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# Ratio limit theorems and applications to ergodic theory

by

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**Abstract.** Let $f$ be a strongly Lebesgue measurable function from the interval $(0, +\infty)$ to a Banach space $(X, |\cdot|)$ and $g$ a non-negative extended real valued Lebesgue measurable function on $(0, +\infty)$. Theorem 1 states that if $e^{-\lambda t}f(t)$ is Bochner integrable with respect to Lebesgue measure on the interval $(0, +\infty)$ for all $0 < \lambda < +\infty$, and if the limit

$$\lim_{b\to+\infty}\Big(\int_0^b f(t)\,dt\Big)\Big/\Big(\int_0^b g(t)\,dt\Big) = x.$$

exists, then

$$\lim_{\lambda\to+0}\Big(\int_0^\infty e^{-\lambda t}f(t)\,dt\Big)\Big/\Big(\int_0^\infty e^{-\lambda t}g(t)\,dt\Big) = x.$$

Theorem 2 states that if $\int_0^\infty e^{-\lambda t}|f(t)|\,dt < +\infty$ and $\int_0^\infty e^{-\lambda t}g(t)\,dt < +\infty$ for some $0 < \lambda < +\infty$, and if the limit

$$\lim_{b\to+0}\Big(\int_0^b f(t)\,dt\Big)\Big/\Big(\int_0^b g(t)\,dt\Big) = x$$

exists, then

$$\lim_{\lambda\to+\infty}\Big(\int_0^\infty e^{-\lambda t}f(t)\,dt\Big)\Big/\Big(\int_0^\infty e^{-\lambda t}g(t)\,dt\Big) = x.$$

We apply these theorems to ergodic theory and deduce pointwise Abelian ergodic theorems from the corresponding usual ergodic theorems of the Cesàro type.

**Introduction.** Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space with positive measure $\mu$ and $(X, |\cdot|)$ a Banach space. Let $L_p(\mu, X) = L_p(\Omega, \Sigma, \mu, X)$, $1 \leqslant p < +\infty$, denote the space of all strongly measurable functions $f$ from $\Omega$ to $X$ for which the norm is given by

$$\|f\|_p = \Big(\int_\Omega |f(\omega)|^p\,d\mu\Big)^{1/p} < +\infty;$$

and let $L_\infty(\mu, X) = L_\infty(\Omega, \Sigma, \mu, X)$ denote the space of all strongly measurable functions $f$ from $\Omega$ to $X$ for which the norm is given by

$$\|f\|_\infty = \operatorname*{ess\,sup}_{\omega\in\Omega}|f(\omega)| < +\infty.$$

Many authors have discussed pointwise Abelian ergodic theorems for bounded linear operators $T$ on $L_p(\mu, X)$ or for one-parameter semi-groups $\{T_t\}_{0<t<+\infty}$ of bounded linear operators on $L_p(\mu, X)$. See, for example, Báez-Duarte [1], Edwards [5], [6], Hasegawa, Tsurumi and the author [8], Kopp [9]–[11], McGrath [12]–[15], Rota [16], and the author [17]–[19]. In this paper, however, we shall observe that pointwise Abelian ergodic theorems follow from the corresponding usual ergodic theorems of the Cesàro type. For this purpose we shall first prove the general ratio limit theorems mentioned in Abstract.

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**Ratio limit theorems.**

THEOREM 1. *Let $f$ be a strongly Lebesgue measurable function from the interval $(0, +\infty)$ to a Banach space $(X, |\cdot|)$ and $g$ a non-negative extended real valued Lebesgue measurable function on $(0, +\infty)$. Suppose $e^{-\lambda t}f(t)$ is Bochner integrable with respect to Lebesgue measure on $(0, +\infty)$ for all $0<\lambda<+\infty$, and $\int_0^\infty g(t)\,dt>0$. If the limit*

$$\lim_{b\to+\infty}\Big(\int_0^b f(t)\,dt\Big)\Big/\Big(\int_0^b g(t)\,dt\Big)=x \tag{1}$$

*exists, then*

$$\lim_{\lambda\to+0}\Big(\int_0^\infty e^{-\lambda t}f(t)\,dt\Big)\Big/\Big(\int_0^\infty e^{-\lambda t}g(t)\,dt\Big)=x. \tag{2}$$

(We define $x/+\infty=0$ for all $x\in X$.)

Proof. By Fubini's theorem and Tonelli's theorem (cf. Theorems III.11.9 and III.11.14 in [4]),

$$\int_0^\infty e^{-\lambda t}f(t)\,dt=\lambda\int_0^\infty e^{-\lambda t}\int_0^t f(s)\,ds\,dt\qquad(0<\lambda<+\infty) \tag{3}$$

and

$$\int_0^\infty e^{-\lambda t}g(t)\,dt=\lambda\int_0^\infty e^{-\lambda t}\int_0^t g(s)\,ds\,dt\qquad(0<\lambda<+\infty). \tag{4}$$

Case I. Let $\int_0^\infty e^{-\lambda t}g(t)\,dt<+\infty$ for all $0<\lambda<+\infty$. Then we have, for every constant $0<M<+\infty$,

$$\lim_{\lambda\to+0}\Big|\lambda\int_0^M e^{-\lambda t}\int_0^t f(s)\,ds\,dt\Big|\leqslant\lim_{\lambda\to+0}\lambda M\int_0^M|f(s)|\,ds=0 \tag{5}$$



and

$$\lim_{\lambda\to+0}\lambda\int_0^M e^{-\lambda t}\int_0^t g(s)\,ds\,dt=0. \tag{6}$$

Write $x+\varepsilon(t)=\big(\int_0^t f(s)\,ds\big)/\big(\int_0^t g(s)\,ds\big)$, and given an $\varepsilon>0$, choose a real constant $M>0$ so that $M<t<+\infty$ implies $|\varepsilon(t)|<\varepsilon$. Then we observe that

$$\int_M^\infty e^{-\lambda t}\int_0^t f(s)\,ds\,dt=\int_M^\infty e^{-\lambda t}(x+\varepsilon(t))\int_0^t g(s)\,ds\,dt$$

and that

$$\Big|\int_M^\infty e^{-\lambda t}\varepsilon(t)\int_0^t g(s)\,ds\,dt\Big|\leqslant\varepsilon\int_M^\infty e^{-\lambda t}\int_0^t g(s)\,ds\,dt.$$

(6) implies that

$$\lim_{\lambda\to+0}\Big(\int_0^M e^{-\lambda t}\int_0^t g(s)\,ds\,dt\Big)\Big/\Big(\int_0^\infty e^{-\lambda t}\int_0^t g(s)\,ds\,dt\Big)=0,$$

and hence, by an easy computation together with (5) and with the boundedness of

$$\Big|\int_M^\infty e^{-\lambda t}\int_0^t f(s)\,ds\,dt\Big|\Big/\Big(\int_M^\infty e^{-\lambda t}\int_0^t g(s)\,ds\,dt\Big),$$

we can see that there exists a real number $\lambda_0>0$ so that $0<\lambda<\lambda_0$ implies

$$\Big|x-\Big(\int_0^\infty e^{-\lambda t}f(t)\,dt\Big)\Big/\Big(\int_0^\infty e^{-\lambda t}g(t)\,dt\Big)\Big|<2\varepsilon.$$

(Here we used equations (3) and (4).) Since $\varepsilon$ is arbitrary, this proves (2).

Case II. Let $\int_0^\infty e^{-\lambda_0 t}g(t)\,dt=+\infty$ for some $0<\lambda_0<+\infty$, but $\int_0^b g(t)\,dt<+\infty$ for all $0<b<+\infty$. It is then enough to show that $x=0$. If this is not the case, then there would exist a real constant $M>0$ such that $M<t<+\infty$ implies $|x+\varepsilon(t)|>|x|/2\,(>0)$. Then, for every $0<\lambda<\lambda_0$,

$$\begin{aligned}\infty=(|x|/2)\int_M^\infty e^{-\lambda t}\int_0^t g(s)\,ds\,dt&\leqslant\int_M^\infty e^{-\lambda t}|x+\varepsilon(t)|\int_0^t g(s)\,ds\,dt\\&\leqslant\int_0^\infty e^{-\lambda t}\int_0^t|f(s)|\,ds\,dt=(1/\lambda)\int_0^\infty e^{-\lambda t}|f(t)|\,dt<+\infty,\end{aligned}$$

which is a contradiction.

Case III. Let $\int_0^b g(t)\,dt=+\infty$ for some $0<b<+\infty$. In this case, the proof is trivial.

The proof is complete.

THEOREM 2. *Let $f$ be a strongly Lebesgue measurable function from the interval $(0, +\infty)$ to a Banach space $(X, |\cdot|)$ and $g$ a non-negative extended real valued Lebesgue measurable function on $(0, +\infty)$. Suppose* $\int_0^\infty e^{-\lambda_0 t}|f(t)|\,dt < +\infty$ *and* $\int_0^\infty e^{-\lambda_0 t} g(t)\,dt < +\infty$ *for some* $0 < \lambda_0 < +\infty$, *and* $\int_0^b g(t)\,dt > 0$ *for every* $0 < b < +\infty$. *If the limit*

$$\lim_{b\to+0} \Big(\int_0^b f(t)\,dt\Big)\Big/\Big(\int_0^b g(t)\,dt\Big) = x \tag{7}$$

*exists, then*

$$\lim_{\lambda\to+\infty} \Big(\int_0^\infty e^{-\lambda t} f(t)\,dt\Big)\Big/\Big(\int_0^\infty e^{-\lambda t} g(t)\,dt\Big) = x. \tag{8}$$

Proof. We shall first prove that, for all $0 < L < +\infty$ and all $0 < M < +\infty$,

$$\lim_{\lambda\to+\infty} \Big(\int_L^\infty e^{-\lambda t} f(t)\,dt\Big)\Big/\Big(\int_0^M e^{-\lambda t} g(t)\,dt\Big) = 0 \tag{9}$$

and

$$\lim_{\lambda\to+\infty} \Big(\int_L^\infty e^{-\lambda t} g(t)\,dt\Big)\Big/\Big(\int_0^M e^{-\lambda t} g(t)\,dt\Big) = 0. \tag{10}$$

To see this, we may and do assume without loss of generality that $0 < M < L < +\infty$. Then, since

$$\Big|\int_L^\infty e^{-\lambda t} f(t)\,dt\Big| \leqslant \int_L^\infty e^{-\lambda t}|f(t)|\,dt \leqslant e^{-(\lambda-\lambda_0)L}\int_L^\infty e^{-\lambda_0 t}|f(t)|\,dt < +\infty$$

for every $\lambda$, with $\lambda_0 < \lambda < +\infty$, and since

$$\int_0^M e^{-\lambda t} g(t)\,dt \geqslant e^{-\lambda M}\int_0^M g(t)\,dt \qquad (0 < \lambda < +\infty),$$

it follows that

$$\limsup_{\lambda\to+\infty} \Big|\int_L^\infty e^{-\lambda t} f(t)\,dt\Big|\Big/\Big(\int_0^M e^{-\lambda t} g(t)\,dt\Big) \leqslant \limsup_{\lambda\to+\infty}\left[e^{-\lambda(L-M)} e^{\lambda_0 L}\frac{\int_L^\infty e^{-\lambda_0 t}|f(t)|\,dt}{\int_0^M g(t)\,dt}\right] = 0.$$

This proves (9) and hence (10).

Let us write, for $0 < t < +\infty$,

$$\tilde f(t) = \int_0^t f(s)\,ds \quad\text{and}\quad \tilde g(t) = \int_0^t g(s)\,ds.$$



We then obtain, using (3) and (4) and applying (9) and (10) to $\tilde f$ and $\tilde g$ instead of $f$ and $g$, respectively, that, for every constant $0 < a < +\infty$,

$$\lim_{\lambda\to+\infty} \Big(\int_a^\infty e^{-\lambda t}\tilde f(t)\,dt\Big)\Big/\Big(\int_0^a e^{-\lambda t}\tilde g(t)\,dt\Big) = 0 \tag{11}$$

and

$$\lim_{\lambda\to+\infty} \Big(\int_a^\infty e^{-\lambda t}\tilde g(t)\,dt\Big)\Big/\Big(\int_0^a e^{-\lambda t}\tilde g(t)\,dt\Big) = 0. \tag{12}$$

Put $x + \varepsilon(t) = \big(\int_0^t f(s)\,ds\big)\big/\big(\int_0^t g(s)\,ds\big)$ for $0 < t < +\infty$, and given an $\varepsilon > 0$, choose a real constant $a > 0$ so that $0 < t < a$ implies $|\varepsilon(t)| < \varepsilon$. Then we have

$$\int_0^a e^{-\lambda t}\int_0^t f(s)\,ds\,dt = \int_0^a e^{-\lambda t}(x+\varepsilon(t))\int_0^t g(s)\,ds\,dt$$

and

$$\Big|\int_0^a e^{-\lambda t}\varepsilon(t)\int_0^t g(s)\,ds\,dt\Big| \leqslant \varepsilon\int_0^a e^{-\lambda t}\int_0^t g(s)\,ds\,dt.$$

Therefore, using (3) and (4) and applying (11) and (12), we see, as in the proof of Theorem 1, that there exists a real number $\beta > \lambda_0$ such that

$$\left|x - \frac{\int_0^\infty e^{-\lambda t} f(t)\,dt}{\int_0^\infty e^{-\lambda t} g(t)\,dt}\right| < 2\varepsilon$$

for all $\lambda$, with $\beta < \lambda < +\infty$.

This completes the proof.

Using essentially the same idea as in the proof of Theorem 1, the following theorem can easily be proved, and we omit the details.

THEOREM 3. *Let $(x_n)$ be a sequence in a Banach space $(X, |\cdot|)$ and $(a_n)$ a sequence of non-negative real numbers, with* $\sum_{n=0}^\infty a_n > 0$. *Suppose* $\sum_{n=0}^\infty |x_n| r^n < +\infty$ *for all* $0 < r < 1$. *If the limit*

$$\lim_{n\to+\infty} \Big(\sum_{i=0}^n x_i\Big)\Big/\Big(\sum_{i=0}^n a_i\Big) = x \tag{13}$$

*exists, then*

$$\lim_{r\to1-0} \Big(\sum_{i=0}^\infty r^i x_i\Big)\Big/\Big(\sum_{i=0}^\infty r^i a_i\Big) = x. \tag{14}$$

**Applications to ergodic theory.** In this section we shall apply the general ratio limit theorems obtained in the preceding section to deduce pointwise Abelian ergodic theorems from the corresponding usual ergodic theorems of the Cesàro type.

Let $1 \leqslant p \leqslant +\infty$ and let $L_p(\mu, X)$ be as in Introduction. Suppose that $t \to f_t$ is a strongly Lebesgue measurable function from the interval $(0, +\infty)$ to $L_p(\mu, X)$ such that, for some non-negative real number $\lambda_0$,

$$\int_0^\infty e^{-\lambda t}\|f_t\|_p\,dt < +\infty \qquad (\lambda_0 < \lambda < +\infty).$$

Then, by Theorem III.11.17 in [4], there exists a function $g(t, \omega)$ from $(0, +\infty)\times\Omega$ to $X$, measurable with respect to the product of Lebesgue measure and $\mu$, such that for almost all $0 < t < +\infty$, $g(t, \omega)$, as a function of $\omega \in \Omega$, belongs to the equivalence class of $f_t$ $(\in L_p(\mu, X))$. Furthermore there exists a measurable subset $E(f)$ of $\Omega$, with $\mu(E(f)) = 0$, such that if $\omega \notin E(f)$, then, for all $0 < b < +\infty$ and all $\lambda_0 < \lambda < +\infty$,

$$\int_0^b |g(t, \omega)|\,dt < +\infty \quad\text{ and }\quad \int_0^\infty e^{-\lambda t}|g(t, \omega)|\,dt < +\infty,$$

and the Bochner integrals $\int_0^b g(t, \omega)\,dt$ and $\int_0^\infty e^{-\lambda t}\, g(t, \omega)\,dt$, as functions of $\omega \in \Omega$, belong to the equivalence classes of the Bochner integrals $\int_0^b f_t\,dt$ $(\in L_p(\mu, X))$ and $\int_0^\infty e^{-\lambda t} f_t\,dt$ $(\in L_p(\mu, X))$, respectively.

In the sequel $g(t, \omega)$ will be denoted by $f_t(\omega)$.

The next theorem is a rather general result.

Theorem 4. *Let $1 \leqslant p \leqslant +\infty$, let $t \mapsto f_t$ be a strongly Lebesgue measurable function from the interval $(0, +\infty)$ to $L_p(\mu, X)$, and let $h_t(\omega)$ be a non-negative extended real valued function on $(0, +\infty)\times\Omega$ which is assumed to be measurable with respect to the product of Lebesgue measure and $\mu$.*

(I) *If $\int_0^\infty e^{-\lambda t}\|f_t\|_p\,dt < +\infty$ for all $0 < \lambda < +\infty$, and if the limit*

$$\lim_{b\to+\infty}\Big(\int_0^b f_t(\omega)\,dt\Big)\Big/\Big(\int_0^b h_t(\omega)\,dt\Big) \tag{15}$$

*exists for $\mu$-almost all $\omega$ in the set $P(h) = \{\omega\colon \int_0^\infty h_t(\omega)\,dt > 0\}$, then the following limit*

$$\lim_{\lambda\to+0}\Big(\int_0^\infty e^{-\lambda t} f_t(\omega)\,dt\Big)\Big/\Big(\int_0^\infty e^{-\lambda t} h_t(\omega)\,dt\Big) \tag{16}$$

*exists and coincides with the limit* (15) *for $\mu$-almost all $\omega \in P(h)$.*



(II) *If $\int_0^\infty e^{-\lambda_0 t}\|f_t\|_p\,dt < +\infty$ for some $0 < \lambda_0 < +\infty$, and if the limit*

$$\lim_{b\to+0}\Big(\int_0^b f_t(\omega)\,dt\Big)\Big/\Big(\int_0^b h_t(\omega)\,dt\Big) \tag{17}$$

*exists for $\mu$-almost all $\omega$ in the set $Q(h) = \{\omega\colon \int_0^b h_t(\omega)\,dt > 0$ for all $0 < b < +\infty$ and $\int_0^\infty e^{-\lambda t}h_t(\omega)\,dt < +\infty$ for some $0 < \lambda < +\infty\}$, then the following limit*

$$\lim_{\lambda\to+\infty}\Big(\int_0^\infty e^{-\lambda t} f_t(\omega)\,dt\Big)\Big/\Big(\int_0^\infty e^{-\lambda t} h_t(\omega)\,dt\Big) \tag{18}$$

*exists and coincides with the limit* (17) *for $\mu$-almost all $\omega \in Q(h)$.*

Proof. Immediate from Theorems 1 and 2.

Let us now consider a one-parameter semigroup $\{T_t\}_{0<t<+\infty}$ (i.e., $T_tT_s = T_{t+s}$ for all positive real numbers $t$ and $s$) of bounded linear operators on $L_p(\mu, X)$, where $1 \leqslant p \leqslant +\infty$ is fixed. $\{T_t\}_{0<t<+\infty}$ is said to be *strongly measurable* if, for each $f \in L_p(\mu, X)$, the function $t \mapsto T_t f$ from the interval $(0, +\infty)$ to $L_p(\mu, X)$ is strongly Lebesgue measurable. It is known (cf. Lemma VIII.1.3 in [4]) that a strongly measurable semigroup $\{T_t\}_{0<t<+\infty}$ is *strongly continuous* (i.e., for each $f \in L_p(\mu, X)$ and each positive real number $s$, we have $\lim_{t\to s}\|T_tf - T_sf\|_p = 0$). A linear operator $T$ on $L_p(\mu, X)$ is said to be a *linear contraction* if $\|T\|_p \leqslant 1$.

The following theorem is a direct consequence of the ergodic theorem obtained in Hasegawa and the author [7] and (I) of Theorem 4.

Theorem 5. *Let $\{T_t\}_{0<t<+\infty}$ be a strongly measurable one-parameter semigroup of linear contractions on $L_1(\mu, K)$, where $K$ denotes the field of complex numbers. Suppose $p_t(\omega)$ is a non-negative extended real valued function on $(0, +\infty)\times\Omega$, measurable with respect to the product of Lebesgue measure and $\mu$, such that $f \in L_1(\mu, K)$ and $|f| \leqslant p_s$ for some $s$ $\mu$-almost everywhere on $\Omega$ imply $|T_tf| \leqslant p_{t+s}$ $\mu$-almost everywhere on $\Omega$ for all $0 < t < +\infty$. Then, for any $f \in L_1(\mu, K)$, the limit*

$$\lim_{\lambda\to+0}\Big(\int_0^\infty e^{-\lambda t} T_tf(\omega)\,dt\Big)\Big/\Big(\int_0^\infty e^{-\lambda t} p_t(\omega)\,dt\Big) \tag{19}$$

*exists and is finite for $\mu$-almost all $\omega$ in the set $\{\omega\colon \int_0^\infty p_t(\omega)\,dt > 0\}$.*

The discrete analogue of Theorem 5, which was originally proved by the author in [17], is also a direct consequence of Chacon's general ratio ergodic theorem ([2], [3]) and Theorem 3.

A one-parameter semigroup $\{T_t\}_{0<t<+\infty}$ of bounded linear operators on $L_p(\mu, X)$ is said to be *strongly integrable over every finite interval* if, for each function $f \in L_p(\mu, X)$, the mapping $t \mapsto T_t f$ is Bochner integrable with respect to Lebesgue measure over every finite interval $(a, b) \subset (0, +\infty)$.

THEOREM 6. *Let $1 \leqslant p \leqslant +\infty$, and let $\{T_t\}_{0<t<+\infty}$ be a one-parameter semigroup of bounded linear operators on $L_p(\mu, X)$ which is assumed to be strongly integrable over every finite interval. Suppose $f \in L_p(\mu, X)$.*

(I) *If $\inf_{0<t<+\infty} (\|T_t\|_p)^{1/t} \leqslant 1$, and if the ergodic limit*

$$\lim_{b \to +\infty} \frac{1}{b} \int_0^b T_t f(\omega)\, dt \tag{20}$$

*exists for $\mu$-almost all $\omega \in \Omega$, then the following Abelian ergodic limit*

$$\lim_{\lambda \to +0} \lambda \int_0^\infty e^{-\lambda t} T_t f(\omega)\, dt \tag{21}$$

*exists and coincides with the ergodic limit* (20) *for $\mu$-almost all $\omega \in \Omega$.*

(II) *If the local ergodic limit*

$$\lim_{b \to +0} \frac{1}{b} \int_0^b T_t f(\omega)\, dt \tag{22}$$

*exists for $\mu$-almost all $\omega \in \Omega$, then the following Abelian local ergodic limit*

$$\lim_{\lambda \to +\infty} \lambda \int_0^\infty e^{-\lambda t} T_t f(\omega)\, dt \tag{23}$$

*exists and coincides with the local ergodic limit* (22) *for $\mu$-almost all $\omega \in \Omega$.*

Proof. Let us write $d = \inf\{(\|T_t\|_p)^{1/t}\colon 0 < t < +\infty\}$. Then, since $\log \|T_{t+s}\|_p \leqslant \log \|T_t\|_p + \log \|T_s\|_p$ for all positive real numbers $t$ and $s$, and since $\sup\{\|T_t\|_p\colon a \leqslant t \leqslant b\} < +\infty$ for all $0 < a < b < +\infty$, by the uniform boundedness principle (cf. Corollary II.3.21 in [4]), it follows from a slight modification of the proof of Lemma VIII.1.4 in [4] that $\lim_{t \to +\infty} (\|T_t\|_p)^{1/t} = d$. Hence we have

$$\int_0^\infty e^{-\lambda t} \|T_t f\|_p\, dt < +\infty$$

for all $0 < \lambda < +\infty$ satisfying $d < \exp(\lambda)$, and therefore Theorem 4 completes the proof of the theorem.



**References**


[1] L. Báez-Duarte, *An ergodic theorem of Abelian type*, J. Math. Mech. 15 (1966), pp. 599–607.

[2] R. V. Chacon, *Operator averages*, Bull. Amer. Math. Soc. 68 (1962), pp. 351–353.

[3] — *Convergence of operator averages*, in: *Ergodic theory* (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), pp. 89–120, Academic Press, New York 1963.

[4] N. Dunford and J. T. Schwartz, *Linear operators.* Part I: *General theory*, Interscience, New York 1958.

[5] D. A. Edwards, *A maximal ergodic theorem for Abel means of continuous-parameter operator semigroups*, J. Functional Analysis 7 (1971), pp. 61–70.

[6] — *On potentials and general ergodic theorems for resolvents*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 20 (1971), pp. 1–8.

[7] S. Hasegawa and R. Sato, *A general ratio ergodic theorem for semigroups*, Pacific J. Math. 62 (1976), pp. 435–437.

[8] —, — and S. Tsurumi, *Vector valued ergodic theorems for a one-parameter semigroup of linear operators*, to appear in Tôhoku Math. J.

[9] P. E. Kopp, *Almost everywhere convergence for Abel means of Dunford–Schwartz operators*, J. London Math. Soc. (2) 6 (1973), pp. 368–372.

[10] — *A pointwise ergodic theorem for pseudo-resolvents*, ibid. (2) 8 (1974), pp. 526–528.

[11] — *Abelian ergodic theorems for vector-valued functions*, Glasgow Math. J. 16 (1975), pp. 57–60.

[12] S. A. McGrath, *An Abelian ergodic theorem for semigroups in $L_p$ space*, Proc. Amer. Math. Soc. 54 (1976), pp. 231–236.

[13] — *On almost everywhere convergence of Abel means of contraction semigroups*, Pacific J. Math. 65 (1976), pp. 405–408.

[14] — *A pointwise Abelian ergodic theorem for $L_p$ semigroups, $1 \leqslant p < \infty$*, J. Functional Analysis 23 (1976), pp. 195–198.

[15] — *Abelian ergodic theorems for contraction semigroups*, Studia Math. 61 (1977), pp. 21–24.

[16] G.-C. Rota, *On the maximal ergodic theorem for Abel-limits*, Proc. Amer. Math. Soc. 14 (1963), pp. 722–723.

[17] R. Sato, *A general ratio ergodic theorem for Abel sums*, Studia Math. 46 (1973), pp. 223–233.

[18] — *Abel-ergodic theorems for subsequences*, Pacific J. Math. 47 (1973), pp. 233–242.

[19] — *An Abel-maximal ergodic theorem for semigroups*, ibid. 51 (1974), pp. 543–547.


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