) = b\left(\frac{z_1+z_2}{2}\right).$$

Next (following the line of thought in [Pee2], [Pee3]) one can think of the case

$$h = P(z)e_{ac} = P(z)e^{az^2/2+cz}$$

where $P(z)$ is a polynomial. These functions also are permuted by $\mathrm{Mp}(2, \boldsymbol{R})$. In particular, it would be nice to have a closer look at the case

$$\beta(z_1, z_2) = (z_1-z_2)^k b\left(\frac{z_1+z_2}{2}\right)$$

mentioned already in [JP].

Remark (on the Gauss–Weierstrass functions). The functions e_{ac} satisfy an eigenvalue equation

$$\left(\frac{\partial}{\partial z} - az\right)e = ce.$$

In particular, they correspond in a natural way to positive isotropic subspaces of maximal dimension (= elements of Siegel's (generalized) upper halfplane). Thus they have a meaning independent of the special representation of the commutation relations used here (viz. the Bargmann–Segal representation).

Similarly, the functions $P \cdot e_{ac}$ may be viewed as "associated" functions, i.e. solutions of

$$\left[\left(\frac{\partial}{\partial z} - az\right) - c\right]^N e = 0.$$

2. Boundedness and smoothness properties of generalized Hankel forms (the case $1 \leqslant p \leqslant \infty$). So far all our considerations have been purely formal. Now we address ourselves to the issue of smoothness of generalized Hankel forms. We begin with the case $1 \leqslant p \leqslant \infty$. The case $0 < p < 1$ will be treated in the following section.

First we settle the question of Hilbert–Schmidt forms.

THEOREM 1. *Assume that $h \cdot b \not\equiv 0$. Then the form $\Gamma_{b,h}$ (if it can be defined in any natural way) is in S_2 (Hilbert–Schmidt class) iff $b, h \in F^{2\alpha,2}(\boldsymbol{C})$.*

Proof. This is easy, as quite generally

$$\|L_\beta\|^2_{\mathrm{H.S.}} = \iint_{C\times C} |\beta(z_1, z_2)|^2\, d\lambda_\alpha(z_1)\, d\lambda_\alpha(z_2).$$

Take now

$$\beta(z_1, z_2) = h\left(\frac{z_1-z_2}{2}\right) b\left(\frac{z_1+z_2}{2}\right).$$

Using the identity

$$2\left(\left|\frac{z_1-z_2}{2}\right|^2 + \left|\frac{z_1+z_2}{2}\right|^2\right) = |z_1|^2 + |z_2|^2, \tag{1}$$

we thus obtain

$$\|\Gamma_{b,h}\|^2_{\mathrm{H.S.}} = \left(\frac{\alpha}{\pi}\right)^2 \iint_{C^2} \left|h\left(\frac{z_1-z_2}{2}\right)\right|^2 e^{2\alpha|(z_1-z_2)/2|^2} \times \left|b\left(\frac{z_1+z_2}{2}\right)\right|^2 e^{2\alpha|(z_1+z_2)/2|^2}\, dE(z_1)\, dE(z_2).$$

As $dE(z_1)\, dE(z_2) = 4dE\left(\frac{z_1+z_2}{2}\right) dE\left(\frac{z_1-z_2}{2}\right)$ it follows that

$$\|\Gamma_{b,h}\|^2_{\mathrm{H.S.}} = \|h\|^2_{F^{2\alpha,2}(C)} \|b\|^2_{F^{2\alpha,2}(C)}. \ \blacksquare$$

Next we investigate when a generalized Hankel form is bounded.

THEOREM 2. (a) *Assume that $h \in F^{2\alpha,1}(\boldsymbol{C})$. Then $b \in F^{2\alpha,\infty}(\boldsymbol{C})$ implies that $\Gamma_{b,h}$ is bounded on $F^{\alpha,2}(\boldsymbol{C})$.*

(b) *Assume that $h \not\equiv 0$. Then $\Gamma_{b,h}$ bounded implies that $b \in F^{2\alpha,\infty}(\boldsymbol{C})$.*

Proof. (a) $b \in F^{2\alpha,\infty}(\boldsymbol{C})$ means that $|b(z)| \leqslant Ce^{\alpha|z|^2}$, and $h \in F^{2\alpha,1}(\boldsymbol{C})$ that $\int_{\boldsymbol{C}} |h(z)|\, d\lambda_\alpha(z) = D < \infty$. Using the former, and once more identity (1), we find

$$|\Gamma_{b,h}(f_1, f_2)| \leqslant C\left(\frac{\alpha}{\pi}\right)^2 \iint_{C^2} \left|h\left(\frac{z_1-z_2}{2}\right)\right| e^{\alpha|(z_1+z_2)/2|^2} \times |f_1(z_1)| e^{-\alpha|z_1|^2} |f_2(z_2)| e^{-\alpha|z_2|^2}\, dE(z_1)\, dE(z_2)$$

$$= C\left(\frac{\alpha}{\pi}\right)^2 \iint_{C^2} \left|h\left(\frac{z_1-z_2}{2}\right)\right| e^{-\alpha|(z_1-z_2)/2|^2} \times |f_1(z_1)| e^{-\alpha|z_1|^2/2} |f_2(z_2)| e^{-\alpha|z_2|^2/2}\, dE(z_1)\, dE(z_2).$$

The last expression is of the form $(u, \varphi * \psi)_{L^2(C)}$, where $*$ is convolution in C and $u \in L^1(C)$, $\varphi, \psi \in L^2(C)$, hence can be estimated by $\|u\|_1 \|\varphi\|_2 \|\psi\|_2$. (Here $\|\cdot\|_p$ serves to denote the norm in $L^p(C)$.) This shows that

$$|\Gamma_{b,h}(f_1, f_2)| \leqslant M \|f_1\|_{F^{\alpha,2}(C)} \|f_2\|_{F^{\alpha,2}(C)}, \tag{2}$$

with $M = 4CD$. ■

(b) Assuming that $\Gamma_{b,h}$ is bounded, i.e. that inequality (2) is fulfilled for some M, and also (without loss of generality) that $h(0) \neq 0$, we take in it $f_1(z) = e^{\alpha\bar{w}z}$, $f_2(z) = e^{\alpha\bar{w}z}$ (where $w \in C$ fixed). As the function $e^{\alpha\bar{w}z}$ is the reproducing kernel in $F^{\alpha,2}(C)$ we have

$$\Gamma_{b,h}(e^{\alpha\bar{w}\cdot}, e^{\alpha\bar{w}\cdot}) = \iint_{C^2} \overline{h\left(\frac{z_1-z_2}{2}\right) b\left(\frac{z_1+z_2}{2}\right)} \times e^{2\alpha\bar{w}(z_1+z_2)/2} d\lambda_\alpha(z_1) d\lambda_\alpha(z_2)$$

$$= \int_C \overline{h\left(\frac{z_1-z_2}{2}\right)} d\lambda_{2\alpha}\left(\frac{z_1-z_2}{2}\right) \times \int_C \overline{b\left(\frac{z_1+z_2}{2}\right)} e^{2\alpha\bar{w}(z_1+z_2)/2} d\lambda_{2\alpha}\left(\frac{z_1+z_2}{2}\right)$$

$$= \int_C \overline{h(z)}\, d\lambda_{2\alpha}(z) \cdot \overline{b(w)} = \overline{h(0)} \cdot \overline{b(w)}.$$

As moreover $\|e^{\alpha\bar{w}\cdot}\|_{F^{\alpha,2}(C)} = e^{\alpha|w|^2/2}$, it follows from (2) that

$$|b(w)| \leqslant M |h(0)|^{-1} e^{\alpha|w|^2},$$

i.e. b is in $F^{2\alpha,\infty}(C)$. ■

THEOREM 3. *Assume that* $h \in F^{2\alpha,1}(C)$. *Then* $b \in F^{2\alpha,1}(C)$ *implies* $\Gamma_{b,h} \in S_1$ (*trace class one*).

Proof. If $b \in F^{2\alpha,1}(C)$ we may write (see [JPR])

$$b(z) = \sum c_i e^{2\alpha\bar{w}_i z - \alpha|w_i|^2}, \quad \sum |c_i| < \infty.$$

Then

$$\Gamma_{b,h}(f_1, f_2) = \sum \bar{c}_i \iint_{C^2} \overline{h\left(\frac{z_1-z_2}{2}\right)} e^{2\alpha w_i \overline{(z_1+z_2)}/2} e^{-\alpha|w_i|^2} \times f_1(z_1) f_2(z_2) d\lambda_\alpha(z_1) d\lambda_\alpha(z_2)$$

$$= \sum \bar{c}_i \int_{C^2} \overline{h(\zeta)} f_1(w_i+\zeta) f_2(w_i-\zeta)\, d\lambda_{2\alpha}(\zeta) \cdot e^{-\alpha|w_i|^2}.$$

Each of the forms

$$f_1 \times f_2 \mapsto f_1(w_i+\zeta) f_2(w_i-\zeta)$$

has rank one and the S_1-norm equals

$$e^{\alpha|w_i+\zeta|^2/2} e^{\alpha|w_i-\zeta|^2/2}.$$

It follows that

$$\|\Gamma_{b,h}\|_{S_1} \leqslant \sum |c_i| \int_C |h(\zeta)| e^{\alpha|w_i+\zeta|^2/2} e^{\alpha|w_i-\zeta|^2/2} e^{-\alpha|w_i|^2} d\lambda_{2\alpha}(\zeta)$$

$$= \sum |c_i| \int_C |h(\zeta)| e^{\alpha|\zeta|^2} d\lambda_{2\alpha}(\zeta) < \infty. \ ■$$

By standard duality and interpolation arguments, going back to Peller's classical paper [Pel] (see also e.g. [JPR], we obtain from Theorem 1-3 (Theorem 3 with $p = 1$) the following final (for $p \geqslant 1$) result.

THEOREM 4. *Assume that* $h \in F^{2\alpha,1}(C)$ *and that* $h \not\equiv 0$. *Then* $\Gamma_{b,h} \in S_p$, *where* $1 \leqslant p \leqslant \infty$, *iff* $b \in F^{2\alpha,p}(C)$. ■

For ordinary Hankel forms ($h \equiv 1$) this result is in [JPR]. In the following section we extend this, thanks to results in [W], to the case $0 < p < 1$.

3. The case $0 < p < 1$. The purpose of this section is to establish the following theorem, where in the proof we follow the same general scheme as in Semmes [S].

THEOREM 5. *Let* $0 < p < 1$. *Assume that* $h \in F^{2\alpha,p}(C)$ *and that* $h \not\equiv 0$. *Then* $\Gamma_{b,h} \in S_p$ *iff* $b \in F^{2\alpha,p}(C)$.

Proof. For the direct proof write (see [W])

$$h(z) = \sum h_i e^{2\alpha\bar{w}_i z - \alpha|w_i|^2}, \quad \sum |h_i|^p < \infty,$$

$$b(z) = \sum b_i e^{2\alpha\bar{w}_i z - \alpha|w_i|^2}, \quad \sum |b_i|^p < \infty.$$

Then

$$\Gamma_{b,h}(f_1, f_2) = \iint_{C^2} \sum_{i,j} \bar{h}_i e^{2\alpha w_i \overline{(z_1-z_2)}/2 - \alpha|w_i|^2} \bar{b}_j e^{2\alpha w_j \overline{(z_1+z_2)}/2 - \alpha|w_j|^2} \times f_1(z_1) f_2(z_2) d\lambda_\alpha(z_1) d\lambda_\alpha(z_2)$$

$$= \sum_{i,j} \bar{h}_i \bar{b}_j \iint_{C^2} e^{\alpha\bar{z}_1(w_i+w_j) - \alpha|w_i+w_j|^2/2} e^{\alpha\bar{z}_2(w_j-w_i) - \alpha|w_j-w_i|^2/2} \times f_1(z_1) f_2(z_2) d\lambda_\alpha(z_1) d\lambda_\alpha(z_2)$$

$$= \sum_{i,j} \bar{h}_i \bar{b}_j f_1(w_i+w_j) e^{-\alpha|w_i+w_j|^2/2} f_2(w_j-w_i) e^{-\alpha|w_i-w_j|^2/2}.$$

It follows that (see the proof of Theorem 2(b))

$$\|\Gamma_{b,h}\|_{S_p} \leqslant C \|h\|_{F^{2\alpha,p}(C)} \|b\|_{F^{2\alpha,p}(C)}.$$

For the converse assume, to begin with, that $b \in F^{2\alpha,p}(C)$ and choose a 1-dense and separated set $\{w_i\} \subset C$ such that

$$\sum_i |b(w_i)e^{-\alpha|w_i|^2}|^p = c_p \|b\|^p_{F^{2\alpha,p}(C)}.$$

(As in the proof of Theorem 2 we may also assume that $h(0) \neq 0$.) Decompose $\{w_i\}$ into finitely many sequences $\{w_i^r\}$, $r = 1, \ldots, c = c(N)$, such that $|w_i - w_j| \geqslant N$, $i \neq j$. Let H^r be an abstract Hilbert space with basis $\{e_i^r\}$. By the decomposition theorem for $F^{\alpha,2}(C)$ (see [JPR]), the relation

$$S^r(\sum_i \lambda_i e_i^r) = \sum \lambda_i f_i^r, \qquad f_i^r(z) = e^{\alpha z \bar{w}_i^r - \alpha|w_i^r|^2/2},$$

defines a bounded map from H^r into $F^{\alpha,2}(C)$. Define a bilinear form B on $H = \bigoplus_{r=1}^c H^r$ by

$$B(f_1, f_2) = \sum_r \Gamma_{b,h}(S^r f_1, S^r f_2).$$

Then clearly

$$\|B\|_{S_p} \leqslant C_{p,N} \|\Gamma_{b,h}\|_{S_p}.$$

Let D be the diagonal part of B and set $F = B - D$. Then

$$\|D\|_{S_p} = \sum_{i,r} |\Gamma_{b,h}(f_i^r, f_i^r)|^p$$

$$= \sum_{i,r} \left| \iint_{C^2} \bar{h}\left(\frac{z_1 - z_2}{2}\right) \bar{b}\left(\frac{z_1 + z_2}{2}\right) f_i^r(z_1) f_i^r(z_2)\, d\lambda_\alpha(z_1)\, d\lambda_\alpha(z_2) \right|^p$$

$$= \sum_{i,r} \left| \iint_{C^2} \bar{h}\left(\frac{z_1 - z_2}{2}\right) \bar{b}\left(\frac{z_1 + z_2}{2}\right) e^{\alpha(z_1+z_2)\bar{w}_i^r - \alpha|w_i^r|^2}\, d\lambda_\alpha(z_1)\, d\lambda_\alpha(z_2) \right|^p$$

$$= \left|\int_C h(z)\, d\lambda_{2\alpha}(z)\right|^p \sum_{i,r} |b(w_i^r) e^{-\alpha|w_i^r|^2}|^p$$

$$= \left|\int_C h(z)\, d\lambda_{2\alpha}(z)\right|^p c_p \|b\|^p_{F^{2\alpha,p}(C)}.$$

If we can also prove that

$$\|F\|_{S_p} = o(\|b\|_{F^{2\alpha,p}(C)}) \qquad \text{as } N \to \infty,$$

we can thus conclude that

$$\|b\|_{F^{2\alpha,p}(C)} \leqslant C \|\Gamma_{b,h}\|_{S_p}.$$

Indeed, one has

$$\|F\|^p_{S_p} \leqslant \sum_{|w_i - w_j| \geqslant N} \left| \iint_{C^2} \bar{h}\left(\frac{z_1 - z_2}{2}\right) \bar{b}\left(\frac{z_1 + z_2}{2}\right) \times e^{\alpha(z_1 - z_2)(\bar{w}_i - \bar{w}_j)/2} e^{\alpha(z_1 + z_2)(\bar{w}_i + w_j)/2} \times e^{-\alpha|(w_i - w_j)/2|^2 - \alpha|(w_i + w_j)/2|^2}\, d\lambda_\alpha(z_1)\, d\lambda_\alpha(z_2) \right|^p$$

$$= \sum_{|w_i - w_j| \geqslant N} \left| h\left(\frac{w_i - w_j}{2}\right) e^{-\alpha|(w_i - w_j)/2|^2} b\left(\frac{w_i + w_j}{2}\right) e^{-\alpha|(w_i + w_j)/2|^2} \right|^p$$

$$= o(\|h\|^p_{F^{2\alpha,p}(C)} \|b\|^p_{F^{2\alpha,p}(C)}) \qquad \text{as } N \to \infty.$$

Finally, assuming only $\Gamma_{b,h} \in S_p$, let us put $b_\zeta(z) = b(\zeta z)$ for $|\zeta| \leqslant 1$. Then $\zeta \mapsto \Gamma_{b_\zeta,h}$ is a holomorphic S_p-valued function. By the maximum principle [K] we have

$$\|\Gamma_{b_\zeta,h}\|_{S_p} \leqslant \|\Gamma_{b,h}\|_{S_p}.$$

Since

$$\|b\|_{F^{2\alpha,p}(C)} = \lim_{\zeta \to 1} \|b_\zeta\|_{F^{2\alpha,p}(C)},$$

the theorem is proved. ∎

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Received May 16, 1988 (2440)