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Locally analytically pseudo-convex topological vector spaces

by

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Abstract. A notion of (analytically) locally pseudo-convex topological vector space is introduced. In particular, we show that an Orlicz modular space L_φ is locally pseudo-convex in this sense if and only if there exist a convex function ψ and numbers α and β such that $\beta\varphi(\alpha x) \leqslant \psi(\log x) \leqslant \varphi(x)$.

0. Introduction. It appears that there is not yet in the literature an adequate treatment of (analytic) pseudo-convexity[1] in the context of general not necessarily locally convex topological vector spaces.

In this note we therefore propose (Sec. 1) two definitions of locally pseudo-convex topological vector spaces. In particular, we consider the case of function spaces. More precisely, we find (Sec. 2) that a function space is locally pseudo-convex essentially if and only if it is "locally logarithmically convex". This corresponds to a classical result for Reinhardt domains in several complex variables. In particular, our result can be applied to the case of Orlicz modular spaces. It turns out (Sec. 3) that such a space L_φ is locally pseudo-convex if and only if there exists a convex function ψ and positive numbers α and β such that $\beta\varphi(\alpha x) \leqslant \psi(\log x) \leqslant \varphi(x)$. Moreover, as remarked by the referee, by virtue of a result by Matuszewska and Orlicz [5] if L_φ is locally bounded then L_φ is automatically locally pseudo-convex. On the other hand, not all locally bounded topological vector spaces are pseudo-convex as shows a most ingenious example (Sec. 4) constructed by the referee.

Not surprisingly the author was led to the present investigation from the point of view of interpolation spaces. This connection is briefly indicated in Sec. 5.

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(1) The word "pseudo-convex" has already an established meaning in the theory of topological vector spaces. Therefore we have (following a suggestion of the referee) appended in the title the extra qualification "analytic". However, for simplicity we will not use it in the paper proper, since in the present context no confusion can arise.

most valuable criticism which lead to a considerable expansion of an earlier version of this note, only dealing with the Orlicz situation. As we have seen several of the most striking results of the note originate from him.

1. Locally pseudo-convex topological vector spaces. Let E be a separated topological vector space over $\boldsymbol{C}$. If E is not locally convex it is not clear what one should mean by a holomorphic function with values in E. We argue that at least for a function f, defined in an open subset ("domain") D of $\boldsymbol{C}$, whose range is contained in a *finite dimensional* subspace L of E, there is no ambiguity: f is holomorphic if and only if all functions of the type $\lambda\circ f$, λ a linear functional on L, are holomorphic in the usual sense. Such a function f will be termed *finite-holomorphic* for short. We therefore adopt the following point of view. We declare a function f defined in D with values in E to be *holomorphic* if it is in the closure of the finite-holomorphic functions in the compact open topology, i.e. if U is any neighborhood of 0 in E and K any compact subset of D there should exist a finite-holomorphic function f_1 such that $f_1(t)-f(t)\in U$ for $t\in K$.

A real-valued upper semicontinuous function h defined in an open subset A of E can now be called *plurisubharmonic* if $h\circ f_1$ is subharmonic whenever f_1 is finite-holomorphic with range contained in A. For example, convex functions are plurisubharmonic in this sense.

It is easy to see that then the maximum principle applies to $h\circ f$ if f is holomorphic in the above sense.

Regarding pseudo-convexity we shall work with the following two conditions:

(I) For every neighborhood U of 0 in E there exists an open starshaped neighborhood V of 0 contained in U and a plurisubharmonic function h defined in V such that $h(0)<1$ and such that the closure of the set $\{z\mid h(z)<1\}$ is contained in V.

(II) For every neighborhood U of 0 in E there exists a neighborhood V_1 of 0 with the following property. Let f be any function defined and continuous in $\bar{D}$ and holomorphic in D, where D is any relatively compact open subset of C with closure $\bar{D}$ and boundary ∂D. Then $f(\partial D)\subset V_1$ implies $f(D)\subset U$.

We say that E is *locally* (I)-*pseudo-convex* if (I) is fulfilled and *locally* (II)-*pseudo-convex* if (II) is fulfilled.

PROPOSITION. *Every locally* (I)-*pseudo-convex space is locally* (II)-*pseudo-convex.*

Proof. Let U be any neighborhood of 0 in our space E and pick up V and h fulfilling the requirements of condition (I). We shall show that the requirements of condition (II) are fulfilled with $V_1=\{z\mid z\in V,\ h(z)$

$<1\}$. Clearly, V_1 is a neighborhood of 0 such that $V_1\subset\bar{V}_1\subset V\subset U$. Let f and D be as in condition (II). Assuming $f(\partial D)\subset V_1$ we have to prove $f(D)\subset U$. For $0\leqslant\varepsilon\leqslant 1$, $t\in\bar{D}$ we define $f_\varepsilon(t)=\varepsilon f(t)$. Then f_ε satisfies the same conditions as f. Set

$$S=\{\varepsilon\mid 0\leqslant\varepsilon\leqslant 1,\ f_\varepsilon(\bar{D})\subset V\}.$$

Then we have:

(i) S is an interval containing 0: Obvious since V is starshaped.

(ii) S is open: Indeed, let $\varepsilon_0\in S$. Then every $t_0\in\bar{D}$ has a neighborhood N such that for some $\eta>0$ holds $f_\varepsilon(t)\in V$ for $t\in N$ and $|\varepsilon-\varepsilon_0|<\eta$. Since $\bar{D}$ is compact we can cover $\bar{D}$ with sets $N_1,\ldots,N_k$ corresponding to numbers $\eta_1,\ldots,\eta_k>0$. So we get $f_\varepsilon(\bar{D})\subset V$ for $|\varepsilon-\varepsilon_0|<\min(\eta_1,\ldots\ldots,\eta_k)$.

(iii) S is closed: This is really the point. Let $\varepsilon\in\bar{S}$. Then $h\circ f(\partial D)<1$. Therefore by the maximum principle $h\circ f_\varepsilon(\bar{D})<1$ so that $f_\varepsilon(\bar{D})\subset V_1\subset\bar{V}_1\subset V$. Next let ε_k be a sequence in S tending to ε. Now if $t\in\bar{D}$ we have $f_{\varepsilon_k}(t)\in\bar{V}_1$. Therefore $f_\varepsilon(t)\in\bar{V}_1$ and we have shown that $f_\varepsilon(\bar{D})\subset V$ which again implies $\varepsilon\in S$.

From (i)–(iii) finally follows $1\in S$ and $f(D)\subset V\subset U$.

Remark. The above proof as well as the definition of pseudo-convexity are modelled on classical ideas in several complex variables (see e.g. [2], Chap. 3).

We do not know if the converse of the proposition is true.

If E is locally convex (in the usual sense) it is clear that E is locally (I)-pseudo-convex and thus also locally (II)-pseudo-convex. Indeed, (I) is fulfilled if we take for V an open starshaped neighborhood of 0 contained in U and set $h=2p_V$, p_V the corresponding semi-norm.

In Sec. 3 we will see that there exist topological vector spaces which are neither locally (I)-pseudo-convex nor locally (II)-pseudo-convex.

Assume that E is locally bounded. Then the topology can be defined by a quasi-norm which always can be assumed to be continuous. More precisely, we will then have a basis of neighborhoods of 0 of the type $U_a=\{z\mid z\in E,\ \|z\|<a\}$. The above conditions (I) and (II) now take the form:

(I′) There exists a plurisubharmonic function h defined in U such that $h(0)<1$ and such that the closure of the set $\{z\mid h(z)<1\}$ is contained in U_1.

(II′) Let f be any function defined and continuous in $\bar{D}$ and holomorphic in D with values in E. There exists a number $a>0$ such that $f(\partial D)\subset U_a$ implies $f(D)\subset U_1$. In other words, with $C=1/a$ holds the inequality

$$\sup_{t\in D}\|f(t)\|\leqslant C\sup_{t\in\partial D}\|f(t)\|.$$

In Sec. 4 we show that not all locally bounded spaces fulfil condition (II′).

2. The case of general function spaces. For the sake of simplicity we confine our attention to the case when the underlying measure space is the interval $(0, \infty)$, with the usual measure dx.

By a *function space* we now mean a topological vector space E the elements of which are complex-valued measurable functions on $(0, \infty)$, with the usual definition of the algebraic structure, possessing the following property:

(*) Every neighborhood U of 0 contains a neighborhood U' such that if $z \in U' \cap E^+$, w — a measurable function with $|w| \leqslant z$, then $w \in U$. (Here and in the sequel E^+ denotes the set of positive elements in E.)

This corresponds to the definition of Reinhardt domain in several complex variables.

For technical reasons we are also going to assume that simple functions are dense in E.

The following results too extend to our situation classical results in several complex variables. We begin with

Proposition 1. *Let E be a function space as above. If E is locally (II)-pseudo-convex then*

(**) *Every neighborhood U of 0 in E contains a neighborhood U' such (that if $z, w \in U' \cap E^+$ then $z^{1-\lambda}w^{\lambda} \in U$ whenever $\lambda \in (0,1)$.*

(A space satisfying condition (**) might be called "locally logarithmically convex".)

Remark. By iteration (**) of course gives: If $z_1, \ldots, z_n \in U' \cap E^+$ then $z_1^{\lambda_1} \ldots z_n^{\lambda_n} \in U$ whenever $\lambda_1 + \ldots + \lambda_n = 1$, $\lambda_1, \ldots, \lambda_n \geqslant 0$.

Proof. Let thus E be locally (II)-pseudo-convex and consider any neighborhood U of 0 in E. Let V_1 be the corresponding neighborhood provided by condition (II). We wish to show that condition (**) is fulfilled with a suitable choice of U', namely the same neighborhood U' which is obtained if we in condition (*) replace U by V_1. With no loss of generality we can also take U to be closed but U' open. We let $D = \{t \mid t \in C, |t| < 1\}$ (unit disk) and take f of the special form

$$f(t) = \sum_{k=1}^{n} e^{g_k(t)} \chi_k$$

where each χ_k is the characteristic set of an interval $I_k \subset (0, \infty)$ (so that $f(t)$ for every $t \in D$ is a simple function) and $g_k = u_k + iv_k$ is holomorphic in the ordinary sense (g_k is a scalar function!). Now the condition $f(\partial D)$

$\subset V_1$ clearly (in view of (*)!) is fulfilled if

$$\sum_{k=1}^{n} e^{u_k(e^{i\theta})} \chi_k \in U'$$

for all $\theta \in [0, 2\pi)$. Similarly if $f(D) \subset U$ then follows in particular

$$f(0) = \sum_{k=1}^{n} \exp\left(\frac{1}{2\pi}\int_0^{2\pi} u_k(e^{i\theta})\,d\theta\right) \chi_k \in U.$$

Next consider the simple functions $z = \sum_{k=1}^{n} a_k \chi_k$ and $w = \sum_{k=1}^{n} b_k \chi_k$ in U' where we can assume that $a_k, b_k > 0$. We define u_k by the formula

$$u_k(e^{i\theta}) = \begin{cases} \log a_k & \text{if} \quad \theta \in [0, 2\pi(1-\lambda)), \\ \log b_k & \text{if} \quad \theta \in [2\pi(1-\lambda), 2\pi) \end{cases}$$

where $0 < \lambda < 1$. The corresponding f is then not continuous on ∂D but using an approximation argument we still can draw the conclusion allowed by condition (II). We conclude that

$$f(0) = \sum_{k=1}^{n} a_k^{1-\lambda} b_k^{\lambda} \chi_k = z^{1-\lambda} w^{\lambda} \in U.$$

So we have verified the property required in condition (**), for simple functions z, w. However, since we have assumed that simple functions are dense in E and that U is closed and U' open it is easy to see that it holds in general. Thus E is "locally logarithmically convex" and the proof is complete. ∎

In formulating the result in the opposite direction we are faced with a slight complication, mainly on the notational level. Since we have obtained a definite result only in the case of Orlicz modular spaces (Sec. 3) — and in that case there is no problem — we will not unduly stress on this point now. The problem is that we have to take the logarithm of elements in E^+, thereby introducing measurable functions which in general also take the value $-\infty$ on a set of positive measure. Thus $\log E^+$ cannot be imbedded in a vector space. However, it is clear what we shall mean by the convex hull of $\log E^+$ or a subset thereof. This is obtained by taking functions of the form $\sum \lambda_k \log z_k$ with $z_k \in E^+$, $\lambda_k \geqslant 0$, $\sum \lambda_k = 1$. Likewise it is clear how to define the notion of convex function on $\log E^+$ or a subset thereof. This is a function u such that $u((1-\lambda)a + \lambda b) \leqslant (1-\lambda)u(a) + \lambda u(b)$ for $0 < \lambda < 1$.

We can now formulate

Proposition 2. *Let again E be a function space and assume that*

(**′) *For every neighborhood U of 0 in E there exists an open star-shaped neighborhood V of 0 contained in U and a convex function u defined in the convex hull of* $\log|V|$ *such that the composition* $u\circ\log$ *is upper semi-continuous (in the topology of E) and such that the closure of the set* $\{z|\ u(\log|z|)\leqslant 1\}$ *is contained in V.*

Then E is locally (I)*-pseudo-convex (and thus a fortiori, by the proposition of Sec.* 1, *locally* (II)*-pseudo-convex).*

Proof. We take of course $h = u\circ\log$ in condition (I). There remains to check that h is indeed plurisubharmonic. That is (cf. Sec. 1), we have to verify that $u(\log|f_1|)$ is subharmonic provided f_1 is finite-holomorphic. This follows of course from the classical fact that the logarithm of the modulus of a holomorphic function is subharmonic. ∎

Remark. The plausible thing to expect is of course that E is locally pseudo-convex, in either sense, if and only if E is "locally logarithmically convex", i.e. satisfies condition (**) of Prop. 1. However we have not been able to prove it so the present arrangement is a bad temporary (?) compromise. What is of course easy to see is that condition (**′) entails (**). (Just take $U' = \{u(\log|z|)\leqslant 1\}$ and use the convexity of u.) In Sec. 3 we shall see that the converse is true at least in the case of Orlicz modular spaces. On the other hand, as for giving examples of spaces satisfying condition (**) the following lemma is useful.

Lemma. *Assume that for some $p\in(0,1)$ holds:*

($\overset{*}{*}$) *Every neighborhood U of 0 in E contains a neighborhood U' such that if*

$$|z|^p \leqslant \lambda_1|z_1|^p+\ldots+\lambda_n|z_n|^p,$$

$1=\lambda_1+\ldots+\lambda_n$ *where* $z\in U'$ *and* $\lambda_i\geqslant 0$ $(i=1,\ldots,n)$ *then* $z\in U$.

Then E is "locally logarithmically convex", i.e. satisfies condition (**) *of Prop.* 1.

Notice that such a space is locally bounded, in fact even p-normable.

3. The case of Orlicz modular space (for definitions see e.g. [6] or [3]). Let φ be an increasing function on $[0,\infty)$ with $\varphi(\tau)>0$ if $\tau\neq 0$, $\lim\limits_{\tau\to 0}\varphi(\tau) = \varphi(0) = 0$. L_φ is the space of measurable complex-valued functions z on $(0,\infty)$ such that for some $a>0$ holds

$$\int_0^\infty \varphi\left(\frac{|z(x)|}{a}\right)dx<\infty.$$

Clearly, L_φ is a vector space and we get a natural structure of topological vector space by taking as neighborhoods of 0 the sets

$$U_{\alpha\beta} = \left\{z\,\Big|\int_0^\infty \varphi\left(\frac{|z(x)|}{a}\right)dx<\beta\right\}\quad \text{with}\quad \alpha,\beta>0.$$

(One readily verifies that $U_{\alpha\beta}+U_{\alpha\beta}\subset U_{\alpha_1\beta_1}$ if $\beta_1\geqslant 2\beta$, $\alpha_1\geqslant 2\alpha$ which proves the continuity of addition.) Clearly condition (*) of Sec. 2 is fulfilled so we have a function space in the sense of Sec. 2. It is possible to prove that L_φ is complete (which by definition entails separated). Since we can obviously do with a denumerable subset of the $U_{\alpha\beta}$, we thus also have a complete metric topological vector space.

Let φ_1 be another function satisfying the same assumptions as φ. One can show (see [5]) that one has a topological linear imbedding $L_\varphi\subset L_{\varphi_1}$ if and only if for some α,β holds $\varphi_1(\tau)\leqslant\beta\varphi(\alpha\tau)$ for all $\tau\in[0,\infty)$. In particular, L_φ and L_{φ_1} coincide as topological vector spaces if and only if for some $\alpha,\beta,\alpha',\beta'$ holds $\beta'\varphi(\alpha'\tau)\leqslant\varphi_1(\tau)\leqslant\beta\varphi(\alpha\tau)$ for all $\tau\in[0,\infty)$. We then say that φ and φ_1 are *equivalent functions*.

Recall (see e.g. [4]), that φ satisfies the Δ_2*-condition* if for some $c>0$ holds $\varphi(2\tau)\leqslant c\varphi(\tau)$ for all $\tau\in[0,\infty)$. By iteration we then also have $\varphi(2^n\tau)\leqslant c\varphi(\tau)$ $(n=1,2,\ldots)$ The following lemma is classical and will not be proven here.

Lemma. *If φ satisfies the Δ_2-condition then the set of simple functions in L_φ is dense in L_φ.* ∎

We can now state the main positive result of this paper.

Theorem. *Consider the Orlicz modular space L_φ. Then L_φ is locally pseudo-convex in either sense if and only if there exists a convex function ψ on $R=(-\infty,\infty)$ and numbers $\alpha,\beta>0$ such that*

$$(\#)\qquad \beta\varphi(\beta x)\leqslant\psi(\log x)\leqslant\varphi(x).$$

That is, φ is equivalent to the composition $\psi\circ\log$.

Proof. (i) Assume that L_φ is locally (II)-pseudo-convex. In condition (**) of Sec. 2, Prop. 1 take $U=U_{11}$, $U'=U_{\alpha\beta}$ (with the appropriate α,β) and $z_j=\sum\limits_{k=1}^n e^{\sigma_{j+k}}\chi_k$, $\lambda_j=1/n$ $(j=1,\ldots,n)$ where χ_k $(k=1,\ldots$ $\ldots,n)$ are the characteristic functions of a family of disjoint intervals $I_k\subset(0,\infty)$, each of measure $m>0$. Then clearly

$$\sum_{j=1}^n\varphi(e^{\sigma_j}/a)m\leqslant\beta\Rightarrow\varphi(e^{(1/n)\Sigma\sigma_j})mn\leqslant 1.$$

This gives upon elimination of m

$$\psi_2\Big((1/n)\sum_{j=1}^n\sigma_j\Big)\leqslant(1/n)\sum_{j=1}^n\psi_1(\sigma_j)$$

where we have defined ψ_1 and ψ_2 by the formulas $\varphi(x)=\psi_1(\log x)$ and $\beta\varphi(\alpha x)=\psi_2(\log x)$, respectively. Using the fact that φ is increasing it is not hard to see that then

$$\psi_2\Big(\sum_{j=1}^{n}\lambda_j\sigma_j\Big)\leqslant\sum_{j=1}^{n}\lambda_j\psi_1(\sigma_j)\quad\text{if}\quad\sum_{j=1}^{n}\lambda_j=1,\ \lambda_j\geqslant 0.$$

Let finally ψ be the largest convex minorant of ψ_1, i.e. $\psi(\sigma)=\inf\sum_{j=1}^{n}\lambda_j\psi_1(\sigma_j)$ where $\sigma=\sum_{j=1}^{n}\lambda_j\sigma_j$, $1=\sum_{j=1}^{n}\lambda_j$, $\lambda_j\geqslant 0$. Then we get $\psi_2(\sigma)\leqslant\psi(\sigma)$. Since obviously $\psi(\sigma)\leqslant\psi_1(\sigma)$ we have thereby established (#).

(ii) Assume that (#) is fulfilled. Indeed, without loss of generality we may assume that we have equality, i.e. $\varphi(x)=\psi(\log x)$. We can also take $U=U_{11}$ in condition (**′) of Sec. 2, Prop. 2. It is clear how to choose the function u. Namely, we just take $u(\zeta)=\int_0^\infty\varphi(\zeta(x))\,dx$, i.e. the corresponding function h in condition (I) is $h(z)=\int_0^\infty\psi(\log|z(x)|)\,dx$. ∎

Now we fix attention to the locally bounded case. Then one has the following rather surprising result.

Proposition (due to the referee). *Every locally bounded Orlicz modular space is locally pseudo-convex.*

Proof. In fact due to a result by Matuszewska and Orlicz [5] (compare Turpin [9], p. 77) one can, upon passing to an equivalent function if necessary, take φ of the form $\varphi(\tau)=\omega(\tau^p)$ where ω is a convex function and $p\in(0,1)$. Then we can apply our theorem with $\psi(x)=\omega(e^{px})$ which obviously is a convex function, too. Alternatively we could have used the lemma of Sec. 2. ∎

Remark. The result of Matuszewska and Orlicz [5] referred to above in the proof of the Proposition can be obtained along the following lines: It is easy to see that L_φ is locally bounded if and only if for every $\beta>0$ there exists $\lambda>0$ such that $\varphi(\lambda\tau)\leqslant\beta\varphi(\tau)$ for all $\tau\in[0,\infty)$. Using submultiplicativity it then follows that there exist a $p>0$ such that $\varphi(\lambda\tau)\leqslant C\lambda^p\varphi(\tau)$ for all $\tau\in[0,\infty)$. Introducing the function

$$\omega(\tau)=\inf\sum\lambda_j\varphi(\tau_j^{1/p})$$

where $\tau=\sum\lambda_j\tau_j$, $1=\sum\lambda_j$, $\lambda_j\geqslant 0$, which plainly is convex, we see that φ indeed is equivalent to a function of the desired type. (Regarding this construction see [7].)

On the other hand, returning to the abstract situation (Sec. 1) not all locally bounded topological vector spaces are locally pseudo-convex, as we will see in Sec. 4.

4. Construction of a locally bounded space which is not locally pseudo-convex (by the referee). Fix $p\in(0,1)$. Consider the class $\mathscr{K}$ of all p-normable quasi-Banach spaces, i.e. quasi-Banach spaces with a quasi-norm

satisfying $\|z_1+z_2\|^p\leqslant\|z_1\|^p+\|z_2\|^p$. If every E in $\mathscr{K}$ is locally (II)-pseudo-convex then the inequality of condition (II′) holds with a universal constant C. For if X_n is a sequence in $\mathscr{K}$ with constants C_n tending to ∞ consider the l_p sum of the X_n.

Now let E be a Banach space with normalized basis e_n $(n=0,1,\ldots)$; e.g. $E=l^2$ will do. Denote the unit ball of E by B. Define

$$f(t)=\sum_{n=0}^{\infty}2^{-n}t^ne_n\qquad(t\in D).$$

Let F be the closed p-convex hull of the set $\{f(t)\mid t\in\partial D\}$. For each $\eta>0$ let $\|\ \|_\eta$ be the quasi-norm on E whose unit ball is the closed absolutely p-convex hull of $F\cup\eta B$. Then $\|f(0)\|_\eta\leqslant C$ for each $\eta>0$. Letting $\eta\to 0$ we obtain $f(0)\in CF$. Thus for each n there exist finite sequences $\{a_{nk}\}_1^{N(n)}$ in C and $\{t_{nk}\}_1^{N(n)}$ such that

$$\sum_{k=1}^{N(n)}|a_{nk}|^p\leqslant C^p,$$

$$\Big\|\sum_{k=1}^{N(n)}a_{nk}f(t_{nk})-f(0)\Big\|\leqslant 1/n.$$

Consider finally the measures μ_n on ∂D given by $\mu_n=\sum_{k=1}^{N(n)}a_{nk}\delta_{t_{nk}}$. Then the sequence μ_n has a weak*-cluster point ν and ν is purely atomic, in fact ν has finite p-variation $\leqslant C$. Also

$$\int_{\partial D}d\nu=1,\qquad\int_{\partial D}t^n\,d\nu=0\quad\text{if}\quad n\geqslant 1,$$

since $f(0)=e_0$. By the F. and M. Riesz theorem this is impossible.

5. Interpolation of locally pseudo-convex spaces. We restrict attention to the locally bounded case. Let thus $\bar{A}=(A_0,A_1)$ be any quasi-Banach couple (see e.g. [1]). If A_0 and A_1 are locally pseudo-convex (in some sense) there arises the question whether there exist non-trivial interpolation spaces with respect to $\bar{A}$ which again are locally pseudo-convex. To this end let us make the following additional hypothesis:

(𝔥) The K-functional is pseudo-convex in the sense that if f is a holomorphic function with values in $\Sigma(\bar{A})$ then we have

$$\sup_{s\in\bar{D}}K(s,f(t);\bar{A})\leqslant C\sup_{s\in\partial D}K(s,f(t);\bar{A})\qquad\text{for}\qquad s\in(0,\infty)$$

with C depending on $\bar{A}$ and where D is the strip $0<\operatorname{Re}s<1$. (Recall that

$$K(s,a;\bar{A})=\inf_{a=a_0+a_1}(\|a_0\|_{A_0}+s\|a_1\|_{A_1})$$

when $s \in (0, \infty)$ and $a \in \Sigma(\bar{A}) = A_0 + A_1$.) Then it is easy to see that the (real) interpolation spaces $\bar{A}_{\theta q}$ $(0 < \theta < 1,\ 0 < q \leqslant \infty)$ are locally (II)-pseudo-convex. Furthermore, regarding the complex interpolation spaces $\bar{A}_{[\theta]}$ $(0 < \theta < 1)$ one can prove that $\bar{A}_{[\theta]} \subset \bar{A}_{\theta\infty}$. (The opposite inclusion $A_{\theta q} \subset A_{[\theta]}$ is always true, with some $q > 0$ depending on the moduli of concavity of A_0 and A_1.) The hypothesis ($\mathfrak{h}$) is fulfilled e.g. in the cases (L^{p_0}, L^{p_1}) and (H^{p_0}, H^{p_1}), as a consequence of the more or less explicit expressions for the K-functional in these cases (see [1]).

Remark. Complex interpolation of p-convex spaces has been considered by Rivière [8].

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