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# Lower $s$-numbers and their asymptotic behaviour

by

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**Abstract.** We introduce geometric characteristics of Banach space operators, analogous to the $s$-numbers, which are suitable for the lower part of the spectrum. For Hilbert space operators these quantities coincide with the eigenvalues below the bottom of the essential spectrum of the modulus. In general, their asymptotic behaviour corresponds to the distribution of the jumps of the minimum index in the semi-Fredholm domain. The paper is a continuation of [14].

**1. Lower approximation numbers.** Let $T$ be a bounded linear operator on a complex Banach space $X$. Let $U$ denote the closed unit ball of $X$. Let

$$m(T) = \inf\{\|Tx\|:\ \|x\| = 1\}$$

be the minimum modulus of $T$, and let

$$q(T) = \sup\{\varepsilon \geqslant 0:\ TU \supset \varepsilon U\}$$

be the surjection modulus of $T$. We note that both $m(T)$ and $q(T)$ are positive if and only if $T$ is invertible, and in this case $m(T) = q(T) = \|T^{-1}\|^{-1}$.

For each $r = 1, 2, \ldots, \infty$ we define the following lower analogues of the approximation numbers:

$$m_r(T) = \sup\{m(T+F):\ \operatorname{rank} F < r\},$$

$$q_r(T) = \sup\{q(T+F):\ \operatorname{rank} F < r\},$$

$$g_r(T) = \max\{m_r(T),\ q_r(T)\}.$$

We note that $g_\infty(T) > 0$ if and only if $T$ is a semi-Fredholm operator, i.e. either the null space $N(T)$ is finite-dimensional and the range $R(T)$ is closed, or the codimension of $R(T)$ is finite. For such operators it will be useful to consider the index

$$\operatorname{ind} T = \dim N(T) - \operatorname{codim} R(T),$$

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and also the minimum index

$$\operatorname{min.ind} T = \min\{\dim N(T),\ \operatorname{codim} R(T)\},$$

which is always finite for semi-Fredholm operators.

It was shown in [14], Theorem 8.3 that

$$s(T) = \lim_k g_\infty(T^k)^{1/k}$$

is the semi-Fredholm radius of $T$, i.e. the supremum of all $\varepsilon \geqslant 0$ such that $T-\lambda I$ is semi-Fredholm for $|\lambda| < \varepsilon$. In this paper we intend to study the asymptotic behaviour of the finer characteristics $g_r(T)$ for $r = 1, 2, \ldots$ It turns out that this is closely related to the distribution of the jumps of the function $\operatorname{min.ind}(T-\lambda I)$ in the disk $|\lambda| < s(T)$. These jumping points were also studied in Section 7 of [14], and we develop further the underlying technique.

It is well known that the function $\operatorname{min.ind}(T-\lambda I)$ is constant everywhere in the disk $|\lambda| < s(T)$ except possibly for a discrete subset $E$. We denote by $n(T)$ this constant, and call it the *stability index* of the semi-Fredholm operator $T$. For $\omega$ in $E$ we have $\operatorname{min.ind}(T-\omega I) > n(T)$, and $X$ decomposes into the direct sum of two closed $T$-invariant subspaces $Y_\omega$ and $Z_\omega$, where $Z_\omega$ is finite-dimensional and $T-\omega I$ is nilpotent on it, while the restriction of $T-\lambda I$ to $Y_\omega$ has constant minimum index on a neighbourhood of $\omega$. This is the Kato decomposition [4], Theorem 4; an elementary exposition has recently been given by West [10]. Consistently with the matrix case we define the (algebraic) *multiplicity of the jumping point* $\omega$ to be $\dim Z_\omega$ in the corresponding Kato decomposition (cf. [2], Theorem 2.21). In other words, it is the dimension of the salient of the generalized kernel $\left(\bigcup_k N((T-\omega I)^k)\right)^-$ on the generalized range $\bigcap_k R((T-\omega I)^k)$ (cf. [4], [10]). Thus the points in $E$ can be ordered in such a way that

$$|\omega_1(T)| \leqslant |\omega_2(T)| \leqslant \ldots < s(T),$$

where each jump appears consecutively according to its multiplicity. If there are only $p$ $(= 0, 1, 2, \ldots)$ such jumps, we put $|\omega_{p+1}(T)| = |\omega_{p+2}(T)| = \ldots = s(T)$. Our main result can be stated as follows.

1.1. THEOREM. *Let $T$ be a semi-Fredholm operator. For each $r = 1, 2, \ldots$ we have*

$$|\omega_r(T)| = \lim_k g_{kn+r}(T^k)^{1/k}, \tag{1}$$

*where $n = n(T)$ is the stability index of $T$.*

In [5] and [14] analogous formulas were derived for the eigenvalues of $T$ having absolute value greater than the essential spectral radius of $T$. However, the corresponding $s$-number function depended only on the index of the eigenvalue. This is not surprising in the light of Theorem 1.1, because on punctured neighbourhoods of such eigenvalues we have $T-\lambda I$ invertible so that the stability index is zero in that case.



The proof of Theorem 1.1 will be given in Section 3. As in [14] the difficult part lies in the lower estimate showing that the quantities in question converge sufficiently far; the essential tool is analytic: [15], Theorem 2. Although the desired upper estimate can be obtained directly, it seems useful to compare the $g$-numbers with a natural extension of the Bernstein and Mityagin numbers introduced in Section 2. This way is almost equally simple and, moreover, it reveals a link between the Riesz eigenvalues and the Kato jumps in the spectrum of a general operator. The connection becomes particularly transparent for Hilbert space operators for which we identify, in Section 4, the geometric characteristics with the eigenvalues of the modulus, obtaining thus a symmetric counterpart to the classical characterization of the $s$-numbers.

**2. Lower Bernstein and Mityagin numbers.** If $W$ is a closed subspace of $X$, we denote by $J_W$ the embedding map of $W$ into $X$, and by $Q_W$ the canonical map of $X$ onto the quotient space $X/W$. The process started in [7] and continued in [14] admits a further extension in the case of Bernstein and Mityagin sequences. For each $r = 1, 2, \ldots, \infty$ let us define

$$B_r(T) = \sup\{m(TJ_W)\colon \operatorname{codim} W < r\},$$

$$M_r(T) = \sup\{q(Q_V T)\colon \dim V < r\}.$$

Clearly, $B_r(T) > 0$ if and only if $T$ is a semi-Fredholm operator with $\dim N(T) < r$. Similarly, $M_r(T) > 0$ if and only if $T$ is semi-Fredholm with $\operatorname{codim} R(T) < r$.

It is easy to see that $m_r(T) > 0$ if and only if $T$ is semi-Fredholm with $\dim N(T) < r$ and $\operatorname{ind} T \leqslant 0$. Similarly, $q_r(T) > 0$ if and only if $T$ is semi-Fredholm with $\operatorname{codim} R(T) < r$ and $\operatorname{ind} T \geqslant 0$. Consequently, $m_r(T)$ and $q_r(T)$ are equal whenever they are both positive.

It is well known that the set of semi-Fredholm operators having the minimum index less than a fixed $r$ is open in the algebra $B(X)$ of all operators (cf. [1], Theorems 4.2.1 and 4.2.2). If $T$ is in this set, we denote by $d_r(T)$ the radius of the largest open ball centred at $T$ and contained in that set; otherwise we put $d_r(T) = 0$. We recall that the index is constant on any connected set (in particular, on any ball) consisting of semi-Fredholm operators.

2.1. PROPOSITION. *For every operator $T$ and each $r = 1, 2, \ldots, \infty$ we have $m_r(T) \leqslant B_r(T)$, $q_r(T) \leqslant M_r(T)$, and $\max\{B_r(T), M_r(T)\} \leqslant d_r(T)$.*

Proof. Let $\operatorname{rank} F < r$. Then $W = N(F)$ has codimension less than $r$ and $m(T+F) \leqslant m((T+F)J_W) = m(TJ_W)$, hence $m_r(T) \leqslant B_r(T)$. Also $V$

$= R(F)$ has dimension less than $r$ and $q(T+F) \leqslant q(Q_V(T+F)) = q(Q_V T)$, hence $q_r(T) \leqslant M_r(T)$.

Let $S$ be any operator with $\|S\| < B_r(T)$. By the definition of the latter quantity there is a subspace $W$ of codimension less than $r$ such that $\|S\| < m(TJ_W)$. Since

$$|m((T+S)J_W) - m(TJ_W)| \leqslant \|SJ_W\| \leqslant \|S\| < m(TJ_W),$$

we conclude that $m((T+S)J_W) > 0$. This implies that $T+S$ is semi-Fredholm with $\dim N(T+S) < r$, hence $B_r(T) \leqslant d_r(T)$. Similarly one can show that also $M_r(T) \leqslant d_r(T)$.

2.2. Lemma. *Suppose that $T$ is semi-Fredholm and $X$ decomposes into the direct sum of two closed $T$-invariant subspaces $Y$ and $Z$. Let* $\operatorname{min.ind}(TJ_Y) = n$ *in $Y$, and* $\dim Z \geqslant r$. *Let $P$ be the projection of $X$ onto $Z$ along $Y$. Then*

$$d_{n+r}(T) \leqslant \|P\| \cdot \|TJ_Z\|.$$

Proof. Let $S$ be the operator which is zero on $Y$ and $-T$ on $Z$. Then $\operatorname{min.ind}(T+S) \geqslant n+r$ and for any $x = Px + (I-P)x$ we have $Sx = SPx = -TPx$, hence $\|S\| \leqslant \|TJ_Z\| \cdot \|P\|$. This proves the lemma.

**3. Proof of Theorem 1.1.** I. Let $r$ be a positive integer such that $|\omega_r(T)| < s(T)$. Let $Z$ be the direct sum of the finite-dimensional parts in the Kato decompositions corresponding to the points $\omega_1(T), \ldots, \omega_r(T)$; we note that the parts corresponding to different jumps have zero intersection because the restriction of $T-\lambda I$ to these parts is either invertible or nilpotent. Hence $\dim Z \geqslant r$ by the definition of the multiplicity. Denoting by $Y$ the intersection of the corresponding Kato complements, we have $\operatorname{min.ind}((T-\lambda I)J_Y) = n$ in $Y$, on a neighbourhood of zero in the complex plane; this implies that $\operatorname{min.ind}(T^k J_Y) = kn$ for $k = 1, 2, \ldots$ (cf. [15], p. 139). Hence Lemma 2.2 yields

$$d_{kn+r}(T^k) \leqslant \|P\| \cdot \|T^k J_Z\|, \quad k = 1, 2, \ldots$$

Since $\|T^k J_Z\|^{1/k}$ tends to the spectral radius of $TJ_Z$, which is $|\omega_r(T)|$, we conclude that

$$\limsup_k d_{kn+r}(T^k)^{1/k} \leqslant |\omega_r(T)|. \tag{2}$$

If $|\omega_r(T)| = s(T)$, then the last inequality is a consequence of [14], Theorem 8.2. By Proposition 2.1 this proves one inequality for (1).

II. To prove the other inequality we may suppose that $|\omega_r(T)| > 0$. We also omit the trivial case when $X$ is finite-dimensional and $r > \dim X$, because then both sides in (1) are clearly infinite. Let $p$ denote the total multiplicity of the jumps having absolute value less than $|\omega_r(T)|$. So we have $0 \leqslant p < r$. As in part I we consider the direct sum $Z$ of the finite-dimensional summands at the points $\omega_1(T), \ldots, \omega_p(T)$. Now $\dim Z = p$. Let $Y$ be the complement to $Z$ from the Kato decomposition.



Let $F_1 = \alpha J_Z$ with $\alpha$ so large that $(T+F_1-\lambda)J_Z$ be invertible on $|\lambda| < |\omega_r(T)|$. We extend $F_1$ by zero on $Y$. Thus $\operatorname{rank} F_1 = p$.

Next we let $F_2$ be zero on $Z$. We note that the minimum index of $(T-\lambda)J_Y$ is constant on $|\lambda| < |\omega_r(T)|$, being equal to the stability index $n = n(T)$. Let $0 < \varepsilon < |\omega_r(T)|$. By [15], Theorem 2 we can construct $F_2$ in $B(Y)$ of rank $n$ such that the minimum index of $(T+F_2-\lambda)J_Y$ in $Y$ is zero on $|\lambda| \leqslant |\omega_r(T)| - \varepsilon$. (If $\operatorname{ind} T \leqslant 0$, this construction is shown in [15], p. 138. If $\operatorname{ind} T \geqslant 0$, we choose $n$ functionals linearly independent on every $N((T-\lambda)J_Y)$ for $|\lambda| \leqslant |\omega_r(T)| - \varepsilon$, cf. [15], p. 141, and then apply the argument in [11], Theorem 3.10.)

Let $A = T + F_1 + F_2$ on $X$. Since $A - \lambda I$ is bounded from below or surjective on the disk $|\lambda| \leqslant |\omega_r(T)| - \varepsilon$, we get by [6]

$$\lim_k g_1(A^k)^{1/k} \geqslant |\omega_r(T)| - \varepsilon. \tag{3}$$

Since $F_1$ commutes with both $T$ and $F_2$, we can write

$$A^k = (T+F_1+F_2)^k = (T+F_2)^k + B_k,$$

where $R(B_k)$ is contained in $R(F_1)$, hence $\operatorname{rank} B_k \leqslant p < r$. Next, we have

$$(T+F_2)^k = T^k + C_k,$$

where $R(C_k)$ is contained in the sum $R(F_2) + TR(F_2) + \ldots + T^{k-1}R(F_2)$, hence $\operatorname{rank} C_k \leqslant kn$. Consequently, we get $A^k = T^k + S_k$, where $S_k = B_k + C_k$ has rank less than $kn+r$. Hence $g_1(A^k) \leqslant g_{kn+r}(T^k)$ by the definition of the $g$-numbers. Thus (3) actually proves the remaining inequality for (1).

Proposition 2.1 and inequality (2) yield the following corollary of Theorem 1.1.

3.1. Corollary. *Let $T$ be a semi-Fredholm operator. For each $r = 1, 2, \ldots$ we have*

$$|\omega_r(T)| = \lim_k d_{kn+r}(T^k)^{1/k}, \tag{4}$$

*where $n = n(T)$ is the stability index of $T$. If* $\operatorname{ind} T \leqslant 0$, *then also*

$$|\omega_r(T)| = \lim_k B_{kn+r}(T^k)^{1/k} \tag{5}$$

*for all $r$, and if* $\operatorname{ind} T \geqslant 0$, *then*

$$|\omega_r(T)| = \lim_k M_{kn+r}(T^k)^{1/k} \tag{6}$$

*for all $r$.*

We note that (5) does not generally hold if $\operatorname{ind} T > 0$, because $\dim N(T^k)$ may be too large which makes the $B$-numbers small: for instance, if $\operatorname{ind} T = +\infty$, then all the $B$-numbers are zero. Similarly for (6).

One could also establish a number of analogues of the axiomatic properties of the $s$-numbers stated in [7], Section 1, but we omit these questions because they are not difficult and we do not need them in this paper. On the other hand, we are able to prove directly the Hilbert space characterization in the next section.

**4. Hilbert space operators.** For operators on Hilbert spaces we show that the geometric characteristics studied in the preceding sections have a natural eigenvalue interpretation analogous to the classical characterization of the $s$-numbers.

We recall that $T$ is a semi-Fredholm operator with finite-dimensional null space if and only if the bottom $\mu_\infty(T)$ of the essential spectrum of the modulus $|T| = (T^* T)^{1/2}$ is positive, and $\mu_\infty(T) = d_\infty(T)$ in that case [12]. Let

$$\mu_1(T) \leqslant \mu_2(T) \leqslant \ldots$$

be the sequence of the eigenvalues of $|T|$ less than $\mu_\infty(T)$, counted according to their algebraic multiplicities. If there are only $p$ $(= 0, 1, 2, \ldots)$ such eigenvalues, we put $\mu_{p+1}(T) = \mu_{p+2}(T) = \ldots = \mu_\infty(T)$. We also define $\nu_r(T) = \mu_r(T^*)$ for all $r$.

Now we can state the following refinement of the aforementioned result of [12] and of that in [3], Proposition 6.10(i).

4.1. THEOREM. *Let $T$ be an operator on a Hilbert space. Then $B_r(T) = \mu_r(T)$, $M_r(T) = \nu_r(T)$, and $g_r(T) = \max\{B_r(T), M_r(T)\} = d_r(T)$ for each $r = 1, 2, \ldots, \infty$. More precisely, if $T$ is not semi-Fredholm of positive index, then $m_r(T) = B_r(T) = d_r(T) = \mu_r(T)$ for all $r$, and if $T$ is not semi-Fredholm of negative index, then $q_r(T) = M_r(T) = d_r(T) = \nu_r(T)$ for all $r$.*

Proof. If $T$ is not semi-Fredholm, then all the above quantities are zero. So suppose in the rest that $T$ is semi-Fredholm. If $\operatorname{ind} T \leqslant 0$, then an argument similar to [12], p. 226 shows that $d_r(T) \leqslant \mu_r(T)$. If $\operatorname{ind} T \geqslant 0$, then by duality we get $d_r(T) \leqslant \nu_r(T)$. Hence $d_r(T) \leqslant \max\{\mu_r(T), \nu_r(T)\}$ for all $r$.

The equality $B_r(T) = \mu_r(T)$ is the minimax principle ([9], Theorem XIII.1), and $M_r(T) = \nu_r(T)$ follows by duality as in [14], Theorem 4.2(iii).

By virtue of Proposition 2.1 and duality it remains to show that $m_r(T) \geqslant \mu_r(T)$, provided that $\operatorname{ind} T \leqslant 0$. To this end we may suppose that $\mu_r(T) > 0$. Let $0 < \varepsilon < \mu_r(T)$. By the minimax principle ([9], Theorem XIII.1) there is a closed subspace $W$ of codimension less than $r$ such that $m(TJ_W) > \mu_r(T) - \varepsilon$. We shall construct an operator $F$ of rank less than $r$ such that $m(T+F) = m(TJ_W)$. This will prove the claim.



Let $V$ be the orthogonal complement to $W$. Let $Z$ be a complement to the direct (but not necessarily orthogonal) sum $W \oplus N(T)$. Since $T$ is one-to-one on $W \oplus Z$, we have $R(T) = R(TJ_W) \oplus R(TJ_Z)$. Since $\operatorname{ind} T \leqslant 0$, we have $\operatorname{codim} R(T) \geqslant \dim N(T)$. Hence

$$\operatorname{codim} R(TJ_W) \geqslant \dim N(T) + \dim Z = \operatorname{codim} W = \dim V.$$

This inequality ensures that there exists a one-to-one map $S$ from $V$ into the orthogonal complement of $R(TJ_W)$, and we may require that $m(SJ_V)$ be equal to a prescribed positive number, say $m(TJ_W)$. Let

$$Fv = (S - T)v \quad \text{for } v \text{ in } V,$$
$$Fw = 0 \quad \text{for } w \text{ in } W,$$

so that $\operatorname{rank} F < r$.

Let $x = v + w$ be a unit vector with $v$ in $V$ and $w$ in $W$. Hence $\|v\|^2 + \|w\|^2 = 1$. Since $(T+F)x = Sv + Tw$, where $Sv$ and $Tw$ are orthogonal, we have

$$\|(T+F)x\|^2 = \|Sv\|^2 + \|Tw\|^2 \geqslant m(TJ_W)^2 (\|v\|^2 + \|w\|^2) = m(TJ_W)^2,$$

hence $m(T+F) \geqslant m(TJ_W)$. This completes the proof.

Corollary 3.1 and Theorem 4.1 show that among the various $s$-numbers studied in [7] the Bernstein and Mityagin sequences appear to be the most universal ones, giving a unified approach to the subtle structure of the spectrum of a general operator. The following corollary, which generalizes [13], Corollary 2, explains the relationships between the stability index, the Kato jumps, and the eigenvalues of the modulus.

4.2. COROLLARY. *Let $T$ be a semi-Fredholm operator on a Hilbert space. Let $r = 1, 2, \ldots$ If $\operatorname{ind} T \leqslant 0$, then*

$$|\omega_r(T)| = \lim_k \mu_{kn+r}(T^k)^{1/k},$$

*and if $\operatorname{ind} T \geqslant 0$, then*

$$|\omega_r(T)| = \lim_k \nu_{kn+r}(T^k)^{1/k},$$

*where $n = n(T)$ is the stability index of $T$.*

4.3. COROLLARY. *Let $T$ be a normal operator on a Hilbert space. Then $|\omega_r(T)| = \mu_r(T)$ for all $r$.*

Proof. It is well known and easy to show (by using the property $\|Tx\| = \|T^* x\|$) that normal operators have finite ascent (actually, $N(T) = N(T^2)$). This implies that the stability index of such operators can only be zero (cf. [10], Proposition 2.6; in other words, this is the well known fact that for normal operators the Weyl and Browder essential spectra coincide). Hence in Corollary 4.2 we now have $n(T) = 0$, and the result follows by the spectral mapping theorem.

One could also consider analogous perturbations of the reduced minimum modulus

$$\gamma(T) = \{\inf\{\|Tx\|: \operatorname{dist}(x, N(T)) = 1\}$$

by letting

$$\gamma_r(T) = \sup\{\gamma(T+F): \operatorname{rank} F < r\}$$

for $r = 1, 2, \ldots, \infty$. With the functions $\gamma_1$ and $\gamma_\infty$ asymptotic formulas were obtained for the stability radius ([15], Theorem 1) and the semi-Fredholm radius ([14], Theorem 8.2). However, the refinements $\gamma_r$ for $r = 2, 3, \ldots$ are too crude in general. For instance, considering the matrices

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad F = \begin{pmatrix} -1 & c \\ 0 & 0 \end{pmatrix}$$

as operators on the two-dimensional Hilbert space, we have $\gamma(T+F) = (|c|^2+4)^{1/2}$, hence $\gamma_2(T) = \infty$. The example can be modified to the infinite-dimensional Hilbert space by letting $T = 3I$ on the orthogonal extension; then $\gamma_2(T) = 3 = s(T)$. So there is no hope of obtaining the eigenvalue 2 of $T$ in terms of the functions $\gamma_r$.

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