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Invariant sub-σ-algebras for substitutions of constant length

by

MIECZYSŁAW K. MENTZEN (Toruń)

Abstract. For automorphisms arising from substitutions of constant length all factors are described.

Introduction. Let $T: (X, \mathcal{B}, \mu) \circlearrowleft$ be an ergodic automorphism of a Lebesgue space. Then an automorphism $\tau: (Y, \mathcal{C}, \nu) \circlearrowleft$ is called a *factor* of T whenever there exists a measure-preserving $f: (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ such that $\tau f = fT$. It is a general problem in ergodic theory to determine all factors (up to measure-theoretic isomorphism) for a given T.

In the present paper we deal with this problem for the class of substitutions of constant length [1], [3], [8]. From the ergodic theory point of view the automorphisms generated by substitutions are finite extensions of automorphisms with rational pure point spectra, i.e. $X = X' \times Z_n$, $\mu = \mu' \times \nu_n$ $(\nu_n(i) = 1/n)$, $T = T'_\psi$, where $\psi: X' \to S(n)$ is measurable and $T': (X', \mathcal{B}', \mu') \circlearrowleft$ has (rational) pure point spectrum, $T(x', i) = T'_\psi(x', i) = (T'(x'), \psi(x')(i))$. Therefore T' is a factor of T. This factor has discrete spectrum. An interesting question is whether there are other factors but with partly continuous spectrum. In 1972 Kamae [3] introduced an algorithm to produce some new substitutions starting with a fixed one. It turns out that his procedure yields some nontrivial factors. The point is that, in fact, it gives all factors with partly continuous spectrum (Theorem 8).

It is well known that to determine all factors (up to m.t. isomorphism) it is enough to compute all T-invariant sub-σ-algebras $\mathcal{C} \subset \mathcal{B}$. However, it is then possible that for some different $\mathcal{C}, \mathcal{C}' \subset \mathcal{B}$ the corresponding factors $T: (X, \mathcal{C}, \mu) \circlearrowleft$ and $T: (X, \mathcal{C}', \mu) \circlearrowleft$ are isomorphic. If this is not the case we say that all factors are *canonical* [6]. We prove that the number of $\mathcal{C} \subset \mathcal{B}$ inducing factors with partly continuous spectrum is finite.

Preliminaries. Let $\mathcal{X} = (X, \mathcal{B}, \mu, T)$, where T is an automorphism of the Lebesgue space $(X, \mathcal{B}, \mu)$, be an ergodic dynamical system. If $\mathcal{Y}$

$= (Y, \mathscr{C}, \nu, S)$ is another dynamical system then we call $\mathscr{Y}$ a *factor* of $\mathscr{X}$ if there is a measure-preserving map φ from X onto Y satisfying $\varphi T = S\varphi$. If $\mathscr{Y}$ is a factor of $\mathscr{X}$ then there exists a Lebesgue space $(Z, \mathscr{F}, \alpha)$ such that $\mathscr{X}$ is isomorphic to the system $(Y\times Z, \mathscr{C}\otimes\mathscr{F}, \nu\times\alpha, \bar{T})$ with $\bar{T}$ defined by

$$\bar{T}(y, z) = (S(y), T_y(z)),$$

where $\{T_y\}_{y\in Y}$ is a measurable familly of automorphisms of Z (see [6]). In what follows we will write T instead of $\bar{T}$. Moreover, if no confusion can arise we will omit $\mathscr{B}$, μ and write $\mathscr{X} = (X, T)$.

Denote by $L^2(\mathscr{X})$ the Hilbert space of complex-valued square integrable functions on X with inner product $\langle f, g\rangle = \int f\bar{g}\,d\mu$. We define U_T: $L^2(\mathscr{X}) \to L^2(\mathscr{X})$ by the formula $U_T(f) = f\circ T$. Then U_T is a unitary operator on $L^2(\mathscr{X})$. Denote by $S_p(T)$ the discrete part of the spectrum of U_T. We will say "spectrum of T" instead of "spectrum of U_T" and "discrete part of the spectrum of T" instead of "discrete part of the spectrum of U_T". If the set of all eigenfunctions of T is linear dense in $L^2(\mathscr{X})$ then we call $\mathscr{X}$ (or T) a *system with discrete spectrum*, or a *system with pure point spectrum*. If $S_p(T) \neq \{1\}$ and T is not with discrete spectrum then T is said to be a *system with partly continuous spectrum*.

DEFINITION 1 ([6]). A factor $\mathscr{Y}$ of an ergodic dynamical system $\mathscr{X}$ is said to be *canonical* if any two homomorphisms φ, ψ: $\mathscr{X}\to\mathscr{Y}$ satisfy

$$\varphi^{-1}(\varepsilon_Y) = \psi^{-1}(\varepsilon_Y),$$

where ε_Y denotes the point partition of Y.

A *canonical system* is a dynamical system $\mathscr{Y}$ such that for any ergodic system $\mathscr{X}$, if $\mathscr{Y}$ is a factor of $\mathscr{X}$ then $\mathscr{Y}$ is a canonical factor of $\mathscr{X}$.

LEMMA 2 ([6], [4]). *An ergodic dynamical system $\mathscr{Y}$ is a canonical system if and only if $\mathscr{Y}$ has discrete spectrum.*

DEFINITION 3. If $\mathscr{D} = (D, \mathscr{C}, \nu, \tau)$ is a factor with discrete spectrum of an ergodic dynamical system $\mathscr{X}$ via a homomorphism φ, and a natural number c satisfies

$$c = \operatorname{card}\varphi^{-1}(d) \quad \text{for a.e. } d\in D$$

then $\mathscr{X}$ is called a *c-extension* of $\mathscr{D}$.

LEMMA 4 ([7]). *If $\mathscr{X}$ is an ergodic dynamical system and $\mathscr{X}$ is a c-extension of its maximal factor with discrete spectrum then the maximal sequence entropy of $\mathscr{X}$ is equal to* $\log(c)$. ∎

LEMMA 5. *Let $\mathscr{X} = (X, T)$ be an ergodic c-extension of its maximal factor $\mathscr{D} = (D, \tau)$ with discrete spectrum. If $\mathscr{Y} = (Y, S)$ is a factor of $\mathscr{X}$ then:*

(i) *$\mathscr{Y}$ is a b-extension of its maximal factor with discrete spectrum and $b\leqslant c$.*

(ii) *There exists a factor $\mathscr{X}_0$ of $\mathscr{X}$ such that $\mathscr{X}_0$ is a b-extension of $\mathscr{D}$ and $\mathscr{Y}$ is a factor of $\mathscr{X}_0$.*

Proof. Assertion (i) is an immediate consequence of Lemma 4. To prove (ii) let us assume that $X = D\times Z$, $Y = E\times V$ and $T(d, z) = (\tau(d), T_d(z))$, $S(e, v) = (\gamma(e), S_e(v))$, where (E, γ) is a maximal factor of $\mathscr{Y}$ with discrete spectrum.

Define a T-invariant partition ξ on X by the formula

$$\xi = (\varepsilon_D\times\nu_Z)\vee\varphi^{-1}(\varepsilon_Y),$$

where φ is a homomorphism from X onto Y. Let $X_0 = X|_\xi$, $T_0 = T|_\xi$, $\mathscr{X}_0 = (X_0, T_0)$. Obviously, $\mathscr{X}_0$ is a factor of $\mathscr{X}$, $\mathscr{Y}$ is a factor of $\mathscr{X}_0$ and $\mathscr{X}_0$ is a c_0-extension of $\mathscr{D}$ for some $c_0\leqslant c$. By Lemma 4

$$c_0 \geqslant b. \tag{1}$$

On the other hand, the partitions $\varepsilon_D\times\nu_Z$ and $\varphi^{-1}(\varepsilon_E\times\nu_V)$ satisfy

$$\varepsilon_D\times\nu_Z > \varphi^{-1}(\varepsilon_E\times\nu_V).$$

It follows that for each $d\in D$ there exists $e\in E$ such that $\{d\}\times Z\subset \varphi^{-1}(\{e\}\times V)$. Thus

$$\{d\}\times Z = (\{d\}\times Z)\cap\varphi^{-1}(\{e\}\times V) = \bigcup_{v\in V}\big((\{d\}\times Z)\cap\varphi^{-1}(e, v)\big).$$

This implies that almost each atom of the partition $\varepsilon_D\times\nu_Z$ consists of at most $\operatorname{card} V = b$ atoms of the partition ξ. Hence $c_0\leqslant b$. In view of (1), $c_0 = b$. ∎

LEMMA 6. *If (X, T) is an ergodic dynamical system with partly continuous spectrum and X_0 is an ergodic component of T^h, where h is a natural number, then (X_0, T^h) is a system with partly continuous spectrum.*

Proof. Denote by ξ the T-invariant partition of (X, T) corresponding to its maximal factor with discrete spectrum and by ξ_0 the T^h-invariant partition of (X_0, T^h) corresponding to its maximal factor with discrete spectrum. We intend to prove that $\xi|_{X_0} > \xi_0$.

If f_{λ_0} is an eigenfunction of $U_{T^h}|_{L^2(X_0, T^h)}$ corresponding to the eigenvalue λ_0, then we can extend f_{λ_0} to an eigenfunction f of U_T in the following way: Put $\lambda^h = \lambda_0$. If

$$X = X_0\cup TX_0\cup\ldots\cup T^{g-1}X_0, \quad T^iX_0\cap T^jX_0 = \emptyset,\ i\neq j,$$

where g divides h, then we define

$$f(x) = \lambda^i f_{\lambda_0}(T^{-i}x) \quad \text{if } x\in T^iX_0.$$

Then for $x \in T^i X_0$

$$f(Tx) = \lambda^{i+1} f_{\lambda_0}(T^{-i-1}(Tx)) = \lambda \cdot \lambda^i f_{\lambda_0}(T^{-i}x) = \lambda f(x).$$

Hence $f = f_\lambda$ is an eigenfunction of T and $f_\lambda|X_0 = f_{\lambda_0}$.

This implies that each eigenfunction of $U_T|_{L^2(X_0, T^h)}$ is constant on atoms of $\xi|_{X_0}$, so $\xi|_{X_0} > \xi_0$.

Obviously, $\xi|_{X_0} \neq \varepsilon_{X_0}$. Therefore $\xi_0 \neq \varepsilon_{X_0}$ and (X_0, T^h) has partly continuous spectrum.

Now, we define the notion of dynamical system arising from a substitution of constant length.

Let r be a natural number. The set $\{0, 1, \ldots, r-1\}$ will be denoted by N_r. Put $N_r^* = \bigcup_{n \geq 1} N_r^n$. The elements of N_r^* are called *blocks*. If $B \in N_r^*$, $B = (b_0 b_1 \ldots b_{n-1})$, then $B[s, t] = (b_s \ldots b_t)$, $B[s, s] = B[s]$, $0 \leq s \leq t \leq n-1$, and the number $|B| = n$ is called the *length* of B. Similarly we define $x[s, t]$ for $x \in N_r^{\mathbf{Z}}$.

In the sequel we will use the metric d on N_r^n, $n \geq 1$, defined by

$$d(B, C) = \operatorname{card}\{0 \leq t \leq n-1 \colon B[t] \neq C[t]\}/n \quad \text{for } B, C \in N_r^n.$$

We assume that the reader is acquainted with [1]. In this paper we only recall some facts connected with this subject.

Let λ be a natural number, $\lambda \geq 2$, and let $\theta \colon N_r \to N_r^\lambda$. There is a natural extension of θ to a map from N_r^n into $N_r^{\lambda n}$ and also to a map from $N_r^{\mathbf{Z}}$ into itself (denoted by θ as well) given by

$$\theta(B) = \theta(b_0)\theta(b_1)\ldots\theta(b_{n-1}) \quad \text{if } B = (b_0 \ldots b_{n-1}) \in N_r^n,$$

$$\theta(x) = \ldots\theta(b_{-1})\theta(b_0)\theta(b_1)\ldots \quad \text{if } x = \ldots b_{-1} b_0 b_1 \ldots \in N_r^{\mathbf{Z}},$$

where in the latter case by convention the 0th symbol of $\theta(x)$ coincides with the initial symbol of $\theta(b_0)$. Then θ^k denotes the k-fold composition of θ.

Assume that $\theta \colon N_r \to N_r^\lambda$ is a one-to-one map. If there exists a natural number m such that for any $i, j \in N_r$ and for some $0 \leq s \leq \lambda^m - 1$ the equality $\theta^m(i)[s] = j$ holds, then θ is called a *substitution of constant length* λ *on* r *symbols*.

For such a substitution there exist i and j in N_r such that for some t the first symbol of $\theta^t(j)$ is j and the last symbol of $\theta^t(i)$ is i (such symbols are said to be the *cyclic first symbol* and the *cyclic last symbol* respectively).

Let $x_0 \in N_r^{\mathbf{Z}}$ be defined by the formula

$$x_0[-\lambda^{tn}, \lambda^{tn}-1] = \theta^{tn}(ij), \quad n = 1, 2, \ldots$$

Then x_0 is a fixed point of θ^t. If we replace θ^t by θ and λ^t by λ then the number t in the definition of the cyclic first symbol and the cyclic last symbol is equal to 1, the definition of x_0 is

$$x_0[-\lambda^n, \lambda^n-1] = \theta^n(ij), \quad n = 1, 2, \ldots,$$

and x_0 is a fixed point of θ.

Let T be the shift on $N_r^{\mathbf{Z}}$, i.e. T is defined by $T(x) = y$, where $y[n] = x[n+1]$, $n \in \mathbf{Z}$.

Denote by $X(\theta) = \overline{\{T^n(x_0) \colon n \in \mathbf{Z}\}}$ the closure in $N_r^{\mathbf{Z}}$ of the orbit of x_0 via T. Then (see [1]) $(X(\theta), T)$ is a uniquely ergodic dynamical system equipped with a unique T-invariant probability measure μ. Let $\mathscr{X}(\theta) = (X(\theta), \mu, T)$ and $S_p(\theta) = S_p(T)$. Dekking [2] has shown that $S_p(\theta) = \{\exp(2\pi ik/(h\lambda^n)) \colon k \in \mathbf{Z},\ n \in N\}$, where h is a natural number called the *height* of θ and $h \leq r$, g.c.d. $(h, \lambda) = 1$ (Queffelec [8] has shown that h divides $\lambda - 1$). If $h = 1$ then the substitution θ is called *pure*. In this case θ is a c-extension of the system $\mathscr{Z}(\lambda) = (Z(\lambda), \nu, \tau)$ ($Z(\lambda)$ denotes the topological group of λ-adic numbers, ν is the Haar measure and τ denotes the homeomorphism of $Z(\lambda)$ corresponding to the addition of the unit element) for some natural number c (see [2]). We call $c = c(\theta)$ the *column number* of θ.

If $h > 1$ then $\mathscr{X}(\theta)$ is isomorphic to

$$(Z_h \times X(\eta), \sigma), \tag{2}$$

where $Z_h = \mathbf{Z}/h\mathbf{Z}$, η is a pure substitution of length λ and

$$\sigma(g, x) = \begin{cases} (g+1, x) & \text{if } g = 0, 1, \ldots, h-2, \\ (0, Tx) & \text{if } g = h-1. \end{cases} \tag{3}$$

(We will use the symbol T to denote the shift transformation on $X(\eta)$ for any substitution η.)

The substitution η is said to be the *pure base* of θ.

There are algorithms to calculate the height and column number of θ (see [2]). Another definition of the column number of a pure substitution θ of length λ on r symbols is

$$c(\theta) = \min_{n \geq 1} \min_{0 \leq t \leq \lambda^n - 1} \operatorname{card}\{\theta^n(0)[t], \theta^n(1)[t], \ldots, \theta^n(r-1)[t]\}. \tag{4}$$

By the column number of any substitution we mean the column number of its pure base.

If n is a natural number then each block $\theta^n(i)$, where $i \in N_r$, is said to be an *n-symbol*.

It is an easy consequence of the definition of $X(\theta)$ that each $x \in X(\theta)$ is, for any natural number n, an infinite concatenation of n-symbols, i.e.

$$x = \ldots\theta^n(i_{-1})\theta^n(i_0)\theta^n(i_1)\ldots \tag{5}$$

Kamae [3] has shown that $X(\theta)$ is, for any n, the disjoint union of the sets $X_n^k = T^k(\theta^n(X(\theta)))$, $k = 0, 1, \ldots, \lambda^n - 1$. This implies that the representation (5) is unique.

If $x, y \in X(\theta)$ then we say that x and y *have the same structure of n-symbols* if there is $0 \leqslant k \leqslant \lambda^n - 1$ such that $x, y \in X_n^k$ (see Fig. 1).

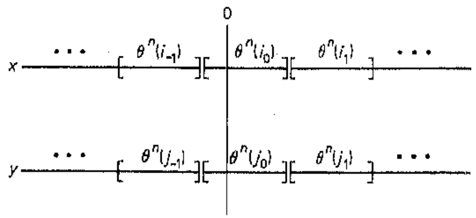

Fig. 1

We will deal with substitutions with partly continuous spectrum. It has been proved in [5] that if θ is such a substitution then for $i, j \in N_r$, $i \neq j$, there is a natural number $\beta > 0$ such that for all $n > 0$

$$d(\theta^n(i), \theta^n(j)) \geqslant \beta. \tag{6}$$

The factors. Assume that θ is a substitution of length λ on r symbols. The construction below (used by Kamae [3] to calculate the height of a substitution) gives a factor $\mathscr{F}$ of $\mathscr{X}(\theta)$ such that either

$$S_p(\mathscr{F}) \supset \{\exp(2\pi i k/\lambda^n)\colon k \in \boldsymbol{Z}, n \in \boldsymbol{N}\}, \quad \text{or}$$

$$S_p(\mathscr{F}) \subset \{\exp(2\pi i k/h)\colon k \in \boldsymbol{Z}\},$$

where h is the height of θ.

DEFINITION 7 ([3]). An equivalence relation $\sim$ on N_r is said to be *θ-consistent* if $i \sim j$ implies $\theta(i)[t] \sim \theta(j)[t]$, $t = 0, 1, \ldots, \lambda - 1$.

Let $\sim$ be a θ-consistent relation on N_r. Assume that $\tau\colon N_s \to N_r/\sim$ is a bijection. Then the substitution $\tilde{\theta}$ on N_s defined by

$$\tilde{\theta}(i) = \tau^{-1}\theta\tau(i) \tag{7}$$

has length λ and $\mathscr{X}(\tilde{\theta})$ is a factor of $\mathscr{X}(\theta)$. If the height of θ is 1 and $s > 1$ then $S_p(\tilde{\theta}) = S_p(\theta)$. We can find examples such that $\mathscr{X}(\tilde{\theta})$ has partly continuous spectrum as well as ones with pure point spectrum. If $h > 1$ then there is a relation $\sim$ such that $\mathscr{X}(\tilde{\theta})$ is a rotation on Z_h ([3]).

The factors of $\mathscr{X}(\theta)$ having pure point spectrum are well known: each of

them is isomorphic to some rotation on the character group $(S_p(\mathscr{F}))^{\wedge}$ of $S_p(\mathscr{F})$, i.e.

$$\mathscr{F} \cong \mathscr{Z}(p) \times \mathscr{Z}_m, \tag{8}$$

where $pq = \lambda$ for some q and m divides $q^t h$ for some t.

THEOREM 8. *Assume that θ is a substitution of constant length λ and the height of θ is equal to h. Then the only factors of $\mathscr{X}(\theta)$ are either substitutions defined by (7) or factors with pure point spectrum of the form (8).*

In order to prove Theorem 8 we will need some further facts. First we describe (see [2]) a construction of the pure base of a substitution θ of height $h > 1$.

Let x_0 be a fixed point of θ. Set $M = \{B \in N_r^h\colon$ there is an integer n such that $B = x_0[nh, (n+1)h - 1]\}$.

Assume that $\psi\colon N_s \to M$ is a bijection. We define the substitution η on N_s putting

$$\eta(j) = \psi^{-1}\theta\psi(j),$$

i.e. if $\psi(j) = j_1 \ldots j_h$ then

$$\eta(j) = \psi^{-1}\theta(j_1 \ldots j_h) = \psi^{-1}(k_1 \ldots k_h k_{h+1} \ldots k_{2h} k_{2h+1} \ldots k_{\lambda h})$$

$$= \psi^{-1}(k_1 \ldots k_h)\psi^{-1}(k_{h+1} \ldots k_{2h}) \ldots \psi^{-1}(k_{(\lambda-1)h+1} \ldots k_{\lambda h}).$$

Then η is the pure base of θ.

The elements of M satisfy ([2]):

(9) If $B, C \in M$ and $B[u] = C[v]$ then $u = v$.

It follows from (9) that each sequence x from $X(\theta)$ is a unique concatenation of blocks from M.

We say that x and y *have the same structure of M-blocks* if there is $0 \leqslant t \leqslant h-1$ such that $x[nh+t, (n+1)h+t-1] \in M$ and $y[nh+t, (n+1)h+t-1] \in M$ for any $n \in \boldsymbol{Z}$.

If we denote by ξ the partition of $X(\theta)$ corresponding to the (canonical) factor $\mathscr{Y}(\lambda)$ then almost each atom of ξ consists of ch points, where h is the height of θ. Take $P \in \xi$. Then

$$\operatorname{card} P[0] = ch \tag{10}$$

($P[0]$ denotes the set $\{x_1[0], \ldots, x_{ch}[0]\}$, where $P = \{x_1, \ldots, x_{ch}\}$). Indeed, (10) easily follows from (9) and [2]. Set

$$\mathscr{A} = \{A \subset N_r\colon \operatorname{card} A = ch \text{ and there is } P \in \xi \text{ such that}$$
$$A = P[0] \text{ and } \mu\{P\colon P[0] = A\} = \mu(A) > 0\}.$$

DEFINITION 9. A pair $(i, j) \in N_r \times N_r$ is called *essential* if there exists a set $A \in \mathscr{A}$ containing i and j.

LEMMA 10. (a) *If $P \in \xi$ then $\theta(P) \in \xi$.*

(b) *A pair (i, j) is essential if and only if for any $n \geq 1$ and any $0 \leq k \leq \lambda^n - 1$, $(\theta^n(i)[k], \theta^n(j)[k])$ is an essential pair.*

Proof. (a) Take $P \in \xi$, $P = \{x_1, \ldots, x_{ch}\}$. Then $x_1, \ldots, x_{ch}$ have the same structure of n-symbols for each $n \geq 1$, and so do $\theta(x_1), \ldots, \theta(x_{ch})$. Since θ is one-to-one, $\operatorname{card}\theta(P) = ch$. Thus $\theta(P) \in \xi$.

(b) Suppose (i, j) is an essential pair. Then there are $x, y \in P$ such that $x[0] = i$, $y[0] = j$. By (a), $\theta(x), \theta(y) \in P \in \xi$ and $\theta(i)[k] = T^k(\theta(x)[0])$, $\theta(j)[k] = T^k(\theta(y)[0])$. The proof of the "if" part is similar.

LEMMA 11. *The right-hand side of (4) is equal to ch.*

Proof. It follows from (9) that each $\theta^n(i)$, $n \in N$, $i \in N_r$, is a unique concatenation of the form

$$\theta^n(i) = B_0 B_1 \ldots B_{s_i}[t, \lambda^n - t - 1],$$

where $B_0, B_1, \ldots, B_{s_i} \in M$ and $t = t(n, i)$ depends on n and i, $0 \leq t \leq h-1$. Set

$$S_t^n = \{i \in N_r\colon\ t(n, i) = t\}.$$

The sets $\{S_t^n\}_{t=0}^{h-1}$ are pairwise disjoint and satisfy $S_0^n \cup S_1^n \cup \ldots \cup S_{h-1}^n = N_r$ for each n.

It is easy to calculate that for a pure substitution η on N_s

$$\lim_{n \to \infty} \operatorname{card}\{0 \leq p \leq \lambda^n - 1\colon\ \operatorname{card}\eta^n(N_s)[p] = c(\eta)\}/\lambda^n = 1. \tag{11}$$

This implies

$$c = \min_{n \geq 1} \min_{h \leq p \leq \lambda^n - h} \operatorname{card}\theta^n(S_t^n)[p] \quad \text{for each } t.$$

By virtue of (9)

$$\min_{n \geq 1} \min_{p \leq \lambda^n - 1} \operatorname{card}\theta^n(N_r)[p] = ch \ \blacksquare$$

LEMMA 12. *There exists $w > 0$ such that for any two distinct symbols i and j from N_r and for n large enough*

$$\operatorname{card}\{0 \leq t \leq \lambda^n - 1\colon\ (\theta^n(i)[t], \theta^n(j)[t]) \text{ is an essential pair}\} \geq w\lambda^n.$$

Proof. Let

$$b = \max\{\beta_{pq}\colon\ \beta_{pq} < 1,\ p \neq q,\ p, q \in N_r\}, \quad \text{where}$$
$$\beta_{pq} = \lim_{k \to \infty} d(\theta^k(p), \theta^k(q)).$$

Let $w = \beta/4$, where β satisfies (6) for all $i \neq j$. Take $i \neq j$. Assume that $d(\theta^n(i), \theta^n(j)) \to \beta_{ij} > 0$. Let $\delta > 0$, $\delta < (1-b)\beta/2$. If n is large enough then

$$\beta_{ij} \leq d(\theta^n(i), \theta^n(j)) < \beta_{ij} + \delta. \tag{12}$$

Set

$$\theta^n(i) = i_1 \ldots i_{\lambda^n}, \quad \theta^n(j) = j_1 \ldots j_{\lambda^n}.$$

By (12), there are indices $s_1, \ldots, s_u$, where $\beta_{ij}\lambda^n \leq u < (\beta_{ij} + \delta)\lambda^n$, such that $i_{s_m} \neq j_{s_m}$, $m = 1, \ldots, u$, and $i_p = j_p$ for $p \notin \{s_1, \ldots, s_u\}$. Let

$$a_m = \lim_{k \to \infty} d(\theta^k(i_{s_m}), \theta^k(j_{s_m})).$$

Then

$$\beta_{ij}\lambda^n \leq \sum_{m=1}^{u} a_m \leq (\beta_{ij} + \delta)\lambda^n.$$

Assume that $a_1 = \ldots = a_v = 1$, $a_{v+1} < 1$, $a_{v+2} < 1, \ldots, a_u < 1$. Let $a = \max\{a_{v+1}, \ldots, a_u\}$. We have

$$\beta_{ij}\lambda^n \leq v + (u-v)a = v(1-a) + ua, \quad \text{i.e.} \quad v(1-a) \geq \beta_{ij}\lambda^n - ua.$$

Therefore

$$\begin{aligned} v &\geq \frac{\beta_{ij}\lambda^n - ua}{1-a} \geq \frac{\beta_{ij}\lambda^n - (\beta_{ij}+\delta)\lambda^n}{1-a} \\ &= \lambda^n(\beta_{ij} - \beta_{ij}a - \delta a)/(1-a) \\ &= \beta_{ij}\lambda^n - \lambda^n\delta a/(1-a) \geq \beta_{ij}\lambda^n/2. \end{aligned}$$

Since $\beta \leq a \leq b$ and $\delta < \beta(1-b)/2$ we have

$$v \geq \beta\lambda^n/2 = 2w\lambda^n.$$

By (11), at least $v/2$ indices t among $s_1, \ldots, s_v$ satisfy

$$\operatorname{card}\theta^n(N_r)[t] = ch.$$

Take $P \in \xi$ such that $P \subset X_n^t$. Then $P[0] = \theta^n(N_r)[t]$. Therefore

$$\{\theta^n(i)[t], \theta^n(j)[t]\} = \{i_t, j_t\} \subset P[0]$$

and (i_t, j_t) is an essential pair.

Since $v \geq 2w\lambda^n$, $v/2 \geq w\lambda^n$. $\blacksquare$

LEMMA 13. *There is an integer N such that for a.e. $P \in \xi$*

$$\{P[0], P[1], \ldots, P[N-1]\} = \mathscr{A}.$$

Proof. Set

$$\mathscr{J} = \{A \subset N_r\colon \operatorname{card} A = ch \text{ and there are } n \in N \text{ and } t,\ 0 \leqslant t \leqslant \lambda^n - 1, \text{ such that } A = \theta^n(N_r)[t]\}.$$

Since $\mathscr{J}$ is a finite set, there is n such that

$$\mathscr{J} = \{A \subset N_r\colon \operatorname{card} A = ch \text{ and there is } t,\ 0 \leqslant t \leqslant \lambda^n - 1, \text{ such that } A = \theta^n(N_r)[t]\}.$$

Take $P \in \xi$, $A \in \mathscr{A}$. Since $\mu(A) > 0$ and (11) holds, we can assume that $A \in \mathscr{J}$. ∎

The proposition below gives the first part of Theorem 8.

Proposition 14. *If θ is a substitution of constant length λ on r symbols and $\mathscr{F}$ is a factor of $\mathscr{X}(\theta)$ with partly continuous spectrum and $S_p(\mathscr{F}) \supset \{\exp(2\pi ik/\lambda^n)\colon k \in \mathbf{Z},\ n \in \mathbf{N}\}$ then $\mathscr{F}$ is isomorphic to $\mathscr{X}(\tilde{\theta})$ for some substitution $\tilde{\theta}$ given by* (7).

Proof. Denote by c and h the column number and height of θ respectively. Then $S_p(\mathscr{F}) = \{\exp(2\pi ik/(g\lambda^n))\colon k \in \mathbf{Z},\ n \in \mathbf{N}\}$, where g divides h. In other words, the maximal factor $\mathscr{G}$ of $\mathscr{F}$ with discrete spectrum is isomorphic to $\mathscr{Z}(\lambda) \times \mathscr{Z}_g$. By Lemma 5, $\mathscr{F}$ is a b-extension of $\mathscr{G}$ and $1 < b \leqslant c$. Since $\mathscr{G}$ is canonical, $\mathscr{F}$ is a bg-extension of $\mathscr{Z}(\lambda)$. Therefore there exist a Lebesgue space V of cardinality bg and a family $\{S_d\}_{d \in Z(\lambda)}$ of automorphisms of V such that $F \cong Z(\lambda) \times V$ and $S(d, v) = (\tau(d), S_d(v))$. Moreover, the partition $\{Z(\lambda) \times \{v\}\}_{v \in V}$ is generating for S. Hence for any $\delta > 0$ there is a finite code φ_δ of length $2m_\delta + 1$ approximating φ, i.e.

$$\varphi_\delta\colon N_r^{2m_\delta+1} \to V, \tag{13}$$
$$\lim_{k\to\infty} d(\varphi_\delta(x)[-k, k], \varphi(x)[-k, k]) < \delta \quad \text{for a.e. } x \in X(0).$$

Since for each natural n, $X(\theta) = X(\theta^n)$ we can assume the following:

(14) The number N from Lemma 13 is less than λ.

(15) For any $x \in X(\theta)$ the sector $x[0, \lambda-1]$ contains each element of N_r and some pairs ij and ij', $j \neq j'$.

(16) For each $i \in N_r$ the initial symbol of $\theta(i)$ is a cyclic first symbol.

Take $\varepsilon > 0$, $\varepsilon < 1/(2\lambda^2)$. Put $\delta = \varepsilon/(4\lambda^2)$. Let φ_δ be a finite code satisfying (13).

First, we establish the following: If (i, j) is an essential pair then either

$$\begin{aligned} &d(\varphi_\delta(\theta^n(i)), \varphi_\delta(\theta^n(j))) < \varepsilon \quad \text{or} \\ &d(\varphi_\delta(\theta^n(i)), \varphi_\delta(\theta^n(j))) > 1-\varepsilon \end{aligned} \tag{17}$$

for n large enough.

Take $P \in \xi$. In other words, P is a set of cardinality ch of elements from $X(\theta)$ having the same structure of n-symbols for each n. Let $x, y \in P$. Then for some $a, b \geqslant 0$ (a, b depend on n)

$$x[-a, b] = \theta^{n+1}(k) = \theta^n(k_1)\ldots\theta^n(k_\lambda),$$
$$y[-a, b] = \theta^{n+1}(l) = \theta^n(l_1)\ldots\theta^n(l_\lambda),$$

and for each t, $1 \leqslant t \leqslant \lambda$, (k_t, l_t) is an essential pair. Assume that n satisfies $m_\delta < \delta\lambda^n$ and

$$d(\varphi_\delta(z)[-k, k-1], \varphi(z)[-k, k-1]) < \delta$$

for any $z \in P$ and $k \geqslant \lambda^{n-2}$.

1. Suppose $\varphi(x) = \varphi(y)$. Then

$$\begin{aligned} d(\varphi_\delta(x)[-a, b], \varphi_\delta(y)[-a, b]) &\leqslant d(\varphi_\delta(x)[-a, b], \varphi(x)[-a, b]) \\ &\quad + d(\varphi(x)[-a, b], \varphi(y)[-a, b]) + d(\varphi(y)[-a, b], \varphi_\delta(y)[-a, b)) \\ &< 2\delta + 2\delta = 4\delta. \end{aligned}$$

This implies that

$$d(\varphi_\delta(\theta^n(k_t)), \varphi_\delta(\theta^n(l_t))) < 4\delta\lambda < \varepsilon, \quad t = 1, \ldots, \lambda.$$

2. If $\varphi(x) \neq \varphi(y)$ then

$$\begin{aligned} 1 = d(\varphi(x)[a, b], \varphi(y)[-a, b]) &\leqslant d(\varphi(x)[-a, b], \varphi_\delta(x)[-a, b]) \\ &\quad + d(\varphi_\delta(x)[-a, b], \varphi_\delta(y)[-a, b]) + d(\varphi_\delta(y)[-a, b], \varphi(y)[-a, b]). \end{aligned}$$

Hence

$$d(\varphi_\delta(x)[-a, b], \varphi_\delta(y)[-a, b]) > 1 - 4\delta,$$
$$d(\varphi_\delta(\theta^n(k_t)), \varphi_\delta(\theta^n(l_t))) > 1 - \varepsilon, \quad t = 1, \ldots, \lambda.$$

It follows from (14) that (17) holds.

We look at Fig. 2 and suppose that

$$(i, j) \text{ is an essential pair and } d(\varphi_\delta(\theta^n(i)), \varphi_\delta(\theta^n(j))) < \varepsilon. \tag{18}$$

The block $\theta^{n+1}(i)$ starts with the block $\theta^n(i')$, where i' is a cyclic first symbol (by (16)), hence $\theta^n(i)$ starts with $\theta^{n-1}(i')$. Obviously, $\theta^n(i')$ starts with $\theta^{n-1}(i')$. The same holds for j. By (18),

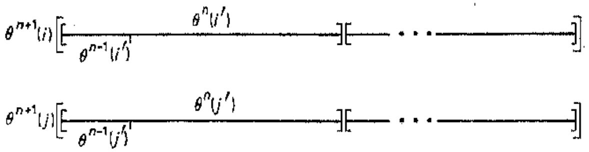

Fig. 2

$$d\big(\varphi_\delta(\theta^{n-1}(i')),\ \varphi_\delta(\theta^{n-1}(j'))\big) < \lambda\varepsilon.$$

Thus

$$d\big(\varphi_\delta(\theta^{n+1}(i)),\ \varphi_\delta(\theta^{n+1}(j))\big) < (\lambda^{n+1} - (\lambda^{n-1} - \lambda^n\varepsilon))/\lambda^{n+1}$$
$$= 1 - (1/\lambda^2 - \varepsilon/\lambda) < 1 - 1/\lambda^2 < 1 - 2\varepsilon < 1 - \varepsilon.$$

By virtue of (17), $d\big(\varphi_\delta(\theta^{n+1}(i)),\ \varphi_\delta(\theta^{n+1}(j))\big) < \varepsilon$. We have obtained

(19) If (18) holds for n then it holds for $n+1$.

Similarly we can show the opposite implication. This forces

(20) $$\varphi(x) = \varphi(y) \quad \text{iff} \quad \varphi(\theta(x)) = \varphi(\theta(y)).$$

Let $\sim$ be the smallest equivalence relation on N_r including the relation R defined as follows:

(21) $$iRj \quad \text{iff} \quad \text{the condition (18) holds.}$$

We will show the following:

(22) The relation $\sim$ is θ-consistent.

Assume that $i \sim j$. Then there are symbols $k^1, \ldots, k^s$, $s \leqslant r$, such that iRk^1, $k^1Rk^2, \ldots, k^sRj$. Set

$$\theta(i) = i_1 \ldots i_\lambda, \qquad \theta(j) = j_1 \ldots j_\lambda,$$
$$\theta(k^p) = k_1^p \ldots k_\lambda^p, \qquad p = 1, \ldots, s.$$

We have to show that for any $1 \leqslant t \leqslant \lambda$, $i_t \sim j_t$. Indeed, by (19), $k_t^p R k_t^{p+1}$, $i_t R k_t^1$, $k_t^s R j_t$. Therefore $i_t \sim j_t$ for $t = 1, \ldots, \lambda$ and $\sim$ is θ-consistent.

Observe that $X(\tilde{\theta})$ is an infinite set, i.e.

(23) $\mathscr{Z}(\lambda)$ is a factor of $\mathscr{X}(\tilde{\theta})$.

Indeed, otherwise $\mathscr{X}(\tilde{\theta}) \cong \mathscr{Z}_{h_1}$, where h_1 divides h, and any two points $x, y \in X(\theta)$ which have the same structure of M-blocks satisfy $x \sim y$. Take $P \in \xi$. Since the number b is greater than 1, there are $x, y \in P$ such that x, y have the same structure of M-blocks and $\varphi(x) \neq \varphi(y)$. By (20), $\varphi(\theta^n(x)) \neq \varphi(\theta^n(y))$. Thus $x \not\sim y$, a contradiction.

To finish the proof of our proposition we will show that $\mathscr{X}(\tilde{\theta})$ is isomorphic to $\mathscr{F}$. Actually we will prove a somewhat stronger fact, namely

(24) If $x, y \in X(\theta)$ then $x \sim y$ iff $\varphi(x) = \varphi(y)$,

where $x \sim y$ iff $x[n] \sim y[n]$ for each $n \in \mathbf{Z}$.

Assume that $\varphi(x) = \varphi(y)$. Then, by the assumption of the proposition, x and y have the same structure of n-symbols for all natural n. Take n for which (18) holds. Then x and y belong to X_n^a for some a. Hence $\theta^n(T^{-a}(x)) \sim \theta^n(T^{-a}(y))$. By (22), $T^{-a}x \sim T^{-a}y$ and $x \sim y$.

Now, assume that $x \sim y$. By (22), $\theta^n(x) \sim \theta^n(y)$. It follows from (23) that x and y have, for all n, the same structure of n-symbols. Therefore $(x[k], y[k])$ is an essential pair, for each k; hence, by (17), $\varphi(\theta^n(x)) = \varphi(\theta^n(y))$. From (23), $\varphi(x) = \varphi(y)$. The fact (24) forces $\mathscr{F}$ to be isomorphic to $\mathscr{X}(\tilde{\theta})$. ∎

PROPOSITION 15. *If $\mathscr{F}$ is a factor of $\mathscr{X}(\theta)$ and the maximal factor of $\mathscr{F}$ with discrete spectrum is equal to $\mathscr{Z}_m \times \mathscr{Z}(p)$, where p divides λ and $p \neq \lambda$, then $\mathscr{F} \cong \mathscr{Z}_m \times \mathscr{Z}(p)$, i.e. $\mathscr{F}$ is a system with discrete spectrum.*

Proof. By virtue of Lemma 6 it is sufficient to show this for a pure substitution θ (since the pure base of any substitution is isomorphic to an ergodic component of its h-power).

Assume that θ is a pure substitution and $\mathscr{F}$ is a factor of $\mathscr{X}(\theta)$ satisfying the assumptions of the proposition. Suppose $\mathscr{F}$ has partly continuous spectrum. By Proposition 14 and Lemma 5 we can assume that $\mathscr{F}$ is a c-extension of its maximal factor with discrete spectrum, where c is the column number of θ.

Let φ_δ be a finite code as in the proof of Proposition 14. Assume that δ and ε are as in that proof and moreover, $\varepsilon < w^3/2000$, with w as in Lemma 12.

If (i, j) is an essential pair then

(25) $$d\big(\varphi_\delta(\theta^n(i)),\ \varphi_\delta(\theta^n(j))\big) > 1 - \varepsilon \quad \text{for } n \text{ large enough.}$$

From this and Lemma 12 we have

(26) There exists a natural t, $t < 2/w$, such that for any distinct i and j from N_r, $d\big(\varphi_\delta(\theta^n(i)),\ \varphi_\delta(\theta^n(j))\big) > 1/t$.

Denote by P the number of all blocks on N_r of length λ (see (14)–(16)). Let W be a finite collection of elements from $X(\theta)$ such that for any $x, y \in W$, $\varphi(x) = \varphi(y)$ and card $W > 2tP$ (for a.e. $z \in F$, card $\varphi^{-1}(z) =$ continuum). Then there are $2t$ elements from W as in Fig. 3.

From $\{x_1, \ldots, x_{2t}\}$ we choose elements x, y such that the number d defined by Fig. 4 satisfies $d \leqslant \lambda^n/(2t)$. Moreover, we can assume that

$$\lambda^n/(4t) \leqslant d \leqslant \lambda^n/(2t).$$

Then each n-symbol $\theta^n(i)$ has the form as in Fig. 5, where $2t \leqslant s \leqslant 4t$ and $d(\varphi_\delta(N_k^i),\ \varphi_\delta(N_l^i)) < s^2\varepsilon \quad (|N_k^i| = d)$.

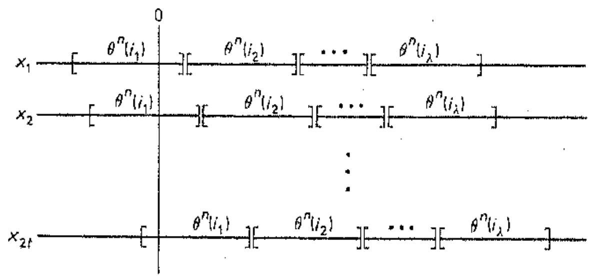

Fig. 3

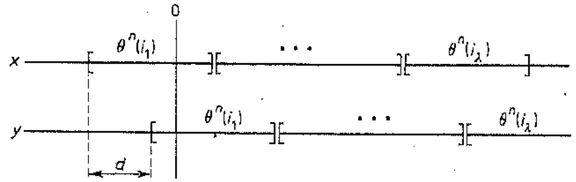

Fig. 4

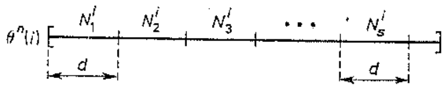

Fig. 5

Indeed,

$$2\delta > d(\varphi_\delta(x)[-\lambda^{n+1},\ \lambda^{n+1}-1],\ \varphi_\delta(y)[-\lambda^{n+1},\ \lambda^{n+1}-1]),$$

hence

$$d(\varphi_\delta(N_k^i),\ \varphi_\delta(N_{k+1}^i)) < 2\delta\cdot 2\lambda s = 4\delta\lambda s < s\varepsilon.$$

Therefore

$$d(\varphi_\delta(N_k^i),\ \varphi_\delta(N_l^i)) \leqslant sd(\varphi_\delta(N_k^i),\ \varphi_\delta(N_{k+1}^i)) < s^2\varepsilon.$$

By virtue of (25),

$$d(\varphi_\delta(N_k^i),\ \varphi_\delta(N_k^j)) > 1/(2t) - 2s^2\varepsilon. \tag{27}$$

To finish the proof let us consider Fig. 6, where $j \neq j'$. By (27),

$$1/(2t) - 2s^2\varepsilon < d(\varphi_\delta(N_1^j),\ \varphi_\delta(N_1^{j'})) < d(\varphi_\delta(A),\ \varphi_\delta(B)) + 2s^2\varepsilon$$
$$< s\varepsilon + 2s^2\varepsilon.$$

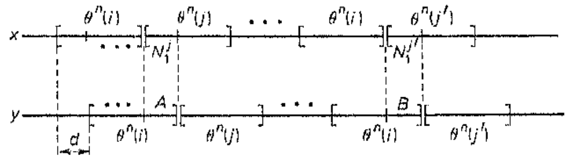

Fig. 6

Therefore $1/(2t) < s\varepsilon + 4s^2\varepsilon < 4t\varepsilon + 4\cdot 16t^2\varepsilon = 4t(1+16t)\varepsilon$ and

$$\varepsilon > 1/(8t^2(1+16t)) > 1/(8t^2\cdot 20t) = 1/(160t^3) > w^3/(200\cdot 8)$$
$$= w^3/1600 > w^3/2000 > \varepsilon,$$

a contradiction. Thus $\mathscr{F}$ is a factor with discrete spectrum. ∎

Proof of Theorem 8. This is a simple consequence of Propositions 14 and 15. ∎

Some properties of factors of substitutions. Theorem 8 says that the only factors with partly continuous spectrum of a substitution θ are among substitutions defined by some θ-consistent relations on the set of symbols. Such relations determine some T-invariant partitions of $X(\theta)$. Moreover, if there are two different θ-consistent relations on N, then the corresponding partitions are different as well. From (23) it follows that if $\mathscr{F}$ is such a factor then it is not only isomorphic to $\mathscr{X}(\tilde{\theta})$, where $\sim$ is defined by (21), but even the partitions of $X(\theta)$ corresponding to $\mathscr{F}$ and $\mathscr{X}(\tilde{\theta})$ are the same. Hence each substitution admits only a finite number of T-invariant sub-σ-algebras corresponding to factors with partly continuous spectrum.

The following example shows that it can happen that there are two factors with partly continuous spectrum and that their maximal common factor has pure point spectrum, i.e. there is no smallest factor with partly continuous spectrum.

Example 1. Let θ be defined as follows:

$$\theta:\ \begin{array}{l} 0 \to 0021 \\ 1 \to 1130 \\ 2 \to 2203 \\ 3 \to 3312 \end{array}$$

There are two θ-consistent relations on N_r:

$$R_1 = \{\{0, 1\}, \{2, 3\}\} =: \{a, b\}, \quad R_2 = \{\{0, 2\}, \{1, 3\}\} =: \{c, d\}.$$

The corresponding substitutions are

$$\eta_1: \begin{array}{l} a \to aaba \\ b \to bbab \end{array} \qquad \eta_2: \begin{array}{l} c \to cccd \\ d \to dddc \end{array}$$

The maximal common factor of η_1 and η_2 is the factor with discrete spectrum $\mathscr{X}(4)$.

Consider the partition ξ of $X(\theta)$ corresponding to the maximal factor of $\mathscr{X}(\theta)$ with discrete spectrum. The atoms of this partition consist of c points, where c is the column number of θ. If η is a substitution such that $\mathscr{X}(\eta)$ is a factor of $\mathscr{X}(\theta)$ then using the canonical system of measures ([9]) on ξ we obtain the following

COROLLARY 16. *The column number of η divides the column number of θ and the height of η divides the height of θ.*

The converse of the above corollary is not true. Namely, if $c(\theta) = ab$, $a, b \neq 1$, then there need not be a θ-consistent relation $\sim$ on the set of symbols of θ such that $c(\tilde{\theta}) = a$. To illustrate this let us consider another example:

EXAMPLE 2.

$$\theta: \begin{array}{l} 0 \to 013 \\ 1 \to 122 \\ 2 \to 230 \\ 3 \to 301 \end{array}$$

The column number of θ is equal to 4, but it has no factors with partly continuous spectrum.

It follows from Theorem 8 that if $\mathscr{F}$ is a factor of θ and $(S_p(\theta):S_p(\mathscr{F})) = \infty$ then $\mathscr{F}$ is a system with discrete spectrum. But we can find a substitution θ of height 2 which has a pure substitution $\tilde{\theta}$ as a factor with partly continuous spectrum:

EXAMPLE 3.

$$\theta: \begin{array}{l} 0 \to 01230 \\ 1 \to 10321 \\ 2 \to 23012 \\ 3 \to 32103 \end{array}$$

There are three θ-consistent relations:

$$R_0 = \{\{0, 2\}, \{1, 3\}\}, \quad R_1 = \{\{0, 1\}, \{2, 3\}\}, \quad R_2 = \{\{0, 3\}, \{1, 2\}\}.$$

The substitutions η_i corresponding to the relations R_i are the following:

$$\eta_0: \begin{array}{l} 0 \to 01010 \\ 1 \to 10101 \end{array}$$

$$\eta_1: \begin{array}{l} 0 \to 00110 \\ 1 \to 11001 \end{array}$$

$$\eta_2: \begin{array}{l} 0 \to 01100 \\ 1 \to 10011 \end{array}$$

η_0 is cyclic: the set $X(\eta_0)$ consists of two points. η_1 and η_2 are pure substitutions with partly continuous spectrum.

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INSTYTUT MATEMATYKI UNIWERSYTETU MIKOŁAJA KOPERNIKA
INSTITUTE OF MATHEMATICS, NICHOLAS COPERNICUS UNIVERSITY
Chopina 12/18, 87-100 Toruń, Poland

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