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Homogeneous Besov spaces on locally compact Vilenkin groups

by

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Abstract. In this paper we shall show the equivalence of various characterizations of the homogeneous Besov spaces defined on certain topological groups G that are the locally compact analogue of the compact groups introduced by Vilenkin in 1947. We then apply some of the results to study the regular extension to $G \times \boldsymbol{Z}$ of the distributions belonging to such Besov spaces.

1. Introduction. For $\alpha > 0$ and $0 < p, q \leqslant \infty$ there exist a large number of equivalent characterizations of the Besov or generalized Lipschitz spaces B_{pq}^{α} on $\boldsymbol{R}^n$. For early results, subject to the restrictions $\alpha > 0$ and $1 \leqslant p, q \leqslant \infty$, see the papers by Besov [2] and Taibleson [13]–[15]. For additional results, see [11] or [20], whereas for the atomic decomposition of Besov spaces on $\boldsymbol{R}^n$, see [6]. In [12] Ricci and Taibleson considered the harmonic extension to the upper half-plane $\boldsymbol{R}_+^2$ of functions belonging to certain Besov spaces on $\boldsymbol{R}$. They introduced a class of function spaces, called A_{pq}^{α}, on $\boldsymbol{R}_+^2$ and showed that the boundary values of the functions in A_{pq}^{α} can be identified as linear functionals on certain Besov spaces. In [3] Bui extended their results to $\boldsymbol{R}^n$. These papers were the motivation for the present paper in which we consider this circle of ideas in the context of a certain class of topological groups instead of $\boldsymbol{R}$ or $\boldsymbol{R}^n$.

We now summarize the content of this paper. In the remainder of this section we describe the topological groups G that will be considered here and we give a brief outline of the distribution theory on these groups. In Section 2 we introduce the inhomogeneous and homogeneous Besov spaces on G. We present several equivalent (quasi-) norms for these spaces and state a duality theorem. In that section we also compare the inhomogeneous and the

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homogeneous Besov spaces with each other. In Section 3 we present a characterization for the homogeneous Besov spaces on G in terms of mean oscillation spaces, whereas in Section 4 we treat the atomic decomposition of the Besov spaces. Finally, in Section 5 we treat the regularization of the distributions belonging to certain homogeneous Besov spaces. We introduce a new class of function spaces and we show that, under suitable restrictions on the parameters involved, their elements can be identified with the elements of the Besov spaces on G.

In this paper G will denote a locally compact Abelian topological group containing a strictly decreasing sequence of open compact subgroups $(G_n)_{-\infty}^{\infty}$ such that

(i) $\bigcup_{n=-\infty}^{\infty} G_n = G$ and $\bigcap_{n=-\infty}^{\infty} G_n = \{0\}$,

(ii) $M := \sup\{\operatorname{order}(G_n/G_{n+1}) \colon n \in \mathbf{Z}\} < \infty$.

Such groups are totally disconnected and they are the locally compact analogue of the groups described by Vilenkin in [21]. Several examples of such groups are given in [5, § 4.1.2]. Additional examples are the p-adic numbers and, more general, the additive group of a local field (see [16] or [19]).

Let Γ denote the dual group of G and for each $n \in \mathbf{Z}$, let Γ_n denote the annihilator of G_n, i.e.,

$$\Gamma_n = \{\gamma \in \Gamma;\ \gamma(x) = 1 \text{ for all } x \in G_n\}.$$

Then (cf. [5, § 4.1.4]) the $(\Gamma_n)_{-\infty}^{\infty}$ form a strictly increasing sequence of open compact subgroups of Γ and we have

(i)* $\bigcup_{n=-\infty}^{\infty} \Gamma_n = \Gamma$ and $\bigcap_{n=-\infty}^{\infty} \Gamma_n = \{1\}$,

(ii)* $\operatorname{order}(\Gamma_{n+1}/\Gamma_n) = \operatorname{order}(G_n/G_{n+1})$.

If we choose Haar measures μ on G and λ on Γ such that $\mu(G_0) = \lambda(\Gamma_0) = 1$ then $\mu(G_n) = \left(\lambda(\Gamma_n)\right)^{-1}$ for all $n \in \mathbf{Z}$; we set $m_n := \lambda(\Gamma_n)$.

We mention here three simple inequalities for the m_n that will be used frequently. For each $\alpha > 0$ and $k \in \mathbf{Z}$ we have

(1) $$2m_k \leqslant m_{k+1} \leqslant M m_k,$$

(2) $$\sum_{n=k}^{\infty} (m_n)^{-\alpha} \leqslant C (m_k)^{-\alpha},$$

(3) $$\sum_{n=-\infty}^{k} (m_n)^{\alpha} \leqslant C (m_k)^{\alpha}.$$

Inequality (1) follows from (ii), whereas (2) and (3) follow easily from (1). Here, like elsewhere in this paper, C denotes a constant whose value may change from one occurrence to the next.

For each $n \in \mathbf{Z}$ we choose elements $z_{l,n} \in G$ $(l \in \mathbf{Z}_+)$ so that the subsets $G_{l,n} := z_{l,n} + G_n$ of G satisfy $G_{k,n} \cap G_{l,n} = \emptyset$ if $k \neq l$ and $\bigcup_{l=0}^{\infty} G_{l,n} = G$; moreover, we choose $z_{0,n}$ so that $G_{0,n} = G_n$.

If we define the function $d\colon G \times G \to \mathbf{R}$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x - y = 0, \\ (m_n)^{-1} & \text{if } x - y \in G_n \setminus G_{n+1}, \end{cases}$$

then d defines a metric on $G \times G$ and the topology on G induced by this metric is the same as the original topology on G. Next, for $x \in G$ we set $\|x\| = d(x, 0)$ if $x \in G_n \setminus G_{n+1}$, and $\|0\| = 0$; then $\|x\| = (m_n)^{-1}$ if and only if $x \in G_n \setminus G_{n+1}$. In a similar way we can define a metric $\tilde{d}$ on $\Gamma \times \Gamma$ such that if we set $\|\gamma\| = \tilde{d}(\gamma, 1)$ then $\|\gamma\| = m_n$ if and only if $\gamma \in \Gamma_n \setminus \Gamma_{n-1}$.

For p with $0 < p \leqslant \infty$ we define p' by $p' = \infty$ if $0 < p \leqslant 1$ and p' is the usual conjugate of p if $1 \leqslant p \leqslant \infty$, i.e., $1/p + 1/p' = 1$. For a given set A we denote its characteristic function by ξ_A. The symbols $\hat{}$ and $\check{}$ will be used to denote the Fourier transform and the inverse Fourier transform, respectively. It is easy to see that if we define the functions Δ_n, $n \in \mathbf{Z}$, on G by

$$\Delta_n(x) = \left(\mu(G_n)\right)^{-1} \xi_{G_n}(x) = m_n \xi_{G_n}(x),$$

then, for $\gamma \in \Gamma$,

$$(\Delta_n)\hat{}\,(\gamma) = \xi_{\Gamma_n}(\gamma),$$

and, for $0 < p \leqslant \infty$, we have

$$\|\Delta_n\|_p = (m_n)^{1-1/p}.$$

To simplify our notation later on, we define the functions φ_n on G by $\varphi_n = \Delta_n - \Delta_{n-1}$, $n \in \mathbf{Z}$.

Following Taibleson's development of a distribution theory on local fields (cf. [16] or [19]), we define $\mathcal{S}(G)$ to be the set of functions on G that have compact support and are constant on the cosets of some G_n in G. A sequence $(\psi_n)_1^{\infty}$ in $\mathcal{S}(G)$ is said to converge to $\psi \in \mathcal{S}(G)$ if all ψ_n, ψ vanish outside a fixed G_k in G, all ψ_n, ψ are constant on the cosets of a fixed G_l in G and $\psi_n(x)$ converges uniformly to $\psi(x)$ on G. Then $\mathcal{S}(G)$ is called the space of test functions on G. The space of linear functionals on $\mathcal{S}(G)$ is denoted by $\mathcal{S}'(G)$, its elements are called distributions on G. We say that a sequence $(f_n)_1^{\infty}$ of elements in $\mathcal{S}'(G)$ converges to $f \in \mathcal{S}'(G)$ if for all $\psi \in \mathcal{S}(G)$ we have

$$\lim_{n\to\infty} \langle f_n, \psi\rangle = \langle f, \psi\rangle.$$

Clearly, similar spaces of test functions and distributions can be defined on Γ. It can be shown that the Fourier transform maps $\mathcal{S}(G)$ one-to-one onto $\mathcal{S}(\Gamma)$. Moreover, if we extend the Fourier transform to $\mathcal{S}'(G)$ in the usual way then the Fourier transform maps $\mathcal{S}'(G)$ one-to-one onto $\mathcal{S}'(\Gamma)$. Additional results for such test functions and distributions in case $G = K^+$, the additive group of a local field K, can be found in [19]. These results easily extend to groups G as considered here and will be used without further comment. In particular, for $f \in \mathcal{S}'(G)$ and $\psi \in \mathcal{S}(G)$ such that ψ is constant on the cosets of G_n in G, the convolution $f * \psi$ is defined both as a distribution in $\mathcal{S}'(G)$ and as a function on G that is constant on the cosets of G_n (cf. [18] or [19, Ch. III, (3.15)]).

2. Homogeneous and inhomogeneous Besov spaces. Before giving the definition of the Besov spaces on G we need to introduce a second space of test functions and distributions. Our approach here is similar to Triebel's approach to defining the homogeneous Besov spaces on $\mathbf{R}^n$ [20], but with appropriate changes to account for the topological structure of G. Let

$$Z(G) = \{\psi \in \mathcal{S}(G) \colon \hat{\psi}(0) = \int_G \psi(t)\, d\mu(t) = 0\},$$

and define convergence in $Z(G)$ like in $\mathcal{S}(G)$. Let $Z'(G)$ be the space of linear functionals on $Z(G)$ with convergence in $Z'(G)$ defined like in $\mathcal{S}'(G)$. If $\mathcal{C}$ denotes the set of constant distributions in $\mathcal{S}'(G)$ then $Z'(G)$ can be identified with $\mathcal{S}'(G)/\mathcal{C}$ in the sense that (i) for each $f \in \mathcal{S}'(G)$ its restriction to $Z(G)$ belongs to $Z'(G)$, (ii) if $f, g \in \mathcal{S}'(G)$ and if $g = f + c$ for some constant $c \in \mathcal{S}'(G)$ then the restrictions of f and g to $Z(G)$ determine the same element of $Z'(G)$, (iii) if $\tilde{f} \in Z'(G)$ then there exists an $f \in \mathcal{S}'(G)$ so that its restriction to $Z(G)$ equals $\tilde{f}$; moreover, modulo constants f is determined uniquely by $\tilde{f}$. These facts are easy to establish and we omit the proof. At times we shall disregard the difference between $\tilde{f} \in Z'(G)$ and a corresponding $f \in \mathcal{S}'(G)$.

We now give the definition of the homogeneous Besov spaces on G.

Definition 1. Let $\alpha \in \mathbf{R}$ and $0 < p,\ q \leqslant \infty$. Then

$$\dot{B}(\alpha, p, q) = \Big\{f \in Z'(G);\ \|f\|_{\dot{B}(\alpha,p,q)} := \Big(\sum_{n=-\infty}^{\infty} \big((m_n)^{\alpha} \|f * \varphi_n\|_p\big)^q\Big)^{1/q} < \infty\Big\},$$

with the usual modification if $q = \infty$.

We first make a simple observation about the distributions belonging to $\dot{B}(\alpha, p, q)$ in case $\alpha > 0$ and $1 \leqslant p \leqslant \infty$.

Proposition 1. *If* $\alpha > 0$, $1 \leqslant p \leqslant \infty$ *and* $0 < q \leqslant \infty$ *then* $\dot{B}(\alpha, p, q) \subset L^1_{\mathrm{loc}}(G)$.

Proof. Take any $f \in \dot{B}(\alpha, p, q)$. Since $f \in \mathcal{S}'(G)$ we have

$$f = f * \Delta_0 + \sum_{n=1}^{\infty} f * \varphi_n,$$

with convergence in $\mathcal{S}'(G)$. If $1 < q \leqslant \infty$, then it follows from the inequalities of Minkowski and Hölder that

$$\begin{aligned}\Big\|\sum_{n=1}^{\infty} f * \varphi_n\Big\|_p &\leqslant \sum_{n=1}^{\infty} (m_n)^{-\alpha} (m_n)^{\alpha} \|f * \varphi_n\|_p \\ &\leqslant \Big(\sum_{n=1}^{\infty} (m_n)^{-\alpha q'}\Big)^{1/q'} \Big(\sum_{n=1}^{\infty} \big((m_n)^{\alpha} \|f * \varphi_n\|_p\big)^q\Big)^{1/q} \\ &\leqslant C \|f\|_{\dot{B}(\alpha,p,q)} < \infty,\end{aligned}$$

by inequality (2). If $0 < q \leqslant 1$, the inclusion relation for sequence spaces, $l^q \hookrightarrow l^1$, implies that

$$\begin{aligned}\Big\|\sum_{n=1}^{\infty} f * \varphi_n\Big\|_p &\leqslant \sum_{n=1}^{\infty} \|f * \varphi_n\|_p \leqslant \sum_{n=1}^{\infty} (m_n)^{\alpha} \|f * \varphi_n\|_p \\ &\leqslant \Big(\sum_{n=1}^{\infty} \big((m_n)^{\alpha} \|f * \varphi_n\|_p\big)^q\Big)^{1/q} \leqslant \|f\|_{\dot{B}(\alpha,p,q)} < \infty.\end{aligned}$$

Thus for $0 < q \leqslant \infty$, we have $\sum_{n=1}^{\infty} f * \varphi_n \in L^p(G)$. Since $f * \Delta_0$ is continuous, we may conclude that $f \in L^1_{\mathrm{loc}}(G)$.

Interlude. In his Ph.D. dissertation Ombe defined and studied the inhomogeneous Besov spaces $B(\alpha, p, q)$ on G. For $\alpha \in \mathbf{R}$ and $0 < p,\ q \leqslant \infty$ these spaces are defined as follows (with the usual modification if $q = \infty$):

$$B(\alpha, p, q) = \Big\{f \in \mathcal{S}'(G);\ \|f\|_{B(\alpha,p,q)} := \Big(\|f * \Delta_0\|_p^q + \sum_{n=1}^{\infty} \big((m_n)^{\alpha} \|f * \varphi_n\|_p\big)^q\Big)^{1/q} < \infty\Big\}.$$

As is to be expected, a number of the results proved in [9] for inhomogeneous Besov spaces extend immediately to homogeneous Besov spaces. We mention here two such results. Since the proofs for the homogeneous case closely resemble the proofs in [9] for the inhomogeneous case, they will be omitted. Observe that Theorem 1(c) shows that the spaces $\dot{B}(\alpha, p, q)$ are generalizations to $\alpha \in \mathbf{R}$ and $0 < p,\ q \leqslant \infty$ of the generalized Lipschitz spaces on G.

THEOREM 1. *Let $\alpha > 0$ and $1 \leqslant p,\ q \leqslant \infty$. For $f \in L^1_{\mathrm{loc}}(G)$ the following quantities define equivalent norms on $\dot{B}(\alpha, p, q)$:*

(a) $\|f\|_{\dot{B}(\alpha,p,q)}$,

(b) $\Big(\sum_{n=-\infty}^{\infty} \big((m_n)^\alpha \|f - f * \Delta_n\|_p\big)^q\Big)^{1/q}$,

(c) $\Big(\int_G \|\tau_y f - f\|_p^q \|y\|^{-(q\alpha+1)}\, d\mu(y)\Big)^{1/q}$,

where $\tau_y f(x) = f(x-y)$ for $x, y \in G$. (For $q = \infty$, replace the expressions in (b) *and* (c) *by $\sup_n (m_n)^\alpha \|f - \Delta_n * f\|_p$ and $\operatorname{ess\,sup}_y \|y\|^{-\alpha} \|\tau_y f - f\|_p$, respectively.)*

A special case of the inhomogeneous version of this theorem, namely, where G is the additive group of a local field, was proved earlier by Taibleson (cf. [19, Ch. VI, Theorem (2.2)]). His proof served as model for Ombe's proof in [9].

The second theorem whose proof can be based on Ombe's work identifies the dual spaces $\dot{B}(\alpha, p, q)^*$ of the homogeneous Besov spaces on G.

THEOREM 2. *Let $\alpha \in \boldsymbol{R}$ and $0 < q < \infty$.*

(a) *If $1 \leqslant p < \infty$ we have $\dot{B}(\alpha, p, q)^* \cong \dot{B}(-\alpha, p', q')$.*
(b) *If $0 < p < 1$ we have $\dot{B}(\alpha, p, q)^* \cong \dot{B}(-\alpha - 1 + 1/p, p', q')$.*

In the remainder of this section we turn briefly to a comparison between the B-spaces and the $\dot{B}$-spaces on G. In [1, Theorem 6.3.2] it was shown that for the Besov spaces on $\boldsymbol{R}^n$ we have $B^\alpha_{pq} = L^p \cap \dot{B}^\alpha_{pq}$ when $\alpha > 0$ and $1 \leqslant p, q \leqslant \infty$. In [10] one of the present authors obtained a similar result for the Besov spaces on G (see also [9]).

THEOREM 3. *Let $\alpha > 0$ and $1 \leqslant p, q \leqslant \infty$. Then*

$$B(\alpha, p, q) = L^p \cap \dot{B}(\alpha, p, q).$$

Proof. If $f \in B(\alpha, p, q)$ then an argument like in the proof of Proposition 1 shows that $f \in L^p(G)$. Moreover,

$$\begin{aligned}
\|f\|^q_{\dot{B}(\alpha,p,q)} &\leqslant \sum_{n=-\infty}^{0} \big((m_n)^\alpha \|f\|_p (\|\Delta_n\|_1 + \|\Delta_{n-1}\|_1)\big)^q \\
&\quad + \sum_{n=1}^{\infty} \big((m_n)^\alpha \|f * \varphi_n\|_p\big)^q \\
&\leqslant 2^q \|f\|_p^q \sum_{n=-\infty}^{0} (m_n)^{\alpha q} + \sum_{n=1}^{\infty} \big((m_n)^\alpha \|f * \varphi_n\|_p\big)^q \\
&\leqslant C\|f\|_p^q + \sum_{n=1}^{\infty} \big((m_n)^\alpha \|f * \varphi_n\|_p\big)^q < \infty.
\end{aligned}$$

Conversely, if $f \in L^p \cap \dot{B}(\alpha, p, q)$ then

$$\begin{aligned}
\|f\|_{B(\alpha,p,q)} &= \Big(\|f * \Delta_0\|_p^q + \sum_{n=1}^{\infty} \big((m_n)^\alpha \|f * \varphi_n\|_p\big)^q\Big)^{1/q} \\
&\leqslant \Big(\|f\|_p^q \|\Delta_0\|_1^q + \sum_{n=-\infty}^{\infty} \big((m_n)^\alpha \|f * \varphi_n\|_p\big)^q\Big)^{1/q} \\
&\leqslant \|f\|_p + \|f\|_{\dot{B}(\alpha,p,q)} < \infty.
\end{aligned}$$

This completes the proof of Theorem 3.

A second result relating the spaces $B(\alpha, p, q)$ and $\dot{B}(\alpha, p, q)$ is given in Theorem 4. This result is similar to a result of Johnson for Besov spaces on $\boldsymbol{R}^n$ (see [7, Theorem 1.1]). In both [7] and [8] several applications of Johnson's theorem are given. We intend to give some applications of Theorem 4 elsewhere.

THEOREM 4. *Let $\alpha > 0$ and $1 \leqslant p, q \leqslant \infty$. Let*

$$\pi\dot{B}(\alpha, p, q) := \{f \in \dot{B}(\alpha, p, q);\ \text{for all } \gamma \in \Gamma \text{ we have}$$
$$\gamma f \in \dot{B}(\alpha, p, q) \text{ and } \|\gamma f\|_{\dot{B}(\alpha,p,q)} \leqslant C(\|\gamma\|^\alpha \|f\|_p + \|f\|_{\dot{B}(\alpha,p,q)})\}.$$

Then $\pi\dot{B}(\alpha, p, q) = B(\alpha, p, q)$.

Proof. First assume $f \in \pi\dot{B}(\alpha, p, q)$. Take any $\gamma \in \Gamma \setminus \Gamma_0$; then $\gamma \in \Gamma_n \setminus \Gamma_{n-1}$ for some $n \geqslant 1$ and we have

$$\gamma(f * \Delta_0) = \gamma f * \gamma\Delta_0 * \varphi_n. \tag{4}$$

To see this, take any $\sigma \in \Gamma$; then

$$\big(\gamma(f * \Delta_0)\big)\hat{}\,(\sigma) = \hat{f}(\sigma - \gamma)\, \hat{\Delta}_0(\sigma - \gamma),$$

whereas

$$(\gamma f * \gamma\Delta_0 * \varphi_n)\hat{}\,(\sigma) = \hat{f}(\sigma - \gamma)\, \hat{\Delta}_0(\sigma - \gamma)\, \hat{\varphi}_n(\sigma).$$

Now observe that $\Delta_0(\sigma - \gamma) \neq 0$ if and only if $\sigma - \gamma \in \Gamma_0$, i.e., if $\sigma \in \gamma + \Gamma_0$. Since $n \geqslant 1$, $\gamma + \Gamma_0 \subset \Gamma_n \setminus \Gamma_{n-1}$. Furthermore, for $\sigma \in \Gamma_n \setminus \Gamma_{n-1}$ we have $\hat{\varphi}_n(\sigma) = 1$. Thus we see that $\big(\gamma(f * \Delta_0)\big)\hat{}\,(\sigma) = (\gamma f * \gamma\Delta_0 * \varphi_n)\hat{}\,(\sigma)$ for all $\sigma \in \Gamma$, which implies (4).

Consequently,

$$\begin{aligned}
\|f * \Delta_0\|_p &= \|\gamma(f * \Delta_0)\|_p = \|\gamma\Delta_0 * (\varphi_n * \gamma f)\|_p \qquad (\text{since } 1 \leqslant p \leqslant \infty) \\
&\leqslant \|\gamma\Delta_0\|_1 \|\varphi_n * \gamma f\|_p \leqslant (m_n)^{-\alpha} \|\gamma f\|_{\dot{B}(\alpha,p,q)}.
\end{aligned}$$

Thus

$$\|f\|_{B(\alpha,p,q)} = \big(\|f*\Delta_0\|_q^p + \sum_{n=1}^{\infty} ((m_n)^{\alpha}\|f*\varphi_n\|_p)^q\big)^{1/q}$$
$$\leqslant \big((m_n)^{-\alpha q}\|\gamma f\|^q_{\dot{B}(\alpha,p,q)} + \|f\|^q_{\dot{B}(\alpha,p,q)}\big)^{1/q} < \infty,$$

i.e., $\pi\dot{B}(\alpha, p, q) \subset B(\alpha, p, q)$. Note that this inclusion is true for all $\alpha \in \boldsymbol{R}$ and $1 \leqslant p \leqslant \infty$ and $0 < q \leqslant \infty$.

Next, assume $f \in B(\alpha, p, q)$. If $\gamma = 1$ then $\gamma f = f$ and $\gamma f \in \dot{B}(\alpha, p, q)$ by Theorem 3. If $\gamma \neq 1$ then $\gamma \in \Gamma_n \setminus \Gamma_{n-1}$ for some $n \in \boldsymbol{Z}$ and, moreover, $\|\gamma\| = m_n$. Now we observe that for all $x, h \in G$ we have

$$(\tau_{-h}(\gamma f) - \gamma f)(x) = \tau_{-h}\gamma(x)(\tau_{-h}f - f)(x) + f(x)\gamma(x)(\gamma(h)-1).$$

Thus,

$$\|\tau_{-h}(\gamma f) - \gamma f\|_p \leqslant \|\tau_{-h}\gamma(\tau_{-h}f - f)\|_p + |\gamma(h)-1|\,\|\gamma f\|_p$$
$$= \|\tau_{-h}f - f\|_p + |\gamma(h)-1|\,\|f\|_p.$$

Therefore, applying Theorem 1(c), we see that

$$\|\gamma f\|_{\dot{B}(\alpha,p,q)} \leqslant C\big(\int_G \|\tau_{-h}(\gamma f) - \gamma f\|_p^q \|h\|^{-\alpha q-1}\,d\mu(h)\big)^{1/q}$$
$$\leqslant C\big(\int_G \|\tau_{-h}f - f\|_p^q \|h\|^{-\alpha q-1}\,d\mu(h)\big)^{1/q}$$
$$+ C\big(\int_G |\gamma(h)-1|^q \|f\|_p^q \|h\|^{-\alpha q-1}\,d\mu(h)\big)^{1/q}$$
$$\leqslant C\|f\|_{\dot{B}(\alpha,p,q)} + 2C\|f\|_p\big(\int_{G\setminus G_n} \|h\|^{-\alpha q-1}\,d\mu(h)\big)^{1/q},$$

since $\gamma(h) = 1$ for $\gamma \in \Gamma_n$ and $h \in G_n$. For the integral in the last term we have

$$\int_{G\setminus G_n} \|h\|^{-\alpha q-1}\,d\mu(h) = \sum_{l=-\infty}^{n-1} \int_{G_l\setminus G_{l+1}} \|h\|^{-\alpha q-1}\,d\mu(h)$$
$$\leqslant \sum_{l=-\infty}^{n-1} (m_l)^{\alpha q+1}(m_l)^{-1} \leqslant C(m_n)^{\alpha q} = C\|\gamma\|^{\alpha q},$$

according to inequality (3). Thus we see that

$$\|\gamma f\|_{\dot{B}(\alpha,p,q)} \leqslant C(\|f\|_{\dot{B}(\alpha,p,q)} + \|\gamma\|^{\alpha}\|f\|_p),$$

which implies that $f \in \pi\dot{B}(\alpha, p, q)$.

3. Homogeneous Besov spaces and mean oscillation spaces. In this section we give a characterization of the spaces $\dot{B}(\alpha, p, q)$ in terms of the so-called mean oscillation spaces on G. The equivalence between homogeneous Besov spaces and mean oscillation spaces on $\boldsymbol{R}$ was observed by Ricci and

Taibleson in [12]; for extensions to $\boldsymbol{R}^n$, see [3] or [4]. Because the groups G considered here have a natural decomposition into $G = \bigcup_{l=0}^{\infty} G_{l,n}$ for each $n \in \boldsymbol{Z}$, there is an obvious way of defining the mean oscillation spaces in this context.

DEFINITION 2. Let $0 < p \leqslant \infty$, $n \in \boldsymbol{Z}$ and $f \in L^1_{\mathrm{loc}}(G)$. Then

$$\mathrm{osc}_p(f, n) := \big(\sum_{l=0}^{\infty} (m_l \int_{G_{l,n}} |f(x) - f_{G_{l,n}}|\,d\mu(x))^p\big)^{1/p},$$

where $f_{G_{l,n}} = m_n \int_{G_{l,n}} f(t)\,d\mu(t) = f*\Delta_n(z_{l,n}) = f*\Delta_n(x)$ for $x \in G_{l,n}$.

DEFINITION 3. For $0 < p, q \leqslant \infty$ and $\alpha \in \boldsymbol{R}$ the space $\mathrm{MO}(\alpha, p, q)$ is defined as the space of equivalence classes of locally integrable functions f modulo constants, i.e., $f \in L^1_{\mathrm{loc}}/\mathscr{C}$, such that

$$\mathrm{MO}(\alpha, p, q)(f) := \big(\sum_{n=-\infty}^{\infty} ((m_n)^{\alpha}\,\mathrm{osc}_p(f, n))^q\big)^{1/q} < \infty.$$

We have the following result.

THEOREM 5. (a) *Let* $1 \leqslant p, q \leqslant \infty$ *and* $\alpha > 0$. *If* $f \in \dot{B}(\alpha, p, q)$ *then* $f \in \mathrm{MO}(\alpha - 1/p, p, q)$ *and*

$$\mathrm{MO}(\alpha - 1/p, p, q)(f) \leqslant C\|f\|_{\dot{B}(\alpha,p,q)}.$$

(b) *Let* $0 < p, q \leqslant \infty$ *and* $\alpha > 0$. *If* $f \in \mathrm{MO}(\alpha - 1/p, p, q)$ *then* $f \in \dot{B}(\alpha, p, q)$ *and*

$$\|f\|_{\dot{B}(\alpha,p,q)} \leqslant C\,\mathrm{MO}(\alpha - 1/p, p, q)(f).$$

Proof. (a) If $1 \leqslant p < \infty$ then Hölder's inequality implies that for each $n \in \boldsymbol{Z}$,

$$\mathrm{osc}_p(f, n) = m_n\big(\sum_{l=0}^{\infty}\big(\int_{G_{l,n}} |f(x) - f*\Delta_n(x)|\,d\mu(x)\big)^p\big)^{1/p}$$
$$\leqslant m_n\big(\sum_{l=0}^{\infty}\int_{G_{l,n}} |f(x) - f*\Delta_n(x)|^p\,d\mu(x)(m_n)^{-p/p'}\big)^{1/p}$$
$$= (m_n)^{1/p}\big(\int_G |f(x) - f*\Delta_n(x)|^p\,d\mu(x)\big)^{1/p}$$
$$= (m_n)^{1/p}\|f - f*\Delta_n\|_p.$$

Hence, for each q with $1 \leqslant q < \infty$ we have

$$\mathrm{MO}(\alpha - 1/p, p, q)(f) = \big(\sum_{n=-\infty}^{\infty} ((m_n)^{\alpha-1/p}\,\mathrm{osc}_p(f, n))^q\big)^{1/q}$$
$$\leqslant \big(\sum_{n=-\infty}^{\infty} ((m_n)^{\alpha}\|f - f*\Delta_n\|_p)^q\big)^{1/q} \leqslant C\|f\|_{\dot{B}(\alpha,p,q)},$$

according to Theorem 1(b). Clearly, the above inequality also holds if $q=\infty$.

If $p=\infty$ we have for each $n\in \boldsymbol{Z}$,

$$\operatorname{osc}_\infty(f,n)=\sup_{l\geqslant 0}\Big(m_n\int_{G_{l,n}}|f(x)-f_{G_{l,n}}|\,d\mu(x)\Big)\leqslant \|f-\Delta_n*f\|_\infty.$$

Consequently, for each q with $1\leqslant q<\infty$,

$$\begin{aligned}\mathrm{MO}(\alpha,\infty,q)(f)&\leqslant\Big(\sum_{n=-\infty}^{\infty}\big((m_n)^\alpha\|f-\Delta_n*f\|_\infty\big)^q\Big)^{1/q}\\&\leqslant C\|f\|_{\dot B(\alpha,\infty,q)},\end{aligned}$$

according to Theorem 1(b). If $q=\infty$ the foregoing argument can be simplified slightly.

(b) For $0<p<\infty$ and $n\in\boldsymbol{Z}$ we have

$$\begin{aligned}\|f*\varphi_n\|_p^p&=\sum_{l=0}^{\infty}\int_{G_{l,n}}|f*\varphi_n(x)|^p\,d\mu(x)\\&=\sum_{l=0}^{\infty}(m_n)^{-1}|f*\Delta_n(z_{l,n})-f*\Delta_{n-1}(z_{l,n})|^p,\end{aligned}$$

because both $f*\Delta_n$ and $f*\Delta_{n-1}$ are constant on the sets $G_{l,n}$. Therefore,

$$\begin{aligned}\|f*\varphi_n\|_p^p&=\sum_{l=0}^{\infty}(m_n)^{-1}\Big|m_n\int_{G_{l,n}}f(x)\,d\mu(x)-f*\Delta_{n-1}(z_{l,n})\Big|^p\\&\leqslant\sum_{l=0}^{\infty}(m_n)^{p-1}\Big(\int_{G_{l,n}}|f(x)-f*\Delta_{n-1}(z_{l,n})|\,d\mu(x)\Big)^p.\end{aligned}$$

We now observe that each coset of G_{n-1} contains $p_n:=m_n/m_{n-1}$ cosets of G_n. Thus there are p_n different elements $z_{l,n}$ such that the cosets $z_{l,n}+G_{n-1}$ coincide. Hence,

$$\begin{aligned}\|f*\varphi_n\|_p^p&\leqslant\sum_{j=0}^{\infty}(m_n)^{p-1}p_n\Big(\int_{z_{j,n-1}+G_{n-1}}|f(x)-f*\Delta_{n-1}(z_{j,n-1})|\,d\mu(x)\Big)^p\\&=\sum_{j=0}^{\infty}\Big(\frac{m_n}{m_{n-1}}\Big)^p\frac{1}{m_{n-1}}\Big(m_{n-1}\int_{G_{j,n-1}}|f(x)-f*\Delta_{n-1}(z_{j,n-1})|\,d\mu(x)\Big)^p\\&=\Big(\frac{m_n}{m_{n-1}}\Big)^p\frac{1}{m_{n-1}}\big(\operatorname{osc}_p(f,n-1)\big)^p,\end{aligned}$$

that is,

$$\|f*\varphi_n\|_p\leqslant C(m_{n-1})^{-1/p}\operatorname{osc}_p(f,n-1).$$

The same inequality holds if $p=\infty$ and its proof is simpler. Consequently,

for each q with $0<q\leqslant\infty$ we have

$$\begin{aligned}\|f\|_{\dot B(\alpha,p,q)}&=\Big(\sum_{n=-\infty}^{\infty}\big((m_n)^\alpha\|f*\varphi_n\|_p\big)^q\Big)^{1/q}\\&\leqslant C\Big(\sum_{n=-\infty}^{\infty}\big((m_{n-1})^{\alpha-1/p}\operatorname{osc}_p(f,n-1)\big)^q\Big)^{1/q}\\&=C\cdot\mathrm{MO}(\alpha-1/p,p,q)(f).\end{aligned}$$

Corollary 1. *If* $1\leqslant p,q\leqslant\infty$ *and* $\alpha>0$ *then* $\mathrm{MO}(\alpha-1/p,p,q)=\dot B(\alpha,p,q)$ *and for* f *belonging to these spaces we have*

$$\|f\|_{\dot B(\alpha,p,q)}\approx\mathrm{MO}(\alpha-1/p,p,q)(f).$$

4. Atomic decomposition of the spaces $\dot B(\alpha,p,q)$. A further description of the homogeneous Besov spaces on G involves an atomic decomposition for the elements of these spaces. On $\boldsymbol{R}^n$ comparable results were obtained by Bui [3, Theorem 7]; for additional results see the paper by Frazier and Jawerth [6] who considered both homogeneous and inhomogeneous Besov spaces on $\boldsymbol{R}^n$.

Definition 4. A function $a\colon G\to\boldsymbol{C}$ is an (s,∞) *atom*, $s\in\boldsymbol{R}$, if

(i) a is supported on a set $z+G_n$ for some $z\in G$ and $n\in\boldsymbol{Z}$,

(ii) $|a(x)|\leqslant\big(\mu(z+G_n)\big)^{-s}=(m_n)^s$,

(iii) $\int_G a(x)\,d\mu(x)=0$.

We have the following result.

Theorem 6. *Let* $0<p,q\leqslant\infty$ *and* $\alpha\in\boldsymbol{R}$. *For each* $f\in\dot B(\alpha,p,q)$ *there exist constants* $\lambda_{l,j}$, $l\in\boldsymbol{Z}_+$ *and* $j\in\boldsymbol{Z}$, *and* $(-\alpha+1/p,\infty)$ *atoms* $a_{l,j}$ *with* $\operatorname{supp}(a_{l,j})\subset z_{l,j}+G_{j-1}$ *such that*

$$f=\sum_{j=-\infty}^{\infty}\sum_{l=0}^{\infty}\lambda_{l,j}a_{l,j}\quad\text{in } \mathcal{S}'/\mathcal{C}.$$

Moreover,

$$\|\lambda\|_{p,q}:=\Big(\sum_{j=-\infty}^{\infty}\Big(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\Big)^{q/p}\Big)^{1/q}\leqslant C\|f\|_{\dot B(\alpha,p,q)}.$$

(b) *Let* $0<p,q\leqslant\infty$ *and* $\alpha<0$. *Let* $\lambda_{l,j}$, $l\in\boldsymbol{Z}_+$ *and* $j\in\boldsymbol{Z}$, *be constants such that* $\|\lambda\|_{p,q}<\infty$. *For each* $l\in\boldsymbol{Z}_+$ *and* $j\in\boldsymbol{Z}$, *let* $a_{l,j}$ *be an* $(-\alpha+1/p,\infty)$ *atom supported on* $z_{l,j}+G_j$. *If*

$$f=\sum_{j=-\infty}^{\infty}\sum_{l=0}^{\infty}\lambda_{l,j}a_{l,j},$$

then $f\in\dot B(\alpha,p,q)$ *and* $\|f\|_{\dot B(\alpha,p,q)}\leqslant C\|\lambda\|_{p,q}$.

Proof. (a) For each $f \in \mathcal{S}'/\mathcal{C}$ we have

$$\begin{aligned} f &= \sum_{n=-\infty}^{\infty} f*\varphi_n(x) = \sum_{n=-\infty}^{\infty} f*\varphi_n*\varphi_n(x) \\ &= \sum_{n=-\infty}^{\infty} \int_G f*\varphi_n(t)\,\varphi_n(x-t)\,d\mu(t) \\ &= \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \int_{G_{l,n}} f*\varphi_n(t)\,\varphi_n(x-t)\,d\mu(t) \\ &= \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} (m_n)^{-1} f*\varphi_n(z_{l,n})\,\varphi_n(x-z_{l,n}) \\ &= \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \lambda_{l,n}\, a_{l,n}(x), \end{aligned}$$

where $\lambda_{l,n} = (m_{n-1})^{\alpha-1/p} f*\varphi_n(z_{l,n})$ and $a_{l,n} = (m_{n-1})^{-\alpha+1/p}(m_n)^{-1}\,\varphi_n(x-z_{l,n})$. For each $a_{l,n}$ we have

(i) $\operatorname{supp}(a_{l,n}) \subset z_{l,n}+G_{n-1}$.

(ii) $|a_{l,n}(x)| \leqslant (m_{n-1})^{-\alpha+1/p}$,

(iii) $\int_G a_{l,n}(x)\,d\mu(x) = 0$.

Thus $a_{l,n}$ is an $(-\alpha+1/p, \infty)$ atom on G with support in $z_{l,n}+G_{n-1}$. Moreover, for the $\lambda_{l,n}$ we have

$$\begin{aligned} \|\lambda\|_{p,q} &= \Big(\sum_{n=-\infty}^{\infty} \Big(\sum_{l=0}^{\infty} |(m_{n-1})^{\alpha-1/p} f*\varphi_n(z_{l,n})|^p\Big)^{q/p}\Big)^{1/q} \\ &\leqslant C\Big(\sum_{n=-\infty}^{\infty} (m_n)^{\alpha q}\Big(\sum_{l=0}^{\infty} \int_{z_{l,n}+G_n} |f*\varphi_n(t)|^p\,d\mu(t)\Big)^{q/p}\Big)^{1/q} \\ &= C\,\|f\|_{\dot{B}(\alpha,p,q)}. \end{aligned}$$

This completes the proof of (a).

(b) Let $a_{l,j}$ be a $(-\alpha+1/p, \infty)$ atom with support in $z_{l,j}+G_j$. We first derive an estimate for $\|a_{l,j}*\varphi_n\|_p$ and we distinguish two cases.

(i) If $j \geqslant n$ then the function φ_n, which is constant on the cosets of G_n, is constant on the cosets of G_j in G. Therefore, for each $x \in G$ we have

$$a_{l,j}*\varphi_n(x) = \int_G a_{l,j}(t)\,\varphi_n(x-t)\,d\mu(t) = C \int_{G_{l,j}} a_{l,j}(t)\,d\mu(t) = 0,$$

since $a_{l,j}$ is an atom. Consequently, for all atoms $a_{l,j}$ with $j \geqslant n$ and all p with $0 < p \leqslant \infty$ we have $\|a_{l,j}*\varphi_n\|_p = 0$.

(ii) If $j < n$ then

$$\begin{aligned} \operatorname{supp}(a_{l,j}*\varphi_n) &\subset \operatorname{supp}(a_{l,j}) + \operatorname{supp}(\varphi_n) \\ &= (z_{l,j}+G_j)+G_{n-1} = z_{l,j}+G_j. \end{aligned}$$

Therefore, if $0 < p \leqslant \infty$,

$$\begin{aligned} \|a_{l,j}*\varphi_n\|_p &\leqslant \|a_{l,j}*\varphi_n\|_\infty \big(\mu(\operatorname{supp}(a_{l,j}*\varphi_j))\big)^{1/p} \\ &\leqslant \|a_{l,j}\|_\infty \|\varphi_n\|_1 (m_j)^{-1/p} \\ &\leqslant 2(m_j)^{-\alpha+1/p-1/p} = 2(m_j)^{-\alpha}. \end{aligned}$$

Next, if $f = \sum_{j=-\infty}^{\infty}\sum_{l=0}^{\infty} \lambda_{l,j}\, a_{l,j}$, we estimate $\|f\|_{\dot{B}(\alpha,p,q)}$ as follows.
(A) If $0 < p \leqslant 1$ then for each $n \in \boldsymbol{Z}$,

$$\|f*\varphi_n\|_p^p \leqslant \sum_{j=-\infty}^{\infty}\sum_{l=0}^{\infty} |\lambda_{l,j}|^p \|a_{l,j}*\varphi_n\|_p^p \leqslant C \sum_{j=-\infty}^{n-1}\sum_{l=0}^{\infty} |\lambda_{l,j}|^p (m_j)^{-\alpha p}.$$

Thus for each q with $0 < q < \infty$ we have

$$\begin{aligned} (5)\qquad \|f\|_{\dot{B}(\alpha,p,q)} &= \Big(\sum_{n=-\infty}^{\infty} \big((m_n)^\alpha \|f*\varphi_n\|_p\big)^q\Big)^{1/q} \\ &\leqslant C\Big(\sum_{n=-\infty}^{\infty} (m_n)^{\alpha q}\Big(\sum_{j=-\infty}^{n-1} (m_j)^{-\alpha p} \sum_{l=0}^{\infty} |\lambda_{l,j}|^p\Big)^{q/p}\Big)^{1/q}. \end{aligned}$$

If $0 < q \leqslant p$, so that $0 < q/p \leqslant 1$, we obtain

$$\begin{aligned} \|f\|_{\dot{B}(\alpha,p,q)} &\leqslant C\Big(\sum_{n=-\infty}^{\infty} (m_n)^{\alpha q} \sum_{j=-\infty}^{n-1} \Big((m_j)^{-\alpha p} \sum_{l=0}^{\infty} |\lambda_{l,j}|^p\Big)^{q/p}\Big)^{1/q} \\ &= C\Big(\sum_{n=-\infty}^{\infty} (m_n)^{\alpha q} \sum_{j=-\infty}^{n-1} (m_j)^{-\alpha q}\Big(\sum_{l=0}^{\infty} |\lambda_{l,j}|^p\Big)^{q/p}\Big)^{1/q} \\ &= C\Big(\sum_{j=-\infty}^{\infty} (m_j)^{-\alpha q} \sum_{n=j+1}^{\infty} (m_n)^{\alpha q}\Big(\sum_{l=0}^{\infty} |\lambda_{l,j}|^p\Big)^{q/p}\Big)^{1/q} \\ &\leqslant C\Big(\sum_{j=-\infty}^{\infty} \Big(\sum_{l=0}^{\infty} |\lambda_{l,j}|^p\Big)^{q/p}\Big)^{1/q} = C\,\|\lambda\|_{p,q}. \end{aligned}$$

If $p < q < \infty$, so that $1 < q/p < \infty$, then we can apply Hölder's inequality to (5) with $r = q/p$ and we obtain

$$\begin{aligned} \|f\|_{\dot{B}(\alpha,p,q)} &\leqslant C\Big(\sum_{n=-\infty}^{\infty} (m_n)^{\alpha q} \sum_{j=-\infty}^{n-1} \Big((m_j)^{-\alpha p/r'} (m_j)^{-\alpha p/r} \sum_{l=0}^{\infty} |\lambda_{l,j}|^p\Big)^r\Big)^{1/q} \\ &\leqslant C\Big(\sum_{n=-\infty}^{\infty} (m_n)^{\alpha q}\Big(\sum_{j=-\infty}^{n-1} (m_j)^{-\alpha p}\Big)^{r/r'} \sum_{j=-\infty}^{n-1} (m_j)^{-\alpha p}\Big(\sum_{l=0}^{\infty} |\lambda_{l,j}|^p\Big)^r\Big)^{1/q} \end{aligned}$$

$$\leqslant C\Big(\sum_{n=-\infty}^{\infty}(m_n)^{\alpha q}(m_{n-1})^{-\alpha pr/r'}\sum_{j=-\infty}^{n-1}(m_j)^{-\alpha p}\Big(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\Big)^r\Big)^{1/q}$$
$$\leqslant C\Big(\sum_{n=-\infty}^{\infty}(m_{n-1})^{\alpha p}\sum_{j=-\infty}^{n-1}(m_j)^{-\alpha p}\Big(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\Big)^{q/p}\Big)^{1/q}\leqslant C\|\lambda\|_{p,q},$$

by the same argument as was used in case $0<q\leqslant p$.

If $q=\infty$ we have

$$\|f\|_{\dot{B}(\alpha,p,\infty)}=\sup_n(m_n)^{\alpha}\|f*\varphi_n\|_p$$
$$\leqslant C\sup_n(m_n)^{\alpha}\Big(\sum_{j=-\infty}^{n-1}(m_j)^{-\alpha p}\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\Big)^{1/p}.$$

Now observe that for each sum in this expression we have

$$\sum_{j=-\infty}^{n-1}(m_j)^{-\alpha p}\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\leqslant\sup_j\Big(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\Big)\sum_{j=-\infty}^{n-1}(m_j)^{-\alpha p}$$
$$\leqslant C(m_n)^{-\alpha p}\sup_j\sum_{l=0}^{\infty}|\lambda_{l,j}|^p.$$

Thus we see that

$$\|f\|_{\dot{B}(\alpha,p,\infty)}\leqslant C\sup_n(m_n)^{\alpha}(m_n)^{-\alpha}\Big(\sup_j\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\Big)^{1/p}=C\|\lambda\|_{p,\infty}.$$

(B) If $1<p<\infty$, we have for each $n\in\boldsymbol{Z}$, $x\in G$ and each τ with $0<\tau<1$,

$$|f*\varphi_n(x)|\leqslant\sum_{j=-\infty}^{n-1}\sum_{l=0}^{\infty}|\lambda_{l,j}|\,|a_{l,j}*\varphi_n(x)|^{\tau+(1-\tau)}$$
$$\leqslant\sum_{j=-\infty}^{n-1}\Big(\sum_{l=0}^{\infty}\big(|\lambda_{l,j}|\,|a_{l,j}*\varphi_n(x)|^{\tau}\big)^p\Big)^{1/p}\Big(\sum_{l=0}^{\infty}|a_{l,j}*\varphi_n(x)|^{(1-\tau)p'}\Big)^{1/p'}.$$

Also, if $j<n$ then $\operatorname{supp}(a_{l,j}*\varphi_n)\subset z_{l,j}+G_j$. Thus the functions in the last sum all have different and disjoint supports, so that for each $x\in G$ this sum consists of (at most) one nonzero term. Hence,

$$|f*\varphi_n(x)|\leqslant C\sum_{j=-\infty}^{n-1}\Big(\sum_{l=0}^{\infty}\big(|\lambda_{l,j}|\,|a_{l,j}*\varphi_n(x)|^{\tau}\big)^p\Big)^{1/p}(m_j)^{(-\alpha+1/p)(1-\tau)}.$$

Thus we see that

$$\|f*\varphi_n\|_p\leqslant C\sum_{j=-\infty}^{n-1}(m_j)^{(-\alpha+1/p)(1-\tau)}\Big\|\Big(\sum_{l=0}^{\infty}\big(|\lambda_{l,j}|\,|a_{l,j}*\varphi_n|^{\tau}\big)^p\Big)^{1/p}\Big\|_p.$$

Again, since the functions in the last sum have disjoint supports we obtain

$$\Big\|\Big(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p|a_{l,j}*\varphi_n|^{\tau p}\Big)^{1/p}\Big\|_p=\Big(\int_G\sum_{l=0}^{\infty}|\lambda_{l,j}|^p|a_{l,j}*\varphi_n(x)|^{\tau p}\,d\mu(x)\Big)^{1/p}$$
$$=\Big(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\int_G|a_{l,j}*\varphi_n(x)|^{\tau p}\,d\mu(x)\Big)^{1/p}$$
$$\leqslant C\Big(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p(m_j)^{(-\alpha+1/p)\tau p}(m_j)^{-1}\Big)^{1/p}$$
$$=C(m_j)^{(-\alpha+1/p)\tau-1/p}\Big(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\Big)^{1/p}.$$

Consequently,

$$\|f*\varphi_n\|_p\leqslant C\sum_{j=-\infty}^{n-1}(m_j)^{(-\alpha+1/p)(1-\tau)}(m_j)^{(-\alpha+1/p)\tau-1/p}\Big(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\Big)^{1/p}$$
$$=C\sum_{j=-\infty}^{n-1}(m_j)^{-\alpha}\Big(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\Big)^{1/p}.$$

From this we see immediately that for $0<q<\infty$ and $\alpha<0$,

$$\|f\|_{\dot{B}(\alpha,p,q)}\leqslant C\Big(\sum_{n=-\infty}^{\infty}(m_n)^{\alpha q}\sum_{j=-\infty}^{n-1}(m_j)^{-\alpha q}\Big(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\Big)^{q/p}\Big)^{1/q}$$
$$=C\Big(\sum_{j=-\infty}^{\infty}(m_j)^{-\alpha q}\Big(\sum_{l=0}^{\infty}|\lambda_{l,j}|^p\Big)^{q/p}\sum_{n=j+1}^{\infty}(m_n)^{\alpha q}\Big)^{1/q}\leqslant C\|\lambda\|_{p,q}.$$

For $q=\infty$ the foregoing proof can be simplified somewhat.

(C) If $p=\infty$, $0<q\leqslant\infty$ and $\alpha<0$ we can still show that $\|f\|_{\dot{B}(\alpha,\infty,q)}\leqslant C\|\lambda\|_{\infty,q}$. The proof is simpler than for the cases considered in (A) and in (B), and will be omitted.

Remark. 1. In [6] Frazier and Jawerth defined their atoms to be functions on $\boldsymbol{R}^n$ whose partial derivatives up to a certain order, depending on α, satisfy certain growth restrictions. Using such atoms they proved a result for the spaces $\dot{B}^{\alpha}_{pq}(\boldsymbol{R}^n)$, like our Theorem 6(b), for all $\alpha\in\boldsymbol{R}$. It would be interesting to obtain a more general condition than condition (ii) in Definition 4 for atoms on G so that we could extend Theorem 6(b) to all $\alpha\in\boldsymbol{R}$.

5. The regularization of Besov spaces and the spaces $A(\alpha, p, q)$. In [17] Taibleson gave convincing reasons for his claim that the functions $\Delta_n(x)$, called $R(x,-n)$ in [17], on a local field K play the same rôle there as the Poisson kernels $P(x,y)$ do on $\boldsymbol{R}^{n+1}_+$. That is, if for a given $f\in\mathcal{S}''(G)$ we define $F\colon K\times\boldsymbol{Z}\to\boldsymbol{C}$ by $F(x,n)=f*\Delta_n(x)$, then the function F plays the rôle of a harmonic function on $K\times Z$. F is called the *regularization* of f and F is an example of a regular function on $K\times Z$. Here a *regular function* is defined to be a function u on $K\times Z$ so that (i) for each $k\in Z$ the function $u_k(x):=u(x,k)$ is constant on the cosets of G_k in G, and (ii) $u_l(x)=u_k*u_l(x)$ for all $x\in G$ and $k, l\in Z$ such that $k\leqslant l$. In [17, Lemma 1] (see also [19, Ch. IV, Lemma (1.5)]) Taibleson proved that every regular function $u(x,k)$ is the regularization of some distribution $f\in\mathcal{S}''(K)$ and that f is the boundary value of u, in the sense that $\lim_{k\to\infty}\mu(x,k)=f$ in $\mathcal{S}''(K)$. Moreover, u is the regularization of its boundary value f. Clearly, we can replace K by G in the foregoing without any difficulty.

In this section we study the regularization of distributions F belonging to certain Besov spaces on G. We now give a definition.

DEFINITION 5. Let $0<p, q\leqslant\infty$ and $\alpha\in R$. For a measurable function $u\colon G\times Z\to C$ we set

$$\|u\|_{\alpha,p,q}=\Big(\sum_{n=-\infty}^{\infty}\big((m_n)^{-\alpha}\|u(\cdot,n)\|_p\big)^q\Big)^{1/q},$$

with the usual modification if $q=\infty$, and we define $A(\alpha,p,q)$ to be the space of all regular functions u on $G\times Z$ for which $\|u\|_{\alpha,p,q}<\infty$.

Remark 2. Since the spaces $A(\alpha,p,q)$ are, in fact, mixed norm spaces the usual inclusion relations hold. Specifically, if $0<p_1\leqslant p_2\leqslant\infty$, $0<q\leqslant q_1\leqslant\infty$ and $\alpha+1/p=\alpha_1+1/p_1$ then $A(\alpha,p,q)\subset A(\alpha_1,p_1,q_1)$, with continuous embedding.

As we shall see, the spaces $A(\alpha,p,q)$ play an important rôle in the study of the regularization of the elements of Besov spaces. We first prove a representation theorem for the functions in $A(\alpha,p,q)$.

THEOREM 7. *Let $0<p, q\leqslant\infty$ and $\alpha>-1/p$. If $u\in A(\alpha,p,q)$ then u can be represented as*

$$u(x,k)=\sum_{j=-\infty}^{k}u(\cdot,j)*\varphi_j(x),$$

with the sum converging absolutely for each $x\in G$.

Proof. We have

$$A:=\sum_{j=-\infty}^{k}|u_j*\varphi_j(x)|\leqslant\sum_{j=-\infty}^{k}\int_G|u(z,j)\,\varphi_j(x-z)|\,d\mu(z).$$

(a) If $1\leqslant p\leqslant\infty$ we can apply Hölder's inequality and we obtain

$$A\leqslant\sum_{j=-\infty}^{k}\Big(\int_G|u(z,j)|^p\,d\mu(z)\Big)^{1/p}\Big(\int_G|\varphi_j(x-z)|^{p'}d\mu(z)\Big)^{1/p'}$$
$$\leqslant\sum_{j=-\infty}^{k}\|u_j\|_p\big(\|\Delta_j\|_{p'}+\|\Delta_{j-1}\|_{p'}\big).$$

Since $u\in A(\alpha,p,q)$ we have for each $j\in Z$, $(m_j)^{-\alpha}\|u_j\|_p\leqslant\|u\|_{\alpha,p,q}$. Thus

$$A\leqslant C\|u\|_{\alpha,p,q}\sum_{j=-\infty}^{k}(m_j)^{\alpha+1/p}<\infty,$$

by inequality (3), which can be applied since $\alpha+1/p>0$.

(b) If $0<p<1$ then

$$A\leqslant\sum_{j=-\infty}^{k}\|u_j\|_1\|\varphi_j\|_\infty.$$

Since u_j is constant on the cosets of G_j we have

$$\|u_j\|_1=\int_G|u(z,j)|\,d\mu(z)=\sum_{l=0}^{\infty}(m_j)^{-1}|u(z_{l,j},j)|$$
$$\leqslant\Big(\sum_{l=0}^{\infty}|(m_j)^{-1}u(z_{l,j},j)|^p\Big)^{1/p}$$
$$=\Big((m_j)^{1-p}\sum_{l=0}^{\infty}(m_j)^{-1}|u(z_{l,j},j)|^p\Big)^{1/p}$$
$$=(m_j)^{1/p-1}\Big(\int_G|u(z,j)|^p\,d\mu(z)\Big)^{1/p}.$$

Hence, if $0<p<1$ then

$$A\leqslant C\sum_{j=-\infty}^{k}(m_j)^{1/p}\|u_j\|_p\leqslant C\|u\|_{\alpha,p,q}\sum_{j=-\infty}^{k}(m_j)^{\alpha+1/p}<\infty.$$

Thus the series $\sum_{j=-\infty}^{k}u(\cdot,j)*\varphi_j(x)$ converges absolutely and it obviously converges to $u(x,k)$.

COROLLARY 2. *Let $0<p, q\leqslant 0$ and $\alpha>-1/p$. If $u\in A(\alpha,p,q)$ then there exist constants $\lambda_{l,j}$ with $l\in Z_+$ and $j\in Z$, so that u can be represented as*

$$u(x,k)=\sum_{j=-\infty}^{k}\sum_{l=0}^{\infty}(m_j)^{\alpha-1+1/p}\lambda_{l,j}\,\varphi_j(x-z_{l,j}).$$

Moreover, $\|\lambda\|_{p,q}\leqslant C\|u\|_{\alpha,p,q}$.

Proof. According to Theorem 7 we have

$$u(x,k)=\sum_{j=-\infty}^{k}\sum_{l=0}^{\infty}\int_{G_{l,j}}u(z,j)\,\varphi_j(x-z)\,d\mu(z)$$
$$=\sum_{j=-\infty}^{k}\sum_{l=0}^{\infty}(m_j)^{\alpha-1+1/p}\lambda_{l,j}\,\varphi_j(x-z_{l,j}),$$

where $\lambda_{l,j}=(m_j)^{-\alpha-1/p}u(z_{l,j},j)$. An easy computation shows that

$$\|\lambda\|_{p,q}=\Big(\sum_{j=-\infty}^{k}\big((m_j)^{-\alpha}\|u_j\|_p\big)^q\Big)^{1/q}\leqslant\|u\|_{\alpha,p,q}.$$

We now state a converse to this corollary. Its proof will be omitted, because it is virtually the same as the proof of Theorem 6(b).

THEOREM 8. *Let* $0<p,\, q\leqslant\infty$ *and* $\alpha>0$. *If* $\lambda_{l,j}$ *satisfies* $\|\lambda\|_{p,q}<\infty$ *and if* $u\colon G\times \mathbf{Z}\to \mathbf{C}$ *is defined by*

$$u(x,k)=\sum_{j=-\infty}^{k}\sum_{l=0}^{\infty}(m_j)^{\alpha-1+1/p}\lambda_{l,j}\varphi_j(x-z_{l,j}),$$

then the series defining $u(x,k)$ *converges absolutely for every* $x\in G$ *and uniformly on each set*

$$E_{k_0}:=\{(x,k);\ x\in G \text{ and } -\infty<k\leqslant k_0\},\qquad k_0\in \mathbf{Z}.$$

Moreover, $\|u\|_{\alpha,p,q}\leqslant C\|\lambda\|_{p,q}$.

Remark 3. Corollary 2 and Theorem 8 are the analogue on G of Theorem (1.10) in [12] or Proposition 4 in [3].

We now turn to the final results of this paper which, in fact, are its *raison d'être*, namely, a description of the regularization of the elements of a Besov space $\dot{B}(\alpha,p,q)$.

THEOREM 9. *Let* $0<p,\, q\leqslant\infty$ *and* $\alpha>0$.

(a) *If* $f\in\dot{B}(-\alpha,p,q)$ *then its regularization* F *belongs to* $A(\alpha,p,q)$ *and* $\|F\|_{\alpha,p,q}\leqslant C\|f\|_{\dot{B}(-\alpha,p,q)}$.

(b) *If* $F\in A(\alpha,p,q)$ *then for each* $k\in\mathbf{Z}$, $F_k\in\dot{B}(-\alpha,p,q)$ *and* $\|F_k\|_{\dot{B}(-\alpha,p,q)}\leqslant C\|F\|_{\alpha,p,q}$. *Moreover, if* f *denotes the boundary value of* F, *i.e., if* $f=\lim F_k$ *in* $\mathscr{S}'(G)$, *then* $f\in\dot{B}(-\alpha,p,q)$ *and* $\|f\|_{\dot{B}(-\alpha,p,q)}\leqslant C\|F\|_{\alpha,p,q}$.

Proof. (a) If $f\in\dot{B}(-\alpha,p,q)$ then, according to Theorem 6(a), there exist constants $\lambda_{l,j}$ such that $\|\lambda\|_{p,q}\leqslant C\|f\|_{\dot{B}(-\alpha,p,q)}$ and there exist $(\alpha+1/p,\infty)$ atoms $a_{l,j}$ with $\operatorname{supp}(a_{l,j})\subset z_{l,j}+G_{j-1}$ such that

$$f=\sum_{j=-\infty}^{\infty}\sum_{l=0}^{\infty}\lambda_{l,j}a_{l,j}.$$

Also (compare the proof of Theorem 6(a), replacing φ_n by Δ_n there), for each $n\in\mathbf{Z}$ we have

$$\|a_{l,j}*\Delta_n\|_p=0\qquad \text{if } j>n,$$
$$\|a_{l,j}*\Delta_n\|_p\leqslant C(m_j)^{-\alpha}\qquad \text{if } j\leqslant n.$$

An argument like in the proof of Theorem 6(b) then shows that for all p,q with $0<p,\, q\leqslant\infty$ we have

$$\Big(\sum_{n=-\infty}^{\infty}\big((m_n)^{\alpha}\|f*\Delta_n\|_p\big)^q\Big)^{1/q}\leqslant C\|\lambda\|_{p,q}.$$

Thus we may conclude that

$$\|F\|_{\alpha,p,q}\leqslant C\|f\|_{\dot{B}(-\alpha,p,q)}.$$

(b) Let $F_k=F(\cdot,k)=f*\Delta_k$. For $n\in\mathbf{Z}$ with $n>k$ we have $F_k*\varphi_n=0$, whereas $F_k*\varphi_n=F_n-F_{n-1}=f*\varphi_n$ if $n\leqslant k$. Therefore,

$$\|F_k\|_{\dot{B}(-\alpha,p,q)}=\Big(\sum_{n=-\infty}^{\infty}\big((m_n)^{-\alpha}\|F_k*\varphi_n\|_p\big)^q\Big)^{1/q}$$
$$=\Big(\sum_{n=-\infty}^{k}\big((m_n)^{-\alpha}\|f*\varphi_n\|_p\big)^q\Big)^{1/q}.$$

Using the representation for the functions F_k given in Corollary 2, a proof like that for Theorem 6(b) shows that for each $k\in\mathbf{Z}$, $\|F_k\|_{\dot{B}(-\alpha,p,q)}\leqslant C\|F\|_{\alpha,p,q}$. Therefore,

$$\|f\|_{\dot{B}(-\alpha,p,q)}=\Big(\sum_{n=-\infty}^{\infty}\big((m_n)^{-\alpha}\|f*\varphi_n\|_p\big)^q\Big)^{1/q}\leqslant C\|F\|_{\alpha,p,q},$$

which completes the proof of Theorem 9.

We can reformulate Theorem 9 as follows.

COROLLARY 3. *Let* $0<p,\, q\leqslant\infty$ *and* $\alpha>0$. *For any distribution* $f\in\mathscr{S}'(G)$ *we have*

$$\|f\|_{\dot{B}(-\alpha,p,q)}=\Big(\sum_{n=-\infty}^{\infty}\big((m_n)^{-\alpha}\|f*\varphi_n\|_p\big)^q\Big)^{1/q}$$
$$\approx\Big(\sum_{n=-\infty}^{\infty}\big((m_n)^{-\alpha}\|f*\Delta_n\|_p\big)^q\Big)^{1/q}.$$

It is easy to see, by means of an example, that Theorem 9(a) does not extend to $\alpha\leqslant 0$. Namely, if we take $f=\Delta_0$ then

$$\|\Delta_0\|_{\dot{B}(-\alpha,p,q)}=\Big(\sum_{k=-\infty}^{\infty}\big((m_k)^{-\alpha}\|\Delta_0*\varphi_k\|_p\big)^q\Big)^{1/q}$$
$$=\Big(\sum_{k=-\infty}^{0}\big((m_k)^{-\alpha}\|\varphi_k\|_p\big)^q\Big)^{1/q}$$
$$\leqslant C\Big(\sum_{k=-\infty}^{0}(m_k)^{(-\alpha+1-1/p)q}\Big)^{1/q}<\infty,$$

in case $-\alpha+1-1/p>0$ and $q>0$. Also, if F denotes the regularization of

Δ_0 then

$$\|F\|_{\alpha,p,q} = \Big(\sum_{k=-\infty}^{\infty} \big((m_k)^{-\alpha}\|\Delta_0 * \Delta_k\|_p\big)^q\Big)^{1/q}$$
$$= \Big(\sum_{k=-\infty}^{0} \big((m_k)^{-\alpha}\|\Delta_k\|_p\big)^q + \sum_{k=1}^{\infty} \big((m_k)^{-\alpha}\|\Delta_0\|_p\big)^q\Big)^{1/q}$$
$$= \Big(\sum_{k=-\infty}^{0} (m_k)^{(-\alpha+1-1/p)q} + \sum_{k=1}^{\infty} (m_k)^{-\alpha q}\Big)^{1/q}$$

and the last series diverges when $\alpha \leqslant 0$ and $q > 0$. Thus if $0 < p \leqslant \infty$, $q > 0$ and $\alpha < \min(0, 1-1/p)$ then $\Delta_0 \in \dot{B}(-\alpha, p, q)$ and $F \notin A(\alpha, p, q)$.

If $1 \leqslant p, q \leqslant \infty$ then Theorem 9(b) can be extended to values of α with $\alpha > -1/p$ in the sense that in this case the boundary value of a given $F \in A(\alpha, p, q)$ can be identified as a bounded linear functional on the mean oscillation space $\mathrm{MO}(\alpha-1/p', p', q')$. We first present a simple lemma.

Lemma 1. *Let $0 < p, q \leqslant \infty$ and $\alpha > -1$. For each $k \in \mathbf{Z}$ the function Δ_k belongs to* $\mathrm{MO}(\alpha, p, q)$ *and* $\mathrm{MO}(\alpha, p, q)(\Delta_k) \leqslant C(m_k)^{\alpha+1}$.

Proof. We first estimate $\mathrm{osc}_p(\Delta_k, n)$ for $n \in \mathbf{Z}$. If $n \leqslant k$ then $G_k \subset G_n$ and we have

$$(\Delta_k)_{G_{0,n}} = m_n \int_{G_n} \Delta_k(t)\,d\mu(t) = m_n,$$

whereas for $l \geqslant 1$ we have

$$(\Delta_k)_{G_{l,n}} = m_n \int_{z_{l,n}+G_n} \Delta_k(t)\,d\mu(t) = 0.$$

A simple computation shows that $\mathrm{osc}_p(\Delta_k, n) \leqslant 2m_n$. If $n > k$ and if $z_{l,n} + G_n \subset G_k$ then $(\Delta_k)_{G_{l,n}} = m_k$, whereas if $(z_{l,n}+G_n) \cap G_k = \emptyset$ we have $(\Delta_k)_{G_{l,n}} = 0$. Thus we see immediately that $\mathrm{osc}_p(\Delta_k, n) = 0$. Therefore,

$$\mathrm{MO}(\alpha, p, q)(\Delta_k) = \Big(\sum_{n=-\infty}^{k} \big((m_n)^{\alpha}\,\mathrm{osc}_p(\Delta_k, n)\big)^q\Big)^{1/q}$$
$$\leqslant 2\Big(\sum_{n=-\infty}^{k} (m_n)^{(\alpha+1)q}\Big)^{1/q} \leqslant C(m_k)^{\alpha+1},$$

because $(\alpha+1)q > 0$.

Theorem 10. *Let $1 \leqslant p, q \leqslant \infty$ and $\alpha > -1/p$ and let $F \in A(\alpha, p, q)$. For each $k \in \mathbf{Z}$ define*

$$\Lambda_k\colon \mathrm{MO}(\alpha-1/p', p', q') \to \mathbf{C}$$

by $\Lambda_k(g) = \langle F_k, g\rangle$. Then Λ_k is a continuous linear functional on $\mathrm{MO}(\alpha-1/p', p', q')$ *and*

$$|\Lambda_k(g)| \leqslant C\|F\|_{\alpha,p,q}\,\mathrm{MO}(\alpha-1/p', p', q')(g).$$

Moreover, if we define $\Lambda(g) = \lim_{k\to\infty}\Lambda_k(g)$, then $\Lambda \in \mathrm{MO}(\alpha-1/p', p', q')^*$ and

$$|\Lambda(g)| \leqslant C\|F\|_{\alpha,p,q}\,\mathrm{MO}(\alpha-1/p', p', q')(g).$$

In addition, we have $F(x, n) = \Lambda(\tau_x \Delta_n)$ for $(x, n) \in G \times \mathbf{Z}$.

Proof. According to Corollary 2, each F_k can be represented as

$$F_k(x) = \sum_{j=-\infty}^{k} \sum_{l=0}^{\infty} \lambda_{l,j}(m_j)^{\alpha-1/p'} \varphi_j(x - z_{l,j}),$$

where $\lambda_{l,j}$ satisfies $\|\lambda\|_{p,q} \leqslant C\|F\|_{\alpha,p,q}$. Thus, for each $g \in \mathrm{MO}(\alpha-1/p', p', q')$ we have

$$|\Lambda_k(g)| = \Big|\sum_{j=-\infty}^{k} \sum_{l=0}^{\infty} \lambda_{l,j}(m_j)^{\alpha-1/p'} \int_G \varphi_j(x-z_{l,j})\,g(x)\,d\mu(x)\Big|$$
$$\leqslant \sum_{j=-\infty}^{k} \sum_{l=0}^{\infty} |\lambda_{l,j}|\,(m_j)^{\alpha-1/p'}\,|\varphi_j * g(z_{l,j})|.$$

Since $1 \leqslant p, q \leqslant \infty$, Hölder's inequality implies that

$$|\Lambda_k(g)| \leqslant \|\lambda\|_{p,q}\Big(\sum_{j=-\infty}^{k} (m_j)^{(\alpha-1/p')q'}\Big(\sum_{l=0}^{\infty} |(\varphi_j * g)(z_{l,j})|^{p'}\Big)^{q'/p'}\Big)^{1/q'}.$$

We now derive an inequality for the inner sum. For each $l \geqslant 0$ we have

$$|\varphi_j * g(z_{l,j})| = |(\Delta_j * g)(z_{l,j}) - (\Delta_{j-1} * g)(z_{l,j})|$$
$$\leqslant m_j \int_{z_{l,j}+G_j} |g(t) - (\Delta_{j-1} * g)(z_{l,j})|\,d\mu(t)$$
$$\leqslant C m_{j-1} \int_{z_{l,j}+G_{j-1}} |g(t) - (\Delta_{j-1} * g)(z_{l,j})|\,d\mu(t).$$

Therefore

$$\Big(\sum_{l=0}^{\infty} |(\varphi_j * g)(z_{l,j})|^{p'}\Big)^{1/p'}$$
$$\leqslant C\Big(\sum_{l=0}^{\infty} \Big(m_{j-1} \int_{z_{l,j}+G_{j-1}} |g(t) - (\Delta_{j-1} * g)(z_{l,j})|\,d\mu(t)\Big)^{p'}\Big)^{1/p'}.$$

Now we observe that each coset of G_{j-1} contains precisely m_j/m_{j-1} different cosets of G_j; hence

$$\Big(\sum_{l=0}^{\infty} |(\varphi_j * g)(z_{l,j})|^{p'}\Big)^{1/p'}$$
$$\leqslant C\left(\frac{m_j}{m_{j-1}} \sum_{s=0}^{\infty} \Big(m_{j-1} \int_{z_{s,j-1}+G_{j-1}} |g(t)-(\Delta_{j-1} * g)(z_{s,j-1})|\, d\mu(t)\Big)^{p'}\right)^{1/p'}$$
$$\leqslant C(m_j/m_{j-1})^{1/p'} \operatorname{osc}_{p'}(g, j-1).$$

Consequently,

$$|\Lambda_k(g)| \leqslant C\|\lambda\|_{p,q}\Big(\sum_{j=-\infty}^{k} \big((m_j)^{\alpha}(m_{j-1})^{-1/p'} \operatorname{osc}_{p'}(g, j-1)\big)^{q'}\Big)^{1/q'}$$
$$\leqslant C\|F\|_{\alpha,p,q} \operatorname{MO}(\alpha-1/p', p', q')(g).$$

Clearly, with Λ as defined in the theorem, Λ satisfies the inequality stated there. Finally, according to Lemma 1, each Δ_n belongs to $\operatorname{MO}(\alpha-1/p', p', q')$ and a straightforward computation shows that

$$\Lambda(\tau_x \Delta_n) = \lim_{k\to\infty} \Lambda_k(\tau_x \Delta_n) = F(x, n)$$

for every $(x, n) \in G \times \boldsymbol{Z}$, which completes the proof of Theorem 10.

Remark 4. In view of the identification of the mean oscillation spaces with certain $\dot{B}$-spaces, as given in Corollary 1, and the duality theorem for $\dot{B}$-spaces, as given in Theorem 2, Theorem 10, when restricted to $\alpha > 0$ and $1 \leqslant p, q < \infty$, agrees with Theorem 9(b).

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