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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

Some restriction theorems for the Heisenberg group

by

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Abstract. We prove two restriction theorems for the Heisenberg group: one analogous to the Stein–Tomas restriction theorem for the Fourier transform on $\mathbf{R}^n$ and the other analogous to Zygmund's theorem for the Fourier transform on $\mathbf{R}^2$.

1. Introduction. On $\mathbf{R}^n$ we can write the Fourier inversion formula in polar coordinates as

$$f(x) = (2\pi)^{-n} \int_0^\infty \Big(\int_{S^{n-1}} e^{i\lambda x\cdot u} \hat{f}(\lambda u)\, d\sigma \Big) \lambda^{n-1}\, d\lambda. \tag{1.1}$$

Then the expression

$$Q_\lambda f(x) = (2\pi)^{-n} \lambda^{n-1} \int_{S^{n-1}} e^{i\lambda x\cdot u} \hat{f}(\lambda u)\, d\sigma \tag{1.2}$$

is an eigenfunction of the Laplacian with eigenvalue $-\lambda^2$. In terms of $Q_\lambda f$ we can write the inversion formula as

$$f(x) = \int_0^\infty Q_\lambda f(x)\, d\lambda. \tag{1.3}$$

The operators Q_λ can also be written as convolutions with Bessel functions, namely

$$Q_\lambda f(x) = (2\pi)^{-n} \lambda^{n-1} \int_{\mathbf{R}^n} f(x-y) (\lambda |y|)^{-n/2+1} J_{n/2-1}(\lambda |y|)\, dy. \tag{1.4}$$

For these operators it has been proved that for $1 \leqslant p \leqslant 2(n+1)/(n+3)$ one has

$$\|Q_\lambda f\|_{p'} \leqslant C_\lambda \|f\|_p. \tag{1.5}$$

As a consequence, one can obtain the Stein–Tomas restriction theorem for the Fourier transform (see [8]):

$$\Big(\int_{S^{n-1}} |\hat{f}(u)|^2\, d\sigma \Big)^{1/2} \leqslant C \|f\|_p, \quad 1 \leqslant p \leqslant \frac{2(n+1)}{n+3}. \tag{1.6}$$

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a measurable partition P we define $P(A) = \bigvee_{g\in A} \Phi^g P$. We put $P_\Phi = P(\mathbf{Z}^d)$, $P_\Phi^- = P(\mathbf{Z}^d_-)$ and $P_n = P(\mathbf{Z}^d_n)$, $1 \leqslant n \leqslant d$. Analogously we define $\mathscr{A}_\Phi$, $\mathscr{A}_\Phi^-$ and $\mathscr{A}_n$ where $\mathscr{A}$ is a σ-subalgebra of $\mathscr{B}$, $1 \leqslant n \leqslant d$.

By the *past σ-algebra* determined by a partition P we mean the σ-algebra $\mathscr{P} = P \vee P_\Phi^-$.

A σ-subalgebra $\mathscr{H}$ which is totally invariant under Φ is called a *factor* of Φ. The factor-action of Φ determined by a factor $\mathscr{H}$ is denoted by $\Phi/\mathscr{H}$. We denote by $\mathscr{N}$ the trivial factor of Φ.

Let $\mathscr{H}$ be a fixed factor of Φ. For a partition $P \in \mathscr{Z}$ we put $h(P, \Phi\,|\,\mathscr{H}) = H(P\,|\,P_\Phi^- \vee \mathscr{H})$. We define the *relative entropy* $h(\Phi\,|\,\mathscr{H})$ of Φ with respect to $\mathscr{H}$ by the formula

$$h(\Phi\,|\,\mathscr{H}) = \sup\{h(P, \Phi\,|\,\mathscr{H});\ P \in \mathscr{Z}\}$$

and the *relative Pinsker σ-algebra* $\pi(\Phi\,|\,\mathscr{H})$ of Φ with respect to $\mathscr{H}$ as the smallest σ-algebra containing all $P \in \mathscr{Z}$ with $h(P, \Phi\,|\,\mathscr{H}) = 0$.

Remark that $h(\Phi\,|\,\mathscr{N})$ and $\pi(\Phi\,|\,\mathscr{N})$ are equal to the entropy $h(\Phi)$ and the Pinsker σ-algebra $\pi(\Phi)$ of Φ respectively (cf. [1]).

If $\mathscr{A} \supset \mathscr{H}$ is a factor of Φ then we write

$$h(\Phi, \mathscr{A}\,|\,\mathscr{H}) \quad \text{and} \quad \pi(\Phi, \mathscr{A}\,|\,\mathscr{H})$$

instead of

$$h(\Phi/\mathscr{A}\,|\,\mathscr{H}) \quad \text{and} \quad \pi(\Phi/\mathscr{A}\,|\,\mathscr{H})$$

respectively. In particular, we define

$$h(\Phi, \mathscr{A}) = h(\Phi, \mathscr{A}\,|\,\mathscr{N}), \qquad \pi(\Phi, \mathscr{A}) = \pi(\Phi, \mathscr{A}\,|\,\mathscr{N}).$$

We put

$$\Gamma_{\mathscr{H}} = \{P \in \mathscr{Z};\ h(P, \Phi\,|\,\mathscr{H}) = h(\Phi\,|\,\mathscr{H}), \qquad \Gamma_\Phi = \Gamma_{\mathscr{N}}.$$

A $\mathbf{Z}^d$-action Φ is said to be a *relative K-action* with respect to $\mathscr{H}$ if $\pi(\Phi\,|\,\mathscr{H}) = \mathscr{H}$ and a *K-action* if the above equality holds for $\mathscr{H} = \mathscr{N}$.

The following definition is equivalent to that of a relatively perfect partition given in [7].

A σ-subalgebra $\mathscr{A} \subset \mathscr{B}$ is said to be *relatively perfect* with respect to $\mathscr{H}$ if

(i) $$\mathscr{H} \subset \mathscr{A}, \quad \Phi^g \mathscr{A} \subset \mathscr{A}, \quad g \in \mathbf{Z}^d_-,$$

(ii) for every Dedekind cut (A, B) of $(\mathbf{Z}^d, <)$ which is a gap

$$\bigvee_{g\in A} \Phi^g \mathscr{A} = \bigcap_{g\in B} \Phi^g \mathscr{A},$$

(iii) $$\mathscr{A}(\mathbf{Z}^d) = \mathscr{B},$$

(iv) $$\bigcap_{g\in \mathbf{Z}^d} \Phi^g \mathscr{A} = \pi(\Phi\,|\,\mathscr{H}),$$

(v) $$h(\Phi\,|\,\mathscr{H}) = H(\mathscr{A}\,|\,\mathscr{A}_\Phi^-).$$

Using the standard automorphisms $T_1, \ldots, T_d$ determined by Φ it is easy to check that the above conditions may be written in the following form:

(i′) $$\mathscr{H} \subset \mathscr{A}, \quad T_k^{-1} \mathscr{A}_k \subset \mathscr{A}, \quad 1 \leqslant k \leqslant d,$$

(ii′) $$\bigcap_{n=0}^{\infty} T_k^{-n} \mathscr{A}_k = T_{k-1}^{-1} \mathscr{A}_{k-1}, \quad 2 \leqslant k \leqslant d,$$

(iii′) $$\bigvee_{n=0}^{\infty} T_1^n \mathscr{A}_1 = \mathscr{B},$$

(iv′) $$\bigcap_{n=0}^{\infty} T_1^{-n} \mathscr{A}_1 = \pi(\Phi\,|\,\mathscr{H}), \quad \mathscr{A}_k = \mathscr{A}(\mathbf{Z}^d_k), \quad 1 \leqslant k \leqslant d,$$

(v′) $$h(\Phi\,|\,\mathscr{H}) = H(\mathscr{A}\,|\,T_d^{-1} \mathscr{A}).$$

The existence of relatively perfect σ-algebras is announced in [6] and proved in [7]. It is clear that a σ-algebra relatively perfect with respect to the trivial σ-algebra $\mathscr{N}$ is perfect ([4]).

It has been shown in [4] that perfect σ-algebras are useful tools for the investigation of mixing and spectral properties of $\mathbf{Z}^d$-actions. On the other hand, the existence of relatively perfect σ-algebras allows one to obtain a functorial characterization of entropy (cf. [7]).

The object of our considerations are generators of perfect σ-algebras, i.e. generators $P \in \mathscr{Z}$ of Φ such that a given perfect σ-algebra $\mathscr{A}$ is the past σ-algebra determined by P.

It is known (cf. [11]) that if Φ is a $\mathbf{Z}^1$-action with finite entropy then any past σ-algebra of Φ determined by a generator of Φ is perfect and vice versa, any perfect σ-algebra is the past σ-algebra determined by some generator of Φ.

It is shown in [9] that, in contrast to the case $d = 1$, there are $\mathbf{Z}^2$-actions Φ with past σ-algebras, determined by generators of Φ, which are not perfect. This fact has stimulated the investigation of the set of those generators of $\mathbf{Z}^d$-actions, $d \geqslant 2$ (called regular in [9]) whose past σ-algebras are perfect.

For this purpose relative ergodic theory has been a useful tool. There are well known interesting applications of this theory to the $\mathbf{Z}^1$-dynamics (see for instance [10], [12], [13]).

We have proved in [5] the existence of countable regular generators with finite entropy of $\mathbf{Z}^2$-actions by a characterization of these generators in terms of the Thouvenot relative K-automorphisms and a relativized Rokhlin generator theorem.

In this paper we sharpen this result. Suppose Φ is a strongly ergodic $\mathbf{Z}^d$-action with $h(\Phi) < \infty$. The main results of this paper are the following.

Applying a relativized generator theorem (Proposition 2) we show that for every finite generator P of Φ there exists a finite regular generator Q which is a refinement of P and that the set of all finite regular generators is dense in Γ_Φ.

Next we prove that in the case $h(\Phi) > 0$, by the use of a relativized Sinai theorem (Theorem A), we may replace regular generators by nonregular ones in the above result. We also show (Proposition 3) that the second result cannot be extended to actions with zero entropy.

2. Results of relative ergodic theory. Let Φ be a fixed $\mathbf{Z}^d$-action on a Lebesgue probability space $(X, \mathscr{B}, \mu)$ and let $\mathscr{H}$ be a fixed factor of Φ.

PROPOSITION 1. *If $\mathscr{A}, \mathscr{C} \subset \mathscr{B}$ are factors of Φ with $\mathscr{C} \subset \mathscr{H} \subset \mathscr{A}$ then*

$$h(\Phi, \mathscr{A} \mid \mathscr{C}) = h(\Phi, \mathscr{A} \mid \mathscr{H}) + h(\Phi, \mathscr{H} \mid \mathscr{C}).$$

Proof. Let $P_k, Q_l \in \mathscr{Z}$, $k, l \geq 1$, be such that $P_k \nearrow \mathscr{H}$ and $Q_l \nearrow \mathscr{A}$. Using a simple generalization of the Pinsker formula ([1]) we have

$$h(P_k \vee Q_l, \Phi \mid \mathscr{C}) = h(P_k, \Phi \mid \mathscr{C}) + H(Q_l \mid Q_l^- \vee (P_k)_\Phi \vee \mathscr{C}),$$

$k \geq 1$. Hence we get

$$h(P_k \vee Q_l, \Phi \mid \mathscr{C}) \geq h(P_k, \Phi \mid \mathscr{C}) + h(Q_l, \Phi \mid \mathscr{H}), \tag{1}$$

$$h(Q_l, \Phi \mid \mathscr{C}) \leq h(P_k, \Phi \mid \mathscr{C}) + H(Q_l \mid Q_l^- \vee (P_k)_\Phi \vee \mathscr{C}), \tag{2}$$

$k, l \geq 1$. Suppose $h(\Phi, \mathscr{A} \mid \mathscr{C}) < \infty$. First taking in (1) the limit as $l \to \infty$ and next as $k \to \infty$ we get

$$h(\Phi, \mathscr{A} \mid \mathscr{C}) \geq h(\Phi, \mathscr{A} \mid \mathscr{H}) + h(\Phi, \mathscr{H} \mid \mathscr{C}). \tag{3}$$

It is clear that (3) is valid also in the case when $h(\Phi, \mathscr{A} \mid \mathscr{C}) = \infty$. The converse inequality can be deduced from (2) in a similar way.

Now assume $h(\Phi, \mathscr{H}) < \infty$. The following corollary shows that in this case our definition of a relative K-action coincides with that of Thouvenot ([14]).

COROLLARY 1. *Φ is a K-action relative to $\mathscr{H}$ iff for every factor $\mathscr{A}$ of Φ with $\mathscr{A} \supset \mathscr{H}$ and $h(\Phi, \mathscr{A}) = h(\Phi, \mathscr{H})$ we have $\mathscr{A} = \mathscr{H}$.*

Proof. $\Rightarrow$ Substituting, in Proposition 1, $\mathscr{C} = \mathscr{N}$ we obtain $h(\Phi, \mathscr{A} \mid \mathscr{H}) = 0$. Hence, by our assumption, $\mathscr{A} \subset \pi(\Phi \mid \mathscr{H}) = \mathscr{H}$ and so $\mathscr{A} = \mathscr{H}$.

$\Leftarrow$ Note that $h(\Phi, \pi(\Phi \mid \mathscr{H}) \mid \mathscr{H}) = 0$. Substituting, in Proposition 1, $\mathscr{A} = \pi(\Phi \mid \mathscr{H})$ and $\mathscr{C} = \mathscr{N}$ we get

$$h(\Phi, \pi(\Phi \mid \mathscr{H})) = h(\Phi, \mathscr{H}).$$

Hence, by our assumption, $\pi(\Phi \mid \mathscr{H}) = \mathscr{H}$, i.e. Φ is a relative K-action with respect to $\mathscr{H}$.

The following two results are shown in the case of $\mathbf{Z}^1$-actions in [13] and [2] respectively. Their proofs for arbitrary $\mathbf{Z}^d$-actions are similar.

Let I be a finite probability vector and $H(I)$ the entropy of I. For a finite ordered partition $P = (P_1, \ldots, P_m)$ of X we denote by $\operatorname{dist} P$ the probability vector

$$\operatorname{dist} P = (\mu(P_1), \ldots, \mu(P_m)).$$

THEOREM A. *If Φ is ergodic with $h(\Phi \mid \mathscr{H}) > 0$ and I is a probability vector such that $H(I) \leq h(\Phi \mid \mathscr{H})$ then there exists a finite partition P of X such that $\operatorname{dist} P = I$, the partitions $\Phi^g P$, $g \in \mathbf{Z}^d$, are independent and the factors P_Φ and $\mathscr{H}$ are independent.*

A measurable partition P of X is said to be an *$\mathscr{H}$-relative generator* of Φ if $P_\Phi \vee \mathscr{H} = \mathscr{B}$. An $\mathscr{N}$-relative generator of Φ is said to be a *generator*.

THEOREM B. *If Φ is ergodic with $h(\Phi) < \infty$ then the set of finite generators of Φ is dense in Γ_Φ.*

The existence of a finite generator of Φ is shown for instance in [13] (see also [12]).

PROPOSITION 2. *If Φ is ergodic and $h(\Phi \mid \mathscr{H}) < \infty$ then there exists a finite $\mathscr{H}$-relative generator of Φ. The set of all finite $\mathscr{H}$-relative generators is dense in $\Gamma_{\mathscr{H}}$.*

Proof. It follows from [8] that there exists an $\mathscr{H}$-relative generator $Q \in \mathscr{Z}$ and that the set of all such generators is dense in $\Gamma_{\mathscr{H}}$. Applying Theorem B to the factor Q_Φ we get the result.

Remark. The first part of Proposition 2 is also shown in [12].

3. Main results. Let Φ be a $\mathbf{Z}^d$-action on $(X, \mathscr{B}, \mu)$ with $h(\Phi) < \infty$, let $\mathscr{H}$ be a fixed factor of Φ and let $T_1, \ldots, T_d$ be the standard automorphisms determined by Φ. For a given partition $P \in \mathscr{Z}$ we denote by $\mathscr{P}$ the past σ-algebra determined by P.

LEMMA 1. *For every partition $P \in \mathscr{Z}$*

$$\bigcap_{n=0}^{\infty} (T_1^{-n} \mathscr{P}_1 \vee \mathscr{H}) \subset \pi(\Phi, P_\Phi \vee \mathscr{H} \mid \mathscr{H}).$$

The above lemma is a consequence of the following property which may be shown similarly to property 8 in [1].

Remark 1. For any partitions $P, Q \in \mathscr{Z}$

$$\lim_{n \to \infty} H(P \mid P_\Phi^- \vee T_1^{-n}(Q_1)_{T_1}^- \vee \mathscr{H}) = h(P, \Phi \mid \mathscr{H}).$$

LEMMA 2 (cf. [7]). *If $\mathscr{A} \supset \mathscr{H}$ is a σ-algebra satisfying the properties* (i′)–(iii′) *of relatively perfect σ-algebras then $\bigcap_{n=0}^{\infty} T_1^{-n} \mathscr{A}_1 \supset \pi(\Phi \mid \mathscr{H})$.*

Now suppose Φ is aperiodic. Since $h(\Phi \mid \mathscr{H}) < \infty$ there exists (cf. [8]) an $\mathscr{H}$-relative generator of Φ belonging to $\mathscr{Z}$.

LEMMA 3. *For every $\mathscr{H}$-relative generator P of Φ*

$$h(\Phi \mid \mathscr{H}) = h(\Phi_1, \mathscr{P}_1 \vee \mathscr{H} \mid T_1^{-1} \mathscr{P}_1 \vee \mathscr{H}).$$

Proof. Let $Q \in \mathscr{Z}$ and $Q \subset \mathscr{P}_1 \vee \mathscr{H}$. Hence $(Q_1)_{T_1}^- \vee \mathscr{H} \subset T_1^{-1}\mathscr{P}_1 \vee \mathscr{H}$ and so

$$\begin{aligned} h(Q, \Phi_1 \mid T_1^{-1}\mathscr{P}_1 \vee \mathscr{H}) &= H(Q \mid Q_{\Phi_1}^- \vee T_1^{-1}\mathscr{P}_1 \vee \mathscr{H}) \\ &\leqslant H(Q \mid Q_{\Phi_1}^- \vee (Q_1)_{T_1}^- \vee \mathscr{H}) \\ &= h(Q, \Phi \mid \mathscr{H}) \leqslant h(\Phi \mid \mathscr{H}). \end{aligned}$$

i.e.

$$h(\Phi_1, \mathscr{P}_1 \vee \mathscr{H} \mid T_1^{-1}\mathscr{P}_1 \vee \mathscr{H}) \leqslant h(\Phi \mid \mathscr{H}).$$

Using the generalized Kolmogorov–Sinai theorem (cf. [8]) we get

$$\begin{aligned} h(\Phi \mid \mathscr{H}) = h(P, \Phi \mid \mathscr{H}) &= H(P \mid P_\Phi^- \vee \mathscr{H}) = H(P \mid P_{\Phi_1}^- \vee T_1^{-1}\mathscr{P}_1 \vee \mathscr{H}) \\ &\leqslant h(\Phi_1, \mathscr{P}_1 \vee \mathscr{H} \mid T_1^{-1}\mathscr{P}_1 \vee \mathscr{H}), \end{aligned}$$

which completes the proof.

Remark 2. Using Proposition 1 it is possible to get the following formula which is a generalization of that given in Lemma 3:

$$h(\Phi_1, T_1^n\mathscr{P}_1 \vee \mathscr{H} \mid T_1^{-1}\mathscr{P}_1 \vee \mathscr{H}) = nh(\Phi \mid \mathscr{H}), \quad n \geqslant 1.$$

Let P be an $\mathscr{H}$-relative generator of Φ. P is said to be *$\mathscr{H}$-regular* if the σ-algebra $\mathscr{P} \vee \mathscr{H}$ is relatively perfect with respect to $\mathscr{H}$. A generator P of Φ which is $\mathscr{N}$-regular is said to be *regular*.

It follows from Lemmas 1 and 2 and the generalized Kolmogorov–Sinai theorem that in the case $d = 1$ every $\mathscr{H}$-relative generator is $\mathscr{H}$-regular.

For $d \geqslant 2$ we have the following

LEMMA 4. *An $\mathscr{H}$-relative generator P of Φ is $\mathscr{H}$-regular iff for every $1 \leqslant k \leqslant d-1$ the factor-action $\Phi_k/\mathscr{P}_k \vee \mathscr{H}$ is a relative K-action with respect to the σ-algebra $T_k^{-1}\mathscr{P}_k \vee \mathscr{H}$.*

Proof. First of all observe that due to Lemmas 1 and 2 the $\mathscr{H}$-regularity of P is equivalent to the fact that the σ-algebra $\mathscr{P} \vee \mathscr{H}$ satisfies (ii′), i.e.

$$\bigcap_{n=0}^{\infty} (T_l^{-n}\mathscr{P}_l \vee \mathscr{H}) = T_{l-1}^{-1}\mathscr{P}_{l-1} \vee \mathscr{H}, \quad 2 \leqslant l \leqslant d. \tag{4}$$

Suppose P is $\mathscr{H}$-regular and $2 \leqslant k \leqslant d$. Using the equalities (4) for $k+1 \leqslant l \leqslant d$ and the equality

$$\bigvee_{n=0}^{\infty} (T_k^n\mathscr{P}_k \vee \mathscr{H}) = \mathscr{P}_{k-1} \vee \mathscr{H}$$

we see that $\mathscr{P} \vee \mathscr{H}$ satisfies the properties (i′)–(iii′) of relatively perfect σ-algebras of Φ_{k-1} acting on the space $(X, \mathscr{P}_{k-1} \vee \mathscr{H}, \mu)$ with respect to the σ-algebra $T_{k-1}^{-1}\mathscr{P}_{k-1} \vee \mathscr{H}$. Therefore applying Lemma 2 we get

$$\bigcap_{n=0}^{\infty} (T_k^{-n}\mathscr{P}_k \vee \mathscr{H}) \supset \pi(\Phi_{k-1}, \mathscr{P}_{k-1} \vee \mathscr{H} \mid T_{k-1}^{-1}\mathscr{P}_{k-1} \vee \mathscr{H}).$$

Substituting in (4) $l = k$ we obtain

$$\pi(\Phi_{k-1}, \mathscr{P}_{k-1} \vee \mathscr{H} \mid T_{k-1}^{-1}\mathscr{P}_{k-1} \vee \mathscr{H}) = T_{k-1}^{-1}\mathscr{P}_{k-1} \vee \mathscr{H},$$

which proves the necessity.

The converse implication follows from the obvious inclusion $\bigcap_{n=0}^{\infty} T_l^{-n}\mathscr{P}_l \supset T_{l-1}^{-1}\mathscr{P}_{l-1}$, $2 \leqslant l \leqslant d$, and the inclusion

$$\bigcap_{n=0}^{\infty} (T_l^{-n}\mathscr{P} \vee \mathscr{H}) \subset \pi(\Phi_{l-1}, \mathscr{P}_{l-1} \vee \mathscr{H} \mid T_{l-1}^{-1}\mathscr{P}_{l-1} \vee \mathscr{H}),$$

which is a consequence of Lemma 1 applied to the action Φ_{l-1} and the σ-algebra $T_{l-1}^{-1}\mathscr{P}_{l-1} \vee \mathscr{H}$, $2 \leqslant l \leqslant d$.

COROLLARY 2. *If P is a regular generator of Φ then the action Φ_1 is a relative K-action with respect to the factor $T_1^{-1}\mathscr{P}_1$.*

Proof. First observe that for every natural number k we have $\pi(\Phi_1, T_1^k\mathscr{P}_1 \mid T_1^{-1}\mathscr{P}_1) = T_1^{-1}\mathscr{P}_1$. Indeed, it follows from Lemma 4 with $\mathscr{H} = \mathscr{N}$ that this equality is true for $k = 0$. If it is true for some $k \geqslant 0$ then it is also true for $k+1$ by Lemma 2 applied to the probability space $(X, T_1^{k+1}\mathscr{P}_1, \mu)$, the action Φ_1, $\mathscr{A} = T_1^{k+1}\mathscr{P}$ and $\mathscr{H} = T_1^{-1}\mathscr{P}_1$. Since P is a generator we have $T_1^k\mathscr{P}_1 \nearrow \mathscr{B}$. Thus the last equality, by the property (h) of [5], implies $\pi(\Phi_1 \mid T_1^{-1}\mathscr{P}_1) = T_1^{-1}\mathscr{P}_1$, which completes the proof.

THEOREM 1. *For any positive integer d, strongly ergodic $\mathbf{Z}^d$-action Φ, factor $\mathscr{H}$ of Φ with $h(\Phi \mid \mathscr{H}) < \infty$, finite $\mathscr{H}$-relative generator P of Φ and $\varepsilon > 0$ there exists a finite $\mathscr{H}$-relative generator $Q \geqslant P$ which is $\mathscr{H}$-regular and $\varrho(P, Q) < \varepsilon$.*

Proof. Since for $d = 1$ every $\mathscr{H}$-relative generator is $\mathscr{H}$-regular our theorem is trivially satisfied in this case.

Now, suppose that our theorem is true for $d-1$. Let Φ, $\mathscr{H}$, P, ε be as in the assumptions of the theorem. Since

$$h(\Phi_1, \pi(\Phi_1 \mid T_1^{-1}\mathscr{P}_1 \vee \mathscr{H}) \mid T_1^{-1}\mathscr{P}_1 \vee \mathscr{H}) = 0$$

the strong ergodicity of Φ and Proposition 2 imply there exists a finite partition $\tilde{Q}$ with $\varrho(\tilde{Q}, T_1^{-1}P) < \varepsilon/2$ and

$$T_1^{-1}\mathscr{P}_1 \vee \mathscr{H} \vee \tilde{Q}_1 = \pi(\Phi_1 \mid T_1^{-1}\mathscr{P}_1 \vee \mathscr{H}).$$

Putting $\bar{Q} = P \vee T_1\tilde{Q}$ and $\bar{\mathscr{H}} = (\bar{Q}_1)_{T_1}^- \vee \mathscr{H}$ we get

$$\bar{\mathscr{H}} = \pi(\Phi_1 \mid T_1^{-1}\mathscr{P}_1 \vee \mathscr{H}), \tag{5}$$

$$\varrho(P, \bar{Q}) \leqslant \varrho(\tilde{Q}, T_1^{-1}P) < \varepsilon/2. \tag{6}$$

From (5) and [5] (the property (e)) we obtain

$$\pi(\Phi_1 \mid \bar{\mathscr{H}}) = \bar{\mathscr{H}}. \tag{7}$$

It follows from Lemma 3 that

$$h(\Phi_1, \bar{Q}_1 \vee \mathscr{H} \mid \mathscr{H}) = h(\Phi \mid \mathscr{H}) < \infty.$$

Applying the induction assumption to the factor-action $\Phi_1/\bar{Q}_1 \vee \mathscr{H}$ and to the σ-algebra $\mathscr{H}$ we get a finite partition $Q \geqslant \bar{Q}$ such that

(8) $$Q_1 \vee \mathscr{H} = \bar{Q}_1 \vee \mathscr{H},$$

(9) the factor-action $\Phi_k/T_k(\bigvee_{l=2}^{k}(Q_l)_{T_l}^{-} \vee \mathscr{H})$ is a relative K-action with respect to the σ-algebra $\bigvee_{l=2}^{k}(Q_l)_{T_l}^{-} \vee \mathscr{H}$, $2 \leqslant k \leqslant d$,

(10) $$\varrho(Q, \bar{Q}) < \varepsilon/2.$$

We shall show that the partition Q satisfies the desired properties.

It is clear that Q is an $\mathscr{H}$-relative generator. Since Q is a refinement of $\bar{Q}$ and $Q \subset \bar{Q}_1 \vee \mathscr{H}$ we have $\mathscr{H} = (Q_1)_{T_1}^{-} \vee \mathscr{H}$. Therefore it follows from (9) that the factor-action $\Phi_k/T_k(\bigvee_{l=1}^{k}(Q_l)_{T_l}^{-}) \vee \mathscr{H}$ is a relative K-action with respect to the σ-algebra $\bigvee_{l=1}^{k}(Q_l)_{T_l}^{-} \vee \mathscr{H}$, $2 \leqslant k \leqslant d$, and (7) implies that this property also holds for $k = 1$. This means, by Lemma 4, that Q is $\mathscr{H}$-regular. It follows from (6) and (10) that $\varrho(P, Q) < \varepsilon$.

Combining the second part of Proposition 2 with Theorem 1 we get at once the following

COROLLARY 3. *The set of all finite $\mathscr{H}$-relative generators of Φ which are $\mathscr{H}$-regular is dense in $\Gamma_{\mathscr{H}}$.*

Substituting $\mathscr{H} = \mathscr{N}$ in Corollary 3 we obtain the first result stated in the abstract.

THEOREM 2. *For any integer $d \geqslant 2$, strongly ergodic $\mathbf{Z}^d$-action Φ such that $0 < h(\Phi) < \infty$, and finite generator P of Φ there exists a nonregular finite generator Q of Φ which is a refinement of P. The set of all finite nonregular generators is dense in Γ_{Φ}.*

Proof. Let $0 < \varepsilon < h(\Phi)$ be arbitrary and let I be a probability vector such that $H(I) = \varepsilon$. It follows from our assumptions and Lemma 3 with $\mathscr{H} = \mathscr{N}$ that the action Φ_1 is ergodic and $h(\Phi_1, \mathscr{P}_1 \mid T_1^{-1}\mathscr{P}_1) = h(\Phi) > 0$. Applying Theorem A to the factor-action $\Phi_1/\mathscr{P}_1$, the σ-algebra $\mathscr{H} = T_1^{-1}\mathscr{P}_1$ and the probability vector I we get a finite partition $\tilde{Q} \subset \mathscr{P}_1$ which generates a Bernoulli factor $\tilde{Q}_1 = \tilde{Q}(\mathbf{Z}_1^d)$ such that $\operatorname{dist}\tilde{Q} = I$ and the factors $\tilde{Q}_1$ and $T_1^{-1}\mathscr{P}_1$ are independent.

It is shown in [3] that Bernoulli shifts are not coalescent. It is not difficult to extend this result to arbitrary Bernoulli $\mathbf{Z}^d$-actions, $d \geqslant 2$. From this it follows that there exists a finite partition $\bar{Q} \subset \tilde{Q}_1$ which generates a Bernoulli factor $\bar{Q}_1$ such that $\bar{Q}_1 \neq \tilde{Q}_1$ and

(11) $$h(\bar{Q}, \Phi_1) = h(\tilde{Q}, \Phi_1).$$

We define

$$\mathscr{H} = \bar{Q}_1 \vee T_1^{-1}\mathscr{P}_1, \quad \tilde{\mathscr{H}} = \tilde{Q}_1 \vee T_1^{-1}\mathscr{P}_1.$$

It is clear that

(12) $$\mathscr{H} \neq \tilde{\mathscr{H}}.$$

Now we will check that

(13) $$\pi(\Phi_1, \tilde{\mathscr{H}} \mid \mathscr{H}) = \tilde{\mathscr{H}}.$$

By the Pinsker formula we have

(14) $$h(\tilde{Q} \vee \bar{Q}, \Phi_1 \mid T_1^{-1}\mathscr{P}_1) = h(\bar{Q}, \Phi_1 \mid T_1^{-1}\mathscr{P}_1) + h(\tilde{Q}, \Phi_1 \mid \mathscr{H}).$$

It follows from (11) and the definition of $\tilde{Q}$ and $\bar{Q}$ that

$$h(\tilde{Q} \vee \bar{Q}, \Phi_1 \mid T_1^{-1}\mathscr{P}_1) = h(\tilde{Q}, \Phi_1 \mid T_1^{-1}\mathscr{P}_1) = h(\bar{Q}, \Phi_1 \mid T_1^{-1}\mathscr{P}_1).$$

Therefore (14) gives $h(\tilde{Q}, \Phi_1 \mid \mathscr{H}) = 0$, i.e. $\tilde{Q} \subset \pi(\Phi_1, \tilde{\mathscr{H}} \mid \mathscr{H})$. Hence $\tilde{\mathscr{H}} \subset \pi(\Phi_1, \tilde{\mathscr{H}} \mid \mathscr{H})$ and so (13) is fulfilled. Combining (12) with (13) we get

(15) $$\pi(\Phi_1 \mid \mathscr{H}) \neq \mathscr{H}.$$

Let $Q = P \vee T_1\bar{Q}$. It is clear that Q is a generator of Φ which refines P. We also have

$$\varrho(P, Q) = H(Q \mid P) \leqslant H(\bar{Q}) = H(\tilde{Q}) = \varepsilon.$$

By (15), Corollary 2 and the equality $\mathscr{H} = (Q_1)_{T_1}^{-}$ the generator Q is nonregular, which proves our theorem.

Remark. By a slight modification of the arguments used in the proof of Theorem 2 one can prove the relative version of this theorem, similar to Theorem 1.

Now we will show that Theorem 2 cannot be extended to actions with zero entropy.

PROPOSITION 3. *There exist $\mathbf{Z}^d$-actions with zero entropy any generator of which is regular.*

Proof. Let $(X, \mathscr{B}, \mu)$ be a nonatomic Lebesgue space and let $T_1, \ldots, T_d$ be commuting and algebraically independent automorphisms of X such that $h(T_1) = 0$ and T_d is ergodic. We define the action Φ on $(X, \mathscr{B}, \mu)$ by the formula

$$\Phi^g = T_1^{n_1} \ldots T_d^{n_d}, \quad g = (n_1, \ldots, n_d) \in \mathbf{Z}^d.$$

It is known (cf. [1]) that $h(\Phi) = 0$. Let P be an arbitrary generator of Φ and let $\mathscr{P}$ be the past σ-algebra determined by P. Since $T_1^{-1}\mathscr{P}_1 \subset \mathscr{P}_1$ and $h(T_1) = 0$ we have $T_1^{-1}\mathscr{P}_1 = \mathscr{P}_1$. Therefore $\mathscr{P}_1 = \mathscr{B}$ and so

$$\pi(\Phi_k, \mathscr{P}_k \mid T_k^{-1}\mathscr{P}_k) = T_k^{-1}\mathscr{P}_k, \quad 1 \leqslant k \leqslant d,$$

i.e. P is regular.

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Generators of perfect σ-algebras of $\mathbf{Z}^d$-actions

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Abstract. Let Φ be a $\mathbf{Z}^d$-action, $d \geqslant 2$, with finite entropy $h(\Phi)$, on a Lebesgue space $(X, \mathscr{B}, \mu)$ and let Γ_Φ be the set of all countable measurable partitions P of X with finite entropy such that the mean entropy $h(P, \Phi)$ equals $h(\Phi)$. It is shown that if Φ is strongly ergodic then the set of all finite partitions of X which generate perfect σ-algebras of Φ is dense in Γ_Φ. If $h(\Phi) > 0$ then it is also a boundary set in Γ_Φ.

1. Introduction and notations. Let $(X, \mathscr{B}, \mu)$ be a Lebesgue probability space and $\mathscr{Z}$ the set of all countable measurable partitions of X with finite entropy. We consider in $\mathscr{Z}$ the Rokhlin metric

$$\varrho(P, Q) = H(P \mid Q) + H(Q \mid P), \qquad P, Q \in \mathscr{Z}.$$

Let $\mathbf{Z}^d$ denote the group of d-dimensional integers and $<$ the lexicographical ordering in $\mathbf{Z}^d$ for $d \geqslant 2$ and the natural ordering for $d = 1$. Let $e^i \in \mathbf{Z}^d$ be the ith unit coordinate vector. We put

$$\mathbf{Z}^d_n = \{g = (m_1, \ldots, m_d) \in \mathbf{Z}^d;\ m_1 = \ldots = m_n = 0\}, \qquad 1 \leqslant n \leqslant d,$$
$$\mathbf{Z}^d_- = \{g \in \mathbf{Z}^d;\ g < (0, \ldots, 0)\}.$$

Let Φ be a $\mathbf{Z}^d$-*action* on $(X, \mathscr{B}, \mu)$, i.e. Φ is an isomorphism of the group $\mathbf{Z}^d$ into the group of all measure-preserving automorphisms of $(X, \mathscr{B}, \mu)$.

The restriction of Φ to $\mathbf{Z}^d_n$ is denoted by Φ_n. We denote by $T_1, \ldots, T_d$ the generators of the group $\Phi(\mathbf{Z}^d)$ which are the images by Φ of the vectors $e^1, \ldots, e^d$ respectively. We call them the *standard automorphisms* determined by Φ.

A $\mathbf{Z}^d$-action Φ is said to be *aperiodic* if

$$\mu(\{x \in X;\ \Phi^g x = x\}) = 0 \quad \text{for every } g \in \mathbf{Z}^d \setminus \{(0, \ldots, 0)\}.$$

Φ is said to be *ergodic* if for any Φ^g-invariant set $A \in \mathscr{B}$ and $g \in \mathbf{Z}^d$ either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. We say that Φ is *strongly ergodic* if the automorphism T_d is ergodic. It is clear that every strongly ergodic action is ergodic.

Let $\mathscr{A}_i$, $i \in I$, be a family of measurable subsets of X. The smallest σ-algebra containing all $\mathscr{A}_i$, $i \in I$, is denoted by $\bigvee_{i \in I} \mathscr{A}_i$. For a given set $A \subset \mathbf{Z}^d$ and

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