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Received March 27, 1987
Revised version July 24, 1987 (2293)

STUDIA MATHEMATICA, T. XCII. (1989)

Weighted inequalities on product domains

by

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Abstract. We prove weighted integral inequalities between the Lusin area functions and the nontangential maximal functions of biharmonic functions on product domains. Furthermore, we study the duality of weighted BMO spaces and weighted H^1 spaces in the two-parameter theory.

§1. Introduction. Let $A(u)$ and $N(u)$ be the Lusin area function and the nontangential maximal function, respectively, of a biharmonic function u on the product space $R_+^{n_1+1} \times R_+^{n_2+1}$, where $R_+^{n_i+1} = R^{n_i} \times (0, \infty)$ $(i = 1, 2)$. In this note we consider a weight function w which satisfies the two-parameter analogue of the Muckenhoupt A_∞ condition and we prove the weighted L^p-"norm" inequalities:

$$\|N(u)\|_{L_w^p} \leqslant c\,\|A(u)\|_{L_w^p} \quad (0 < p < \infty)$$

for biharmonic functions satisfying a reasonable condition (see Theorem 3 in §3). This is an extension to the weighted L^p-spaces of a result of Gundy–Stein [13]. For the proof of Theorem 3, results of Wheeden [23] (see also [11] and [14]) about harmonic majorization and H^p spaces of conjugate harmonic functions are extended to the case of biharmonic functions on the product domains (see Theorems 1 and 2 in §3). These results together with the weighted inequalities for the Lusin functions and the nontangential maximal functions (of the one-parameter theory) proved in [14] are applied to obtain the desired result if we argue as in [13].

As for the converse, we have obtained only a partial result, which we can derive from a weighted analogue of a result of Merryfield [16] (see also [12], [13] and [15] for the unweighted case). We will state these results in §3 without proofs.

Finally, we also study the duality of weighted BMO spaces and weighted H^1 spaces on the product domains (see Theorem 5 in §3).

§ 2. **Preliminaries.** In this section we introduce our basic notation and give some preliminary results.

2.1. Let $x \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} = \boldsymbol{R}^n$ $(n = n_1 + n_2)$. We write $x = (x^{(1)}, x^{(2)})$, $x^{(1)} \in \boldsymbol{R}^{n_1}$, $x^{(2)} \in \boldsymbol{R}^{n_2}$, $x^{(i)} = (x_1^{(i)}, \ldots, x_{n_i}^{(i)})$ $(i = 1, 2)$. When $n_1 = 1$, $n_2 = 1$, we also write $x_1^{(1)} = x_1$, $x_1^{(2)} = x_2$. If $X \in \boldsymbol{R}^{n_1+1} \times \boldsymbol{R}^{n_2+1}$, we write $X = (x^{(1)}, t_1; x^{(2)}, t_2)$, where $x^{(i)} \in \boldsymbol{R}^{n_i}$, $t_i \in \boldsymbol{R}$ $(i = 1, 2)$. We also write $(x^{(1)}, t_1; x^{(2)}, t_2) = (x, t)$, where $x = (x^{(1)}, x^{(2)})$, $t = (t_1, t_2)$. Set $D_i = \boldsymbol{R}_+^{n_i+1} = \{(x^{(i)}, t_i) \in \boldsymbol{R}^{n_i+1}\colon t_i > 0\}$ $(i = 1, 2)$ and $\boldsymbol{D} = \boldsymbol{D}_1 \times \boldsymbol{D}_2$.

2.2. For $a_1, a_2 > 0$, define

$$\Gamma_{a_i}^{(i)}(x^{(i)}) = \{(y^{(i)}, t_i) \in D_i\colon |x^{(i)} - y^{(i)}| < a_i t_i\},$$

and for $a = (a_1, a_2)$, set

$$\Gamma_a(x) = \Gamma_{a_1}^{(1)}(x^{(1)}) \times \Gamma_{a_2}^{(2)}(x^{(2)})$$
$$= \{(y^{(1)}, t_1; y^{(2)}, t_2)\colon (y^{(i)}, t_i) \in \Gamma_{a_i}^{(i)}(x^{(i)}) \ (i = 1, 2)\}.$$

If F is a measurable function on $\boldsymbol{D}$, we define the *nontangential maximal function* by

$$N_a(F)(x) = \sup\{|F(y, t)|\colon (y, t) \in \Gamma_a(x)\},$$

and the *Lusin function* by

$$S_a(F)(x) = \Big(\int_{\Gamma_a(x)} |F(y, t)|^2 t_1^{-n_1} t_2^{-n_2}\, dy \frac{dt}{t_1 t_2}\Big)^{1/2}.$$

We write $N_{(1,1)} = N$, $S_{(1,1)} = S$.

2.3. Let $\partial_j^{(i)} = \partial/\partial x_j^{(i)}$, $j = 1, \ldots, n_i$, $\partial_0^{(i)} = \partial/\partial t_i$ $(i = 1, 2)$. We define

$$|\nabla_i F| = \Big(\sum_{j=0}^{n_i} |\partial_j^{(i)} F|^2\Big)^{1/2}, \qquad |\nabla_1 \nabla_2 G| = \Big(\sum_{j=0}^{n_1}\sum_{k=0}^{n_2} |\partial_j^{(1)} \partial_k^{(2)} G|^2\Big)^{1/2}$$

for suitable functions F and G. We set $A_a(G) = S_a(K)$, where $K(y, t) = t_1 t_2 |\nabla_1 \nabla_2 G(y, t)|$ and write $A_{(1,1)} = A$.

2.4. Let $1 < p < \infty$ and recall that a positive weight w on $\boldsymbol{R}^d$ is said to belong to the *Muckenhoupt class* $A_p(\boldsymbol{R}^d)$ if there is a constant c such that

$$\Big(|I|^{-1} \int_I w(x)\, dx\Big)\Big(|I|^{-1} \int_I w(x)^{-1/(p-1)}\, dx\Big)^{p-1} \leqslant c$$

for all cubes $I \subset \boldsymbol{R}^d$ (see Muckenhoupt [17]). The least such c will be denoted by $c_p(w)$.

Let $w(x^{(1)}, x^{(2)})$ be locally integrable and $0 < w < \infty$. Following Fefferman and Stein [10], we say that w belongs to the class $a_p = a_{p,n_1,n_2}$ $(1 < p < \infty)$ if $w(\cdot, x^{(2)}) \in A_p(\boldsymbol{R}^{n_1})$, $w(x^{(1)}, \cdot) \in A_p(\boldsymbol{R}^{n_2})$ for all $x^{(1)}, x^{(2)}$ and

$$\sup_{x^{(2)} \in \boldsymbol{R}^{n_2}} c_p\big(w(\cdot, x^{(2)})\big) < \infty, \qquad \sup_{x^{(1)} \in \boldsymbol{R}^{n_1}} c_p\big(w(x^{(1)}, \cdot)\big) < \infty.$$

We define the class $a_\infty = a_{\infty,n_1,n_2}$ by $a_\infty = \bigcup_{p>1} a_p$. The following properties of a_p are easily verified.

(2.4.1) $a_p \subset a_q \subset a_\infty$ $\quad (1 < p < q < \infty)$.

(2.4.2) *If* $w \in a_p$ $(1 < p < \infty)$, *then there is* $\varepsilon > 0$ *such that* $w \in a_{p-\varepsilon}$.

(2.4.3) *If* $w \in a_p$ $(1 < p < \infty)$, *then* $w^{-p'/p} \in a_{p'}$, *where* $p' = p/(p-1)$.

(2.4.4) *If* $w \in a_p$ $(1 < p < \infty)$, *then*

$$\int_{\boldsymbol{R}^n} w(y) \prod_{i=1,2} t_i^{n_i p} (t_i + |x^{(i)} - y^{(i)}|)^{-n_i p}\, dy \leqslant c \int_{R(x,t)} w(y)\, dy,$$

where $R(x, t) = B_1(x^{(1)}, t_1) \times B_2(x^{(2)}, t_2)$ $(x = (x^{(1)}, x^{(2)}),\ t = (t_1, t_2))$, $B_i(x^{(i)}, t_i) = \{y^{(i)} \in \boldsymbol{R}^{n_i}\colon |x^{(i)} - y^{(i)}| < t_i\}$ $(i = 1, 2)$.

(2.4.5) *Let* E $(\subset \boldsymbol{R}^{n_2})$ *be a bounded measurable set of positive measure and let* $s > 0$. *If* $w \in a_p$ $(1 < p < \infty)$, *then*

$$W(x^{(1)}) = s \int_E w(x^{(1)}, x^{(2)})\, dx^{(2)} \in A_p(\boldsymbol{R}^{n_1})$$

and there is a constant c *independent of* E *and* s *such that* $c_p(W) \leqslant c$.

Proof. Since $w \in a_p$, we have

$$\int_{\boldsymbol{R}^{n_1}} \{M^{(1)}(f)(x^{(1)})\}^p w(x^{(1)}, x^{(2)})\, dx^{(1)} \leqslant c \int_{\boldsymbol{R}^{n_1}} |f(x^{(1)})|^p w(x^{(1)}, x^{(2)})\, dx^{(1)}, \tag{2.4.6}$$

where $M^{(1)}$ is the *Hardy–Littlewood maximal operator* on $\boldsymbol{R}^{n_1}$:

$$M^{(1)}(f)(x^{(1)}) = \sup_{t_1 > 0} |B_1(x^{(1)}, t_1)|^{-1} \int_{B_1(x^{(1)}, t_1)} |f(y^{(1)})|\, dy^{(1)}$$

and c is independent of $x^{(2)}$. Multiplying both sides of the inequality (2.4.6) by s and integrating them over E, we find

$$\int_{\boldsymbol{R}^{n_1}} \{M^{(1)}(f)(x^{(1)})\}^p W(x^{(1)})\, dx^{(1)} \leqslant c \int_{\boldsymbol{R}^{n_1}} |f(x^{(1)})|^p W(x^{(1)})\, dx^{(1)}. \tag{2.4.7}$$

Let I be a cube in $\boldsymbol{R}^{n_1}$. If we take $f = W^{-1/(p-1)} \chi_I$ in (2.4.7) (for a set S, χ_S denotes its characteristic function), we obtain the desired result. This completes the proof.

If F is a bounded measurable set of positive measure in $\boldsymbol{R}^{n_1}$, then a similar result holds for $V(x^{(2)}) = s\int_F w(x^{(1)}, x^{(2)})\,dx^{(1)}$.

(2.4.8) Let $w \in a_p$ $(1 < p < \infty)$. Then

$$\int_{\boldsymbol{R}^n} M_S(f)^p w\,dx \leqslant c \int_{\boldsymbol{R}^n} |f|^p w\,dx,$$

where M_S is the *strong maximal operator* defined by

$$M_S(f)(x) = \sup_t |R(x, t)|^{-1} \int_{R(x,t)} |f(y)|\,dy.$$

§3. **Statement of results.** First we extend results of Wheeden [23], Gundy and Wheeden [14] to the product spaces (see also [11]). We consider vector-valued functions of the form

$$F(x, t) = \big(u_0(x, t), u_1(x, t), \ldots, u_N(x, t)\big),$$

where $u_j(x, t) = u_j(x^{(1)}, t_1; x^{(2)}, t_2)$ $(j = 0, 1, \ldots, N)$ is biharmonic on $\boldsymbol{D}$, i.e. u_j is twice continuously differentiable on $\boldsymbol{D}$ and $\Delta_i u_j = 0$ $(i = 1, 2)$, where $\Delta_i = \sum_{k=0}^{n_i} (\partial_k^{(i)})^2$ is the Laplacian.

THEOREM 1. *Let F be as above. Suppose that*

$$|F(x, t)|^{p_0} = \Big(\sum_{j=0}^{N} |u_j(x, t)|^2\Big)^{p_0/2} \qquad (p_0 > 0)$$

is bisubharmonic on $\boldsymbol{D}$, i.e. subharmonic in each set of variables $(x^{(i)}, t_i)$ on $\boldsymbol{D}_i$, $i = 1, 2$. Then if $p_0 < p < \infty$ and

$$\sup_t \int_{\boldsymbol{R}^n} |F(x, t)|^p w(x)\,dx < \infty$$

for $w \in a_{p/p_0}$, the following holds:

(a) $\lim_{t\to 0} F(x, t) = F(x, 0)$ *exists for almost every* $x \in \boldsymbol{R}^n$.

(b) $\|F(\cdot, t) - F(\cdot, 0)\|_{L_w^p} \to 0$ *as* $t \to 0$. *(Here* $L_w^p = \{f\colon fw^{1/p} \in L^p\}$ *and* $\|f\|_{L_w^p} = \|fw^{1/p}\|_{L^p}$.*)*

(c) $N(|F|) \leqslant c\,\{M_S(|F(\cdot, 0)|^{p_0})\}^{1/p_0}$.

In order to prove Theorem 1 we need the next theorem.

THEOREM 2. *Let $P_t(x) = \prod_{i=1,2} P_{t_i}^{(i)}(x^{(i)})$, where*

$$P_{t_i}^{(i)}(x^{(i)}) = c_{n_i} t_i (|x^{(i)}|^2 + t_i^2)^{-(n_i+1)/2}$$

is the Poisson kernel associated with $\boldsymbol{D}_i$ $(i = 1, 2)$. Let $1 < p < \infty$ and $w \in a_p$. If s is a nonnegative bisubharmonic function on $\boldsymbol{D}$ satisfying

$$\sup_t \int_{\boldsymbol{R}^n} s(x, t)^p w(x)\,dx < \infty,$$

then for $a = (a_1, a_2)$ $(a_1, a_2 > 0)$ we have

$$s(x, t+a) \leqslant P_t * s(\cdot, a)(x),$$

where $*$ denotes convolution.

Theorem 1 will be used to prove the following assertion.

THEOREM 3. *Let $w \in a_\infty$. If u is biharmonic on $\boldsymbol{D}$ and $u(x, t) \to 0$ $(|t| \to \infty)$ for each x, then*

$$\int_{\boldsymbol{R}^n} N(u)^p w\,dx \leqslant c \int_{\boldsymbol{R}^n} A(u)^p w\,dx \qquad (0 < p < \infty).$$

Next to state a theorem which will be applied to refine Theorem 3, we introduce a maximal operator. Let $F(x, t)$ be a function on $\boldsymbol{D}$ and $\lambda = (\lambda_1, \lambda_2)$ $(\lambda_1, \lambda_2 > 1)$, $h = (h_1, h_2)$ $(h_1, h_2 > 0)$, $r > 0$. We set

$$T_{\lambda,r}(F)(x) = \sup_{h_1,h_2>0} \Big(\int_{S(R(x,h))} |F(y, t)|^r \prod_{i=1,2} h_i^{-\lambda_i n_i} t_i^{(\lambda_i-1)n_i-1}\,dy\,dt\Big)^{1/r},$$

where for an open set U $(\subset \boldsymbol{R}^n)$, $S(U)$ is defined by

$$S(U) = \{(y, t) \in \boldsymbol{D}\colon R(y, t) \subset U\}.$$

(This is a two-parameter analogue of the maximal function introduced by Fefferman–Stein [7].) Then results of Barker [1], Torchinsky [22] and Muckenhoupt–Wheeden [18] are extended to the product domains as follows.

THEOREM 4. *Let F be continuous on $\boldsymbol{D}$. Then*

(a) $T_{(\alpha,\alpha),r}(F) \leqslant c\,[M_S\{N(F)^{r/\alpha}\}]^{\alpha/r}$ $(\alpha > 1,\ r > 0)$;

(b) *if $p\alpha r^{-1} > 1$ and $w \in a_{p\alpha/r}$, then*

$$\int_{\boldsymbol{R}^n} T_{(\alpha,\alpha),r}(F)^p w\,dx \leqslant c \int_{\boldsymbol{R}^n} N(F)^p w\,dx.$$

If u is biharmonic on $\boldsymbol{D}$, we can prove $N_a(u) \leqslant cT_{\lambda,r}(u)$ as in [18, §1] by applying an inequality of Hardy and Littlewood. Therefore (b) of Theorem 4 will be used to obtain a refinement of Theorem 3.

Now we state partial results about the converse of Theorem 3 without proofs (the result (3.2), (3.3)). We confine ourselves to the case $n_1 = 1$, $n_2 = 1$. We first give a weighted analogue of a theorem of Merryfield [16]. We consider weight functions of the form

(3.1) $$w(x) = w_0(x)\, e^{\tilde{b}_1(x_1)} e^{\tilde{b}_2(x_2)} \qquad (x = (x_1, x_2)),$$

where w_0 satisfies $c_1 \leqslant w_0 \leqslant c_2$ for some constants $c_1, c_2 > 0$ and $\tilde{b}_i$ is the modified Hilbert transform of a real-valued function $b_i \in L^\infty(\boldsymbol{R}^1)$ with $\|b_i\|_{L^\infty}$

$<\pi/2$. (The *modified Hilbert transform* $\tilde{b}$ of $b \in L^\infty(\boldsymbol{R}^1)$ is defined by

$$\tilde{b}(x) = \lim_{\varepsilon\to 0} \pi^{-1} \int_{|x-y|>\varepsilon} b(y)\left[\frac{1}{x-y} + \frac{k(y)}{y}\right] dy,$$

where $k(y) = 1$ if $|y| \geqslant 1$ and $k(y) = 0$ if $|y| < 1$.) Let a real-valued function $\varphi \in C_0^\infty(\boldsymbol{R}^1)$ (the class of infinitely differentiable functions with compact support) be such that

$$\operatorname{supp}(\varphi) \subset [-1, 1], \qquad \int_{-\infty}^{\infty} \varphi(x)\,dx = 1.$$

Set $\varphi_{t_i}(x_i) = t_i^{-1} \varphi(t_i^{-1} x_i)$ $(i = 1, 2)$, $\Phi_t(x) = \varphi_{t_1}(x_1)\varphi_{t_2}(x_2)$ $(t = (t_1, t_2),\ x = (x_1, x_2))$. Then we have the following.

(3.2) *Let* $w(x) = w_0(x) e^{\tilde{b}_1(x_1)} e^{\tilde{b}_2(x_2)}$ *be as above. Let* $g \in L_w^2(\boldsymbol{R}^2)$, $u(x, t) = P_t * g(x)$ *and let* $f \in L^\infty(\boldsymbol{R}^2)$ *be such that* $f - c_f \in L_w^2$ *for some constant* c_f. *Suppose that* $N(u)(x) \leqslant 1$ *for all* $x \in \operatorname{supp}(f)$. *Then there is* $\delta > 0$ *such that if* $\|b_i\|_{L^\infty} < \delta$ $(i = 1, 2)$, *then*

$$\int_D |\nabla_1 \nabla_2 u(x, t)|^2 |\Phi_t * f(x)|^2 e^{P(\log w)(x,t)} t_1 t_2\,dx\,dt$$
$$\leqslant c \int_{\boldsymbol{R}^2} |u(x, 0)|^2 |f(x)|^2 w(x)\,dx + c \int_{\boldsymbol{R}^2} |f(x) - c_f|^2 w(x)\,dx,$$

where $P(\log w)(x, t) = P_t * \log w(x)$, $u(x, 0) = g(x)$ (a similar notation will be used below).

We can prove (3.2) by extending [16, Lemma 3.1] to weighted cases. As a consequence of (3.2), we have the following.

(3.3) *Let* u *and* δ *be as in* (3.2). *Suppose that a weight* w *has the form* (3.1) *with* $\|b_i\|_{L^\infty} < \delta$ $(i = 1, 2)$. *Then*

(a) $$m_w(\{x \in \boldsymbol{R}^2\colon A(u)(x) > \lambda\}) \leqslant c\lambda^{-2} \int_{\boldsymbol{R}^2} \{N(u) \wedge \lambda\}^2 w\,dx$$

for $\lambda > 0$, *where* $m_w(E) = \int_E w\,dx$ *for a set* E *and* $\wedge$ *denotes taking the minimum*;

(b) $$\int_{\boldsymbol{R}^2} A(u)^p w\,dx \leqslant c \int_{\boldsymbol{R}^2} N(u)^p w\,dx \qquad (0 < p < 2).$$

It is well known that (b) follows immediately from (a) (see, e.g., [3, Lemma 3.3]).

Finally, we state a result about the duality of weighted BMO spaces and weighted H^1 spaces. (For the one-parameter theory, see Muckenhoupt–Wheeden [19, 20].) Let $\mathscr{S}_0(\boldsymbol{R}^n)$ denote the subspace of the Schwartz class $\mathscr{S}(\boldsymbol{R}^n)$ consisting of $f \in \mathscr{S}(\boldsymbol{R}^n)$ such that $\mathscr{F}f \in C_0^\infty(\boldsymbol{R}^n)$ and

$\operatorname{supp}(\mathscr{F}f) \subset \{x \in \boldsymbol{R}^n\colon x^{(1)} \neq 0,\ x^{(2)} \neq 0\}$ ($\mathscr{F}f$ denotes the Fourier transform of f).

THEOREM 5. *Let* $f \in \mathscr{S}_0(\boldsymbol{R}^n)$ *and* $u(y, t) = P_t * f(y)$. *Let* g *be a measurable function on* $\boldsymbol{R}^n$ *such that*

$$g(x) \prod_{i=1,2} (1 + |x^{(i)}|)^{-n_i - 1} \in L^1(\boldsymbol{R}^n).$$

Let $w \in a_\infty$. *Suppose that there is a constant* $c > 0$ *such that*

$$\int_{S(U)} |\nabla_1 \nabla_2 P(g)(y, t)|^2 e^{-P(\log w)(y,t)} t_1 t_2\,dy\,dt \leqslant c^2 m_w(U)$$

for all open sets U *in* $\boldsymbol{R}^n$. *If we denote the least such* c *by* $\|g\|_{*,w}$, *then*

$$\Big|\int_{\boldsymbol{R}^n} f(x) g(x)\,dx\Big| \leqslant c \|g\|_{*,w} \int_{\boldsymbol{R}^n} A(u)(x) w(x)\,dx.$$

This will be proved as in [16, §6] (see also [5], [6], [8]).

Remark. It is an easy consequence of the definition of a_∞ that if $w \in a_\infty$, then

$$\int_{\boldsymbol{R}^n} |\log w| \prod_{i=1,2} (1 + |x^{(i)}|)^{-n_i - 1}\,dx < \infty.$$

Thus the iterated Poisson integral $P(\log w)$ in Theorem 5 is well defined.

§4. **Proof of Theorem 2.** By the subharmonicity of s on $\boldsymbol{D}_2$, we have

(4.1) $$s(x^{(1)}, t_1; x^{(2)}, t_2) \leqslant c t_2^{-n_2 - 1} \int_0^{2t_2} \int_B s(x^{(1)}, t_1; y^{(2)}, u_2)\,dy^{(2)}\,du_2,$$

where $B = B_2(x^{(2)}, t_2)$. Let $W(x^{(1)}) = \int_B w(x^{(1)}, y^{(2)})\,dy^{(2)}$ for a fixed $(x^{(2)}, t_2)$. Then by Hölder's inequality and the fact that $w \in a_p$, we easily have

(4.2) $$\int_B s(x^{(1)}, t_1; y^{(2)}, u_2)\,dy^{(2)}$$
$$\leqslant c t_2^{n_2} \Big(\int_B s(x^{(1)}, t_1; y^{(2)}, u_2)^p w(x^{(1)}, y^{(2)})\,dy^{(2)}\Big)^{1/p} W(x^{(1)})^{-1/p},$$

where c is independent of $x^{(1)}$. By (4.1) and (4.2) we have

(4.3) $$s(x^{(1)}, t_1; x^{(2)}, t_2)$$
$$\leqslant c t_2^{-1} \int_0^{2t_2} \Big(\int_B s(x^{(1)}, t_1; y^{(1)}, u_2)^p w(x^{(1)}, y^{(2)})\,dy^{(2)}\Big)^{1/p} du_2\, W(x^{(1)})^{-1/p}.$$

Multiplying both sides of the inequality in (4.3) by $W(x^{(1)})^{1/p}$ and using

Hölder's inequality on the right-hand side, we find

$$s(x^{(1)}, t_1; x^{(2)}, t_2) W(x^{(1)})^{1/p} \leqslant c t_2^{-1+1/p'} \Big(\int_0^{2t_2} \int_B s(x^{(1)}, t_1; y^{(2)}, u_2)^p w(x^{(1)}, y^{(2)}) dy^{(2)} du_2 \Big)^{1/p}.$$

Thus

$$\sup_{t_1>0} \int_{R^{n_1}} s(x^{(1)}, t_1; x^{(2)}, t_2)^p W(x^{(1)}) dx^{(1)} \leqslant c \big(\sup_t \int_{R^n} s(x, t)^p w(x) dx\big) < \infty.$$

Since $W \in A_p(R^{n_1})$ (see (2.4.5)), by Wheeden [23, Lemma 1] we have

(4.4) $$s(x^{(1)}, t_1+a_1; x^{(2)}, t_2) \leqslant \int_{R^{n_1}} s(y^{(1)}, a_1; x^{(2)}, t_2) P_{t_1}^{(1)}(x^{(1)}-y^{(1)}) dy^{(1)},$$

for all $a_1 > 0$. Repeating the above argument, we also have

(4.5) $$s(x^{(1)}, t_1; x^{(2)}, t_2+a_2) \leqslant \int_{R^{n_2}} s(x^{(1)}, t_1; y^{(2)}, a_2) P_{t_2}^{(2)}(x^{(2)}-y^{(2)}) dy^{(2)},$$

for all $a_2 > 0$. Combining the inequalities in (4.4) and (4.5), we easily obtain the desired result.

§5. **Proof of Theorem 1.** Let $p_0 < p < \infty$ and set $q = p/p_0$ (>1), $s(y, t) = |F(y, t)|^{p_0} = |F(y, t)|^{p/q}$. Then s is bisubharmonic on D and

$$\sup_t \int s(y, t)^q w(y) dy = \sup_t \int |F(y, t)|^p w(y) dy < \infty.$$

Therefore by compactness there are a sequence $\{t_k\}$ $(k = 1, 2, \ldots)$ with $t_k \to 0$ $(k \to \infty)$ and a function $h \in L_w^q(R^n)$ such that

(5.1) $$\int s(y, t_k) g(y) w(y) dy \to \int h(y) g(y) w(y) dy$$

as $k \to \infty$ for all $g \in L_w^{q'}$. Set $g_0(y) = P_t(x-y) w(y)^{-1}$ for a fixed (x, t). Then since $w^{-q'/q} = w^{1-q'} \in a_{q'}$, it follows that $g_0 \in L_w^{q'}$ (see (2.4.3) and (2.4.4)). Thus taking $g = g_0$ in (5.1) and using Theorem 2, we find that

$$s(x, t) = \lim_{k\to\infty} s(x, t+t_k) \leqslant \lim_{k\to\infty} P_t * s(\cdot, t_k)(x) = P_t * h(x).$$

This implies $|F(x, t)| \leqslant |P_t * h(x)|^{q/p}$. Consequently,

(5.2) $$N(|F|)(x) \leqslant \{N(P(h))(x)\}^{q/p} \leqslant c\{M_S(h)(x)\}^{q/p}.$$

Since $\{M_S(h)\}^{q/p} \in L_w^p$, it follows that $N(|F|)(x) < \infty$ a.e. Thus by Calderón [2] there is a function $F(x, 0)$ such that $F(x, t) \to F(x, 0)$ a.e. as $t \to 0$. This proves (a). (b) follows from (a) and the dominated convergence theorem. Finally, since also $s(x, t_k) \to |F(x, 0)|^{p/q}$ $(k \to \infty)$ in L_w^q, by (5.1) we find that $h(x) = |F(x, 0)|^{p/q}$. Combined with (5.2), this proves (c), which completes the proof of Theorem 1.

§6. **Proof of Theorem 3.** We first prove the following lemma.

Lemma 1. *Let $w \in a_\infty$ and let u be a biharmonic function on D satisfying $u(x, t) \to 0$ as $|t| \to \infty$. Then*

$$\sup_t \int_{R^n} |u(x, t)|^p w(x) dx \leqslant c \int_{R^n} \{A_a(u)(x)\}^p w(x) dx$$

for p, $0 < p < \infty$.

Proof. Let K be a compact set in $R_+^{n_2+1}$. We first remark that if $\mathscr{H}$ is the Hilbert space defined by

$$\mathscr{H} = \{F\colon \|F\| = \big(\int_K |F(y^{(2)}, t_2)|^2 t_2^{1-n_2} dy^{(2)} dt_2\big)^{1/2} < \infty\},$$

then Gundy–Wheeden [14, Theorem 1] extends to the case of harmonic functions with values in $\mathscr{H}$.

Since $u(x, t) \to 0$ as $|t| \to \infty$, it follows that

$$\int_K |\nabla_2 u(y^{(1)}, t_1; y^{(2)}, t_2)|^2 t_2^{1-n_2} dy^{(2)} dt_2 = \||\nabla_2 u(y^{(1)}, t_1; \cdot)|\|^2 \to 0$$

as $t_1 \to \infty$ for all $y^{(1)}$. Thus by the above remark we obtain

$$\int_{R^{n_1}} \Big(\int_{\Gamma_{a_1}^{(1)}(x^{(1)})} \||\nabla_1 \nabla_2 u(y^{(1)}, t_1; \cdot)|\|^2 t_1^{1-n_1} dy^{(1)} dt_1 \Big)^{p/2} w(x) dx^{(1)} \geqslant c \sup_{t_1} \int_{R^{n_1}} \||\nabla_2 u(x^{(1)}, t_1; \cdot)|\|^p w(x) dx^{(1)}.$$

Therefore if we take $K \subset \Gamma_{a_2}^{(2)}(x^{(2)})$ and let $K \to \Gamma_{a_2}^{(2)}(x^{(2)})$, we find

(6.1) $$\int_{R^{n_1}} \{A_a(u)(x)\}^p w(x) dx^{(1)} \geqslant c \sup_{t_1} \int_{R^{n_1}} \{A_{a_2}^{(2)}(u(x^{(1)}, t_1; \cdot))(x^{(2)})\}^p w(x) dx^{(1)},$$

where

$$A_{a_2}^{(2)}(u(x^{(1)}, t_1; \cdot))(x^{(2)}) = \Big(\int_{\Gamma_{a_2}^{(2)}(x^{(2)})} |\nabla_2 u(x^{(1)}, t_1; y^{(2)}, t_2)|^2 t_2^{1-n_2} dy^{(2)} dt_2 \Big)^{1/2}.$$

Integrating both sides of the inequality in (6.1) over R^{n2} and applying [14, Theorem 1] to the Lusin function $A_{a_2}^{(2)}$ on the right-hand side, we easily obtain the desired inequality. This completes the proof of Lemma 1.

Now we give a proof of Theorem 3. Since $w \in a_\infty$, there is q, $1 < q < \infty$, such that $w \in a_q$. For any p, $0 < p < \infty$, let p_0 be a positive number such that $p/p_0 > q$ and let m be a positive integer such that

$$p_0 > \max\{(n_1-1)/(m+n_1-1),\ (n_2-1)/(m+n_2-1)\}.$$

If $\beta^{(i)} = (\beta_1^{(i)}, \ldots, \beta_m^{(i)})$ $(i = 1, 2)$ is a multi-index such that $0 \leqslant \beta_j^{(i)} \leqslant n_i$ $(j = 1, \ldots, m)$, we define a differential monomial $D_i^{\beta^{(i)}}$ of order m by

$$D_i^{\beta^{(i)}} = \partial^{(i)}_{\beta_1^{(i)}} \ldots \partial^{(i)}_{\beta_m^{(i)}}.$$

Next, to prove Theorem 3, we may assume that $\int A(u)^p w\,dx < \infty$. Then it follows from Lemma 1 that $\sup_t \int |u(x, t)|^p w(x)\,dx < \infty$. Thus arguing as in Gundy–Wheeden [14, pp. 119–120], we can define the biharmonic functions

$$u_{\beta^{(1)},\beta^{(2)}}(x, t) = \frac{1}{\{(m-1)!\}^2} \int_{t_1}^{\infty}\int_{t_2}^{\infty} \Big\{\prod_{i=1,2} (s_i - t_i)^{m-1}\Big\} D_1^{\beta^{(1)}} D_2^{\beta^{(2)}} u(x, s)\,ds_2\,ds_1$$

for all multi-indices $\beta^{(1)}, \beta^{(2)}$ of the above form. Note that if $\beta^{(i)} = (0, \ldots, 0)$ $(i = 1, 2)$, then $u_{\beta^{(1)},\beta^{(2)}} = u$.

Let $u_0, u_1, \ldots, u_N$ $(N = (n_1+1)^m(n_2+1)^m - 1)$ be an enumeration of the above biharmonic functions such that $u_0 = u$. Define a vector-valued function F by

$$F(x, t) = \big(u_0(x, t), \ldots, u_N(x, t)\big).$$

Then we first note that $|F(x, t)|^{p_0} = \big(\sum_{j=0}^N |u_j(x, t)|^2\big)^{p_0/2}$ is bisubharmonic on D. (We can prove this by using Calderón–Zygmund [4, Theorem 1]; see also Stein [21, p. 217].)

Next by applying twice the corresponding inequality of the one-parameter theory (see Fefferman–Stein [7, p. 169]), we easily find that $A_{(1/2,1/2)}(u_j) \leqslant cA(u)$ for $j = 1, \ldots, N$. Thus again by Lemma 1 we have

$$\sup_t \int |F(x, t)|^p w(x)\,dx \leqslant c \int \{A(u)(x)\}^p w(x)\,dx \quad (< \infty). \tag{6.2}$$

Since $p/p_0 > q$ and $w \in a_q$, it follows that $w \in a_{p/p_0}$. Therefore by (6.2) and Theorem 1, there is a limit $F(x, 0)$ and

$$N(|F|) \leqslant c\,\{M_S(|F(\cdot, 0)|^{p_0})\}^{1/p_0}.$$

Thus using (2.4.8), we find

$$\begin{aligned}\int N(u)^p w\,dx &\leqslant \int N(|F|)^p w\,dx \leqslant c \int \{M_S(|F(\cdot, 0)|^{p_0})\}^{p/p_0} w\,dx \\ &\leqslant c \int |F(x, 0)|^p w(x)\,dx.\end{aligned} \tag{6.3}$$

Combining the inequalities in (6.2) and (6.3), we conclude the proof of Theorem 3.

§7. Proof of Theorem 4. We first state some properties of Carleson measures on D. Let $\alpha \geqslant 1$ and let μ be a positive measure on D such that

$$\mu(S(U)) \leqslant c\,|U|^\alpha \quad \text{for all open sets } U \subset R^n.$$

Such a measure μ will be called a *Carleson measure of order* α on D. Let F be a continuous function on D and let μ be a Carleson measure of order α on D. Then

$$\mu(\{(y, t) \in D\colon |F(y, t)| > s\}) \leqslant c\,|\{x \in R^n\colon N(F)(x) > s\}|^\alpha \tag{7.1}$$

for $s > 0$;

$$\int_D |F|^\alpha\,d\mu \leqslant c\Big(\int_{R^n} N(F)\,dx\Big)^\alpha; \tag{7.2}$$

$$\int_{S(U)} |F|^\alpha\,d\mu \leqslant c\Big(\int_U N(F)\,dx\Big)^\alpha \quad \text{for all open sets } U \subset R^n. \tag{7.3}$$

((7.1) follows from the definition of a Carleson measure of order α. (7.1) implies (7.2) and (7.3) follows from (7.2); see Barker [1, Lemma 1].)

Now we prove Theorem 4. Since $d\mu_i = t_i^{(\alpha-1)n_i-1}\,dy^{(i)}\,dt_i$ $(\alpha > 1)$ is a Carleson measure of order α on D_i $(i = 1, 2)$ (i.e. $\mu_i(S_i(I_i)) \leqslant c\,|I_i|^\alpha$ for all cubes $I_i \subset R^{n_i}$, where $S_i(I_i) = \{(y^{(i)}, t_i) \in D_i\colon B_i(y^{(i)}, t_i) \subset I_i\}$), it is easy to see that the product measure

$$d\mu = \Big\{\prod_{i=1,2} t_i^{(\alpha-1)n_i-1}\Big\}\,dy\,dt$$

is a Carleson measure of order α on D. Thus by (7.3) we obtain

$$\begin{aligned}T_{(\alpha,\alpha),r}(F)(x) &= \sup_{h_1,h_2>0} \Big(h_1^{-\alpha n_1} h_2^{-\alpha n_2} \int_{S(R(x,h))} |F(y, t)|^r\,d\mu\Big)^{1/r} \\ &\leqslant c \sup_{h_1,h_2>0} \Big\{h_1^{-\alpha n_1} h_2^{-\alpha n_2} \Big(\int_{R(x,h)} N(|F|^{r/\alpha})\,dy\Big)^\alpha\Big\}^{1/r}.\end{aligned}$$

This proves (a) of Theorem 4. (b) follows from (a) and (2.4.8). This completes the proof of Theorem 4.

§8. Proof of Theorem 5. We require the following lemma.

LEMMA 2. *Let* $w \in a_\infty$. *Then*

$$c_1 e^{P(\log w)(y,t)} \leqslant |R(y, t)|^{-1} \int_{R(y,t)} w(x)\,dx \leqslant c_2 e^{P(\log w)(y,t)}.$$

We can prove this by using [20, Lemma 6] and (2.4.5). We omit the details.

Now we prove Theorem 5. For an integer k, let

$$E_k = \{x \in \mathbf{R}^n\colon\ A(u)(x) > 2^k\}, \qquad E_k' = \{x \in \mathbf{R}^n\colon\ M_S(\chi_{E_k})(x) > c_0\},$$

where c_0 is a constant with $0 < c_0 < 1/2$ and will be determined later. Set $A_k = S(E_k') - S(E_{k+1}')$. Then it is easy to see that

$$\bigcup_{k=-\infty}^{\infty} A_k \supset \{(y, t) \in \mathbf{D}\colon\ |\nabla_1 \nabla_2 u(y, t)| \neq 0\}. \tag{8.1}$$

Next we show that

$$\int_{E_k' - E_{k+1}} A(u)^2 w\, dx \geqslant c \int_{A_k} |\nabla_1 \nabla_2 u|^2 e^{P(\log w)} t_1 t_2\, dy\, dt. \tag{8.2}$$

To see this we first note that if $(y, t) \in A_k$, then $R(y, t) \subset E_k'$ and

$$|R(y, t) \cap E_{k+1}| \leqslant c c_0 |R(y, t)|. \tag{8.3}$$

Since $w \in a_\infty$, if c_0 is small enough, by (8.3) we have

$$m_w\big(R(y, t) \cap E_{k+1}\big) \leqslant \tfrac{1}{2} m_w\big(R(y, t)\big)$$

(see Fefferman [9]). Thus

$$\begin{aligned}
\int_{E_k' - E_{k+1}} & A(u)^2 w\, dx \\
&= \int_{\mathbf{D}} |\nabla_1 \nabla_2 u(y, t)|^2 \{m_w\big(R(y, t) \cap E_k'\big) \\
&\qquad - m_w\big(R(y, t) \cap E_{k+1}\big)\} t_1^{1-n_1} t_2^{1-n_2}\, dy\, dt \\
&\geqslant c \int_{A_k} |\nabla_1 \nabla_2 u(y, t)|^2 m_w\big(R(y, t)\big) t_1^{1-n_1} t_2^{1-n_2}\, dy\, dt.
\end{aligned}$$

Combined with Lemma 2, this proves (8.2).

Let $v(y, t) = P(g)(y, t)$. Then since $f \in \mathscr{S}_0(\mathbf{R}^n)$ and

$$g(x) \prod_{i=1,2} (1 + |x^{(i)}|)^{-n_i - 1} \in L^1,$$

we easily see that

$$\int_{\mathbf{R}^n} f(x) g(x)\, dx = c \int_{\mathbf{D}} \{\partial_0^{(1)} \partial_0^{(2)} u(y, t)\} \{\partial_0^{(1)} \partial_0^{(2)} v(y, t)\} t_1 t_2\, dy\, dt.$$

Thus using (8.1) and (8.2), we have

$$\begin{aligned}
\Big|\int_{\mathbf{R}^n} f(x) g(x)\, dx\Big| &\leqslant c \sum_{k=-\infty}^{\infty} \int_{A_k} \{|\nabla_1 \nabla_2 u| e^{P(\log w)/2}\} \\
&\qquad\qquad \times \{|\nabla_1 \nabla_2 v| e^{-P(\log w)/2}\} t_1 t_2\, dy\, dt
\end{aligned}$$

$$\begin{aligned}
&\leqslant c \sum_{k=-\infty}^{\infty} \{\int_{A_k} |\nabla_1 \nabla_2 u|^2 e^{P(\log w)} t_1 t_2\, dy\, dt\}^{1/2} \\
&\qquad\qquad \times \{\int_{A_k} |\nabla_1 \nabla_2 v|^2 e^{-P(\log w)} t_1 t_2\, dy\, dt\}^{1/2} \\
&\leqslant c \sum_{k=-\infty}^{\infty} \{\int_{E_k' - E_{k+1}} A(u)^2 w\, dx\}^{1/2} \\
&\qquad\qquad \times \{\int_{S(E_k')} |\nabla_1 \nabla_2 v|^2 e^{-P(\log w)} t_1 t_2\, dy\, dt\}^{1/2} \\
&\leqslant c \|g\|_{*,w} \sum_{k=-\infty}^{\infty} 2^k m_w(E_k') \leqslant c \|g\|_{*,w} \sum_{k=-\infty}^{\infty} 2^k m_w(E_k) \\
&\leqslant c \|g\|_{*,w} \int A(u) w\, dx.
\end{aligned}$$

Here we have used the inequality $m_w(E_k') \leqslant c m_w(E_k)$, which follows from (2.4.8). This completes the proof.

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Received May 25, 1987
Revised version July 8, 1987 (2317)

STUDIA MATHEMATICA, T. XCII. (1989)

The Denjoy extension of the Bochner, Pettis, and Dunford integrals

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Abstract. In this paper the Denjoy–Dunford, Denjoy–Pettis, and Denjoy–Bochner integrals of functions mapping an interval $[a, b]$ into a Banach space X are defined and studied. Necessary and sufficient conditions for the existence of the Denjoy–Dunford integral are determined. It is shown that a Denjoy–Dunford (Denjoy–Bochner) integrable function on $[a, b]$ is Dunford (Bochner) integrable on some subinterval of $[a, b]$ and that for spaces that do not contain a copy of c_0, a Denjoy–Pettis integrable function on $[a, b]$ is Pettis integrable on some subinterval of $[a, b]$. For measurable functions, the Denjoy–Dunford and Denjoy–Pettis integrals are equivalent if and only if X is weakly sequentially complete. Several examples of functions that are integrable in one sense but not another are included.

The Denjoy integral of a real-valued function is, in the descriptive sense (that is, specifying the properties of the primitive), a natural extension of the Lebesgue integral of a real-valued function. The Bochner, Pettis, and Dunford integrals are generalizations of the Lebesgue integral to Banach-valued functions. In this paper we will study the Denjoy extension of the Bochner, Pettis, and Dunford integrals.

Before embarking on this study a firm foundation must be laid. The reader may wish to begin with Definition 25 and refer to the introductory material as the need arises. We begin with the notions of bounded variation and absolute continuity on a set. Throughout this paper X will denote a real Banach space and X^* its dual.

DEFINITION 1. Let $F\colon [a, b] \to X$ and let E be a subset of $[a, b]$.

(a) The function F is BV on E if $\sup\{\sum_i \|F(d_i)-F(c_i)\|\}$ is finite where the supremum is taken over all finite collections $\{[c_i, d_i]\}$ of nonoverlapping intervals that have endpoints in E.

(b) The function F is AC on E if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_i \|F(d_i)-F(c_i)\| < \varepsilon$ whenever $\{[c_i, d_i]\}$ is a finite collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_i (d_i-c_i) < \delta$.

(c) The function F is BVG on E if E can be expressed as a countable union of sets on each of which F is BV.