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Resolving Banach spaces

by

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Abstract. In this paper we define a property of a projection algebra on a Banach space which we show to be necessary and sufficient for the existence of a resolution of the space taking values in a Banach lattice with order-continuous norm (proper L-space).

0. Introduction. If K is a compact Hausdorff space and $f \mapsto T_f$ an isometric embedding of $C(K)$ into $B(X)$ for some Banach space X, the question arises whether it is possible to find a norm resolution for X over K, i.e. a mapping $x \mapsto [x]$ of X into some Banach function space over K with the properties (a) $\|[x]\| = \|x\|$ for all x in X, (b) $[x+y] \leqslant [x]+[y]$ for x, y in X, (c) $|f|\,[x] = [T_f x]$ for all f in $C(K)$, x in X.

In [4] Cunningham showed the existence of such a resolution in the case where the operators T_f have a lattice property similar to that in M-spaces. He also showed how X can then be represented as a space of vector-valued functions over K, the *function module representation*, in such a way that the operators T_f on X correspond to multiplication by f in the representation. In his doctoral thesis [5] the author showed how to construct a representation of an analogous type, the *integral module representation*, in the case where the embedded copy of $C(K)$ is a strongly closed algebra generated by L^p-projections, that is projections E which have the norm decomposing property

$$\|x\|^p = \|Ex\|^p + \|x-Ex\|^p \quad \text{for all } x, \text{ some } p \in [1, \infty).$$

Attempts to extend the concept of L^p-projections by introducing projections with a more general decomposition property have failed, since it turns out that in all but trivial cases the only projections with the apparently more general property are the L^p-projections themselves (e.g. [3]). In this paper we define a decomposition property, not of individual projections, but of a complete projection algebra and show that this is general enough to completely describe the case where X has a resolution taking values in a Banach lattice with order-continuous norm.

1. Projection algebras. A (linear) *projection* on a Banach space X is a linear mapping $E\colon X \to X$ such that $E^2 = E$. It follows that $(I-E)^2$

$=I-E$ and, if E and F are two commuting projections, that $(EF)^2 = EF$ and $(E+F-EF)^2 = E+F-EF$, so that $I-E$, EF and $E+F-EF$ are also projections.

1.1. DEFINITION. A *Boolean algebra of projections* (*projection algebra*, abbrev.) is a set $\mathfrak{A}$ of commuting projections with the properties:
(i) $0, I \in \mathfrak{A}$, (ii) $E \in \mathfrak{A} \Rightarrow I-E \in \mathfrak{A}$, (iii) $E, F \in \mathfrak{A} \Rightarrow EF \in \mathfrak{A}$.
$\mathfrak{A}$ is a Boolean algebra with the lattice operations

$$E \wedge F = EF \quad \text{and} \quad E \vee F = E+F-EF\ (= I-(I-E)(I-F) \in \mathfrak{A}).$$

We say that a projection algebra $\mathfrak{A}$ on a Banach space X is *complete* if it is complete as a Boolean algebra and, in addition, for each monotone decreasing net $\{E_\gamma\}_{\gamma\in\Gamma}$, $(\inf_\Gamma E_\gamma)x = \lim_\Gamma E_\gamma x$.

Let K be the Stonean space of the projection algebra $\mathfrak{A}$, then there is a natural coorespondence between the elements of $\mathfrak{A}$ and the clopen subsets of K. If f is in $C(K)$ and $f^2 = f$, then f is the characteristic function of some subset of K which, by continuity, must be clopen. Thus there is a natural correspondence between the projections in $\mathfrak{A}$ and the idempotent functions in $C(K)$, a projection corresponding to the characteristic function of the relevant clopen set.

1.2. DEFINITION. A projection E on a Banach space X is said to be *bicontractive* if for all x in X, $\|x\| \geqslant \max\{\|Ex\|, \|x-Ex\|\}$, i.e. if both E and $I-E$ are bounded with norm $\leqslant 1$.

A projection algebra is clearly made up of bicontractive projections if and only if all the projections in it are bounded with norm $\leqslant 1$. In this case the correspondence between the algebra $\mathfrak{A}$ and the idempotent functions in $C(K)$ is norm-preserving. This correspondence can be extended by linearity to an algebra isomorphism between $\operatorname{lin}\mathfrak{A}$ and the step functions in $C(K)$. Even if the projection algebra consists only of bicontractive projections, this algebra isomorphism need not necessarily be an isometry, however we have the following result:

1.3. LEMMA. *Let $\mathfrak{A}$ be a projection algebra on the Banach space X consisting solely of bicontractive projections. Let $f \leftrightarrow T_f$ be the algebra isomorphism between the step functions in $C(K)$ and the operators in $\operatorname{lin}\mathfrak{A}$ (K the Stonean space of $\mathfrak{A}$). Then $\|f\| \leqslant \|T_f\| \leqslant M\cdot\|f\|$ for all step functions f in $C(K)$, whereby $M = \sup_{E\in\mathfrak{A}}\|2E-I\|$.*

Note. Since all E's are bicontractive, we have $1 \leqslant M \leqslant 2$.

Proof. Let f be a positive step function. Then we can write f in the form $f = \sum_1^n \lambda_i \chi_{D_i}$ whereby all λ_i's are positive and $\emptyset \subset D_1 \subset D_2 \subset \ldots \subset D_n$. In this case we clearly have $\|f\| = \sum_1^n \lambda_i$. T_f is then equal to $\sum_1^n \lambda_i E_i$, whereby the projections E_i correspond to the clopen sets D_i in K. Since

all the E_i's have norm $\leqslant 1$, we have immediately $\|T_f\| \leqslant \sum_1^n \lambda_i = \|f\|$. Now suppose that f is an arbitrary step function. Then there is a clopen set D in K such that $(2\chi_D - \mathbf{1})\cdot f$ is positive. Let E be the corresponding projection in $\mathfrak{A}$. We then have

$$\begin{aligned}\|T_f\| &= \|(2E-I)(2E-I)T_f\| \\ &\leqslant \|(2E-I)\|\cdot\|(2E-I)T_f\| \leqslant M\cdot\|(2E-I)T_f\| \\ &= M\cdot\|T_{(2\chi_D-1)f}\| \leqslant M\cdot\|(2\chi_D-\mathbf{1})f\| = M\|f\|.\end{aligned}$$

For the other inequality, let $f = \sum_1^n \lambda_i \chi_{D_i}$ be the representation of a continuous step function in which the D_i's are disjoint and non empty. In this case $\|f\|$ is clearly the maximum of the $|\lambda_i|$'s, say $|\lambda_j|$. Let E_j be the projection in $\mathfrak{A}$ corresponding to D_j. Then for $x \in E_jX$, $x \neq 0$ we have $\|T_f x\| = \|\lambda_j E_j x\| = |\lambda_j|\,\|x\|$ so that $\|T_f\| \geqslant |\lambda_j| = \|f\|$.

Note that the bounds in this lemma are the best possible since the functions corresponding to $2E-I$, $E\in\mathfrak{A}$, all have unit norm.

The result of Lemma 1.3 motivates the following definition:

1.4. DEFINITION. A projection E on a Banach space X is said to be a *mirror-projection* if $2E-I$ is an isometry. Since we have $(2E-I)^2 = 4E^2 - 4E+I = I$, this is equivalent to $\|2E-I\| = 1$.

A mirror-projection is clearly bi-contractive, since

$$2\|Ex\| \leqslant \|x\| + \|2Ex-x\| = \|x\| + \|(2E-I)x\| = 2\|x\|$$

and similarly for $(I-E)x$. Also, if E is a mirror-projection, so is $I-E$. However, the product of commuting mirror-projections need not be one. It has been shown ([2], Sect. 2) that in the classical Banach spaces (L^p-spaces, $C(K)$-spaces, Lindenstrauss spaces) all bicontractive projections are mirror-projections. As an immediate corollary of Lemma 1.3 we now obtain:

1.5. COROLLARY. *Let $\mathfrak{A}$ be a projection algebra on the Banach space X and K the Stonean space of $\mathfrak{A}$. The natural correspondence between the continuous step functions on K and the operators in $\operatorname{lin}\mathfrak{A}$ is an isometry if and only if all projections in $\mathfrak{A}$ are mirror-projections.*

In this case we can extend the isometry to the closure of $\operatorname{lin}\mathfrak{A}$ in $B(X)$ and that of the step functions in $C(K)$. Since K is totally disconnected, the step functions are dense in $C(K)$. We thus have:

1.6. PROPOSITION. *Let $\mathfrak{A}$ be a projection algebra on the Banach space X and K the Stonean space of $\mathfrak{A}$. Then $\overline{\operatorname{lin}\mathfrak{A}}$ is isometrically algebra-isomorphic to $C(K)$ if and only if all projections in $\mathfrak{A}$ are mirror-projections.*

Proof. Since an algebra-isomorphism maps idempotent elements into idempotent elements, it is the natural correspondence of Corollary 1.5 (except perhaps for a homeomorphism of K).

In view of the above results, when $\mathfrak{A}$ consists of mirror-projections, we shall identify $\overline{\operatorname{lin}}\,\mathfrak{A}$ and $C(K)$ and write fx for the action of the operator corresponding to the function f on the element x in X.

2. Decomposition properties. Having in the last section obtained an embedding of $C(K)$ in $B(X)$ for the case where K is the Stonean space of a Boolean algebra of mirror-projections, we now wish to obtain a norm resolution for X over K. In general, such a resolution need not exist since the projections in $\mathfrak{A}$ need not decompose the norms of elements in X in a consistent manner. Thus we shall need to demand some further property of $\mathfrak{A}$ which will guarantee a consistent decomposition. A seemingly weak property of this nature is contained in the following definition:

2.1. Definition. A projection algebra $\mathfrak{A}$ on a Banach space X is said to be *monotone*, if the following condition holds:

If $E_1, E_2, \ldots, E_n$ are pairwise orthogonal elements of $\mathfrak{A}$ with $\bigvee E_i = I$ and for some x, y in X, we have $\|E_i x\| \geqslant \|E_i y\|$ for all i, then also $\|x\| \geqslant \|y\|$,

i.e. if all parts of x are larger (in norm) than the respective parts of y, then x itself is larger than y.

Unfortunately this property is far too stringent for our purpose. Indeed, the author has shown ([6]) that in all but trivial cases a monotone algebra consists only of L^p-projections for some fixed p. We can weaken Definition 2.1 by requiring only that $\|x\| \geqslant \|y\|$ when $\|Ex\| \geqslant \|Ey\|$ for a larger number of projections than merely a single resolution of the identity. Since I itself is in $\mathfrak{A}$, it is clearly vacuous to demand $\|Ex\| \geqslant \|Ey\|$ for all E in $\mathfrak{A}$. The required modification is the subject of the following definition:

2.2. Definition. A subset A of a Boolean algebra $\mathfrak{A}$ is said to be *co-final* in $\mathfrak{A}$ if, for all $F \in \mathfrak{A}$, $F \neq 0$, there is an E in A with $0 < E \leqslant F$.

A projection algebra on a Banach space X is said to be *uniformly decomposing* if and only if $\|x\| \geqslant \|y\|$ whenever the set $\{E \mid \|Ex\| \geqslant \|Ey\|\}$ is co-final in $\mathfrak{A}$, $x, y \in X$.

Note. It then follows that $\|Ex\| \geqslant \|Ey\|$ for all E in $\mathfrak{A}$.

In the case of complete projection algebras there is another formulation of the uniformly decomposing property which turns out to be more useful for the construction of a norm resolution.

2.3. Lemma. *Let $\mathfrak{A}$ be a complete projection algebra on the Banach space X. The following two statements are equivalent:*

(a) *$\mathfrak{A}$ is uniformly decomposing;*

(b) *For all x, y in X there is an E in $\mathfrak{A}$ such that*

$$F \leqslant E \Rightarrow \|Fx\| \geqslant \|Fy\| \quad \textit{and} \quad F \perp E \Rightarrow \|Fx\| \leqslant \|Fy\| \quad \textit{for } F \textit{ in } \mathfrak{A}.$$

Proof. (a)$\Rightarrow$(b) Let x, y be given. Set $E = \bigvee\{F \mid F \in \mathfrak{A},\ \|\hat{F}x\| \geqslant \|\hat{F}y\|$ for $\hat{F} \leqslant F\}$. Since $\mathfrak{A}$ is uniformly decomposing, we clearly have $\|Fx\| \geqslant \|Fy\|$ for all $F \leqslant E$. Suppose $F \perp E$ and $\tilde{F} \in \mathfrak{A}$, $\tilde{F} \neq 0$. Then either there is an $\hat{F} \leqslant \tilde{F}F$ with $\|\hat{F}x\| < \|\hat{F}y\|$ or we have $\tilde{F}F \leqslant E$ which then means $\tilde{F}F = 0$ since $F \perp E$. In the latter case set $\hat{F} = \tilde{F}$. Either way we now have $\hat{F} \leqslant \tilde{F}$, $\hat{F} \neq 0$ with $\|\hat{F}Fx\| \leqslant \|\hat{F}Fy\|$. Thus the set of all $\hat{F} \in \mathfrak{A}$ with $\|\hat{F}Fx\| \leqslant \|\hat{F}Fy\|$ is co-final in $\mathfrak{A}$. Since $\mathfrak{A}$ is uniformly decomposing, we have $\|Fx\| \leqslant \|Fy\|$ as required.

(b)$\Rightarrow$(a) Suppose that $\|x\| < \|y\|$ for some pair x, y in X. Choose $\varepsilon > 0$ with $\|x\| < (1-\varepsilon)\|y\|$ and use (b) to find an E in $\mathfrak{A}$ with $F \leqslant E \Rightarrow \|Fx\| \geqslant \|F(1-\varepsilon)y\|$ and $F \perp E \Rightarrow \|Fx\| \leqslant \|F(1-\varepsilon)y\|$. Let $\hat{E}$ be the carrier projection of x, that is the smallest projection which maps x onto itself. Then $E \operatorname{non} \geqslant \hat{E}$ since $\|x\| < (1-\varepsilon)\|y\|$. Thus $\hat{E} \wedge (I-E) \neq 0$. Let $F \leqslant \hat{E} \wedge (I-E)$, $F \neq 0$. Then $\|Fx\| \neq 0$ and $F \leqslant I-E$ so that $\|Fx\| \leqslant \|F(1-\varepsilon)y\| < \|Fy\|$. Thus the set $\{F \mid F \in \mathfrak{A},\ \|Fx\| \geqslant \|Fy\|\}$ is not co-final in $\mathfrak{A}$.

Thus a complete uniformly decomposing projection algebra contains, for each pair x, y, a projection which divides the space into the part where x is larger (in norm) than y and the part where it is smaller. This is clearly a necessary condition for the existence of a norm resolution whose values lie in a lattice with order-continuous norm. We shall see in the next section that it is also sufficient.

Since Proposition 1.6 refers to projection algebras consisting only of mirror-projections, it would seem that we must in future demand of $\mathfrak{A}$ that it is complete, uniformly decomposing and consists only of mirror-projections. The following simple lemma shows that the latter is redundant.

2.4. Lemma. *Let $\mathfrak{A}$ be a (not necessarily complete) uniformly decomposing projection algebra on the Banach space X. Then $\mathfrak{A}$ consists solely of mirror-projections.*

Proof. Let E be a projection in $\mathfrak{A}$ and x an element in X. For x and $(2E-I)x$ we have:

$$\text{for } F \leqslant E, \qquad \|Fx\| = \|2Fx - Fx\| = \|2FEx - Fx\| = \|F(2E-I)x\|,$$

$$\text{for } F \perp E, \qquad \|Fx\| = \|Fx - 2Fx\| = \|Fx - 2F(I-E)x\| = \|F(2E-I)x\|,$$

and since the set $\{F \mid F \leqslant E \text{ or } F \perp E\}$ is co-final in $\mathfrak{A}$, we have $\|x\| = \|(2E-I)x\|$. Since x was arbitrary, $2E-I$ is an isometry.

Note that although a sub-algebra of an algebra consisting solely of mirror-projections naturally also consists solely of mirror-projections, a sub-algebra of an uniformly decomposing algebra need not be uniformly decomposing. Indeed we have the following proposition.

2.5. PROPOSITION. *If $\mathfrak{A}$ is an uniformly decomposing projection algebra on a Banach space X such that every sub-algebra of $\mathfrak{A}$ is also uniformly decomposing, then $\mathfrak{A}$ is monotone.*

Proof. Let $E_1, E_2, \ldots, E_n$ be pairwise orthogonal projections in $\mathfrak{A}$ with $\bigvee E_i = I$. Let $\mathfrak{A}_1$ be the sub-algebra of $\mathfrak{A}$ generated by the E_i's. $\mathfrak{A}_1$ is atomic and its atoms are the E_i's. Suppose that for some x, y in X $\|E_i x\| \geqslant \|E_i y\|$ for all i. Then $\{E|\ E$ in $\mathfrak{A}_1,\ \|Ex\| \geqslant \|Ey\|\}$ is co-final in $\mathfrak{A}_1$, which is supposed uniformly decomposing. Thus $\|x\| \geqslant \|y\|$, which implies that $\mathfrak{A}$ is monotone.

As already noted, in all but trivial cases a monotone algebra consists solely of L^p-projections for some fixed p and for these algebras the problem of constructing a norm resolution has already been solved.

3. A norm resolution. We now turn to the construction of a norm resolution for the case where $\mathfrak{A}$ is a complete uniformly decomposing projection algebra. The first step in this direction is the following proposition, which relies heavily on Lemma 2.3.

3.1. PROPOSITION. *Let $\mathfrak{A}$ be a complete uniformly decomposing projection algebra on the Banach space X. For each x, y in X, $y \neq 0$, the quotient $\|Ex\|/\|Ey\|$ converges (possibly to ∞) along each ultrafilter U containing the carrier projection of y*

Proof. Let x, y be elements of X and U an ultrafilter in $\mathfrak{A}$ containing the carrier projection of y. Then $\|Ey\| \neq 0$ for all E in U so that the quotient $\|Ex\|/\|Ey\|$ is defined. Let us assume that $\|Ex\|/\|Ey\|$ does not converge along U so that we can find two co-final nets $\{E_\gamma\}_{\gamma\in\Gamma_1}$, $\{E_\gamma\}_{\gamma\in\Gamma_2}$ in U with $\|E_\gamma x\|/\|E_\gamma y\| \xrightarrow[\Gamma_1]{} \lambda_1$, $\|E_\gamma x\|/\|E_\gamma y\| \xrightarrow[\Gamma_2]{} \lambda_2$ whereby $\lambda_1 > \lambda_2$ (λ_1 possibly $= \infty$). Let λ be a finite number with $\lambda_1 > \lambda > \lambda_2$. Then by Lemma 2.3 there is a projection E_λ in $\mathfrak{A}$ such that $\|Fx\| \geqslant \|F(\lambda y)\|$ for $F \leqslant E_\lambda$ and $\|Fx\| \leqslant \|F(\lambda y)\|$ for $F \perp E_\lambda$. There are now two possibilities:

(a) E_λ lies in U, but then $E_\gamma \leqslant E_\lambda$ co-finally for $\gamma \in \Gamma_2$ so that

$$\|E_\gamma x\|/\|E_\gamma y\| = \lambda\|E_\gamma x\|/\|E_\gamma(\lambda y)\| \geqslant \lambda \text{ co-finally}.$$

(b) E_λ does not lie in U, but then $E_\gamma \perp E_\lambda$ co-finally in Γ_1 so that

$$\|E_\gamma x\|/\|E_\gamma y\| = \lambda\|E_\gamma x\|/\|E_\gamma(\lambda y)\| \leqslant \lambda \text{ co-finally}.$$

In either case we have a contradiction, so that we may conclude that our assumption that the quotient does not converge was false.

Note. If K is the Stonean space of $\mathfrak{A}$ the points of K are strictly speaking the ultrafilters in $\mathfrak{A}$, nevertheless we shall write k for a point in K and U_k for the corresponding ultrafilter, in the interest of clarity. For x in X, $\operatorname{supp} x$ will denote the clopen set in K corresponding to the carrier projection of x, this is the same as $\{k|\ U_k$ contains the carrier projection of $x\}$.

The above proposition now allows us to make the following definition:

3.2. DEFINITION. Let $\mathfrak{A}$ be a complete uniformly decomposing projection algebra on the Banach space X and K the Stonean space of $\mathfrak{A}$. For x, y in X, $y \neq 0$ $x/y\colon \operatorname{supp} y \to [0, +\infty]$ is the function defined by $x/y(k)\colon := \lim\limits_{U_k} \|Ex\|/\|Ey\|$ for k in $\operatorname{supp} y$.

The most important elementary properties of these functions are summed up in the following lemma.

3.3. LEMMA. *Let $\mathfrak{A}, X, K$ be as in 3.2, $y \in X$, $y \neq 0$, then x/y is continuous and finite almost everywhere for all $x \in X$ and the mapping $x \mapsto x/y$ from X into $C_{\overline{\mathbf{R}}}(\operatorname{supp} y)$ is sub-linear and absolutely homogeneous with respect to $C(K)$ ($\cong \overline{\operatorname{lin}}\,\mathfrak{A}$).*

Proof. Suppose $x/y(k) = \lambda$, with $0 < \lambda < \infty$. Let ε be arbitrary between 0 and λ. Then by 2.3 there is a projection E_1 in $\mathfrak{A}$ such that

$$F \leqslant E_1 \Rightarrow \|Fx\| \geqslant \|F(\lambda+\varepsilon)y\| \quad \text{and} \quad F \perp E_1 \Rightarrow \|Fx\| \leqslant \|F(\lambda+\varepsilon)y\|$$

and a projection E_2 in $\mathfrak{A}$ with

$$F \leqslant E_2 \Rightarrow \|Fx\| \geqslant \|F(\lambda-\varepsilon)y\| \quad \text{and} \quad F \perp E_2 \Rightarrow \|Fx\| \leqslant \|F(\lambda-\varepsilon)y\|.$$

For all points l for which U_l contains $E_2(I-E_1)$ we now have $\lambda-\varepsilon \leqslant x/y\ (l) \leqslant \lambda+\varepsilon$. These l form a clopen set containing k. For $\lambda = 0$ or ∞, an analogous argument with one projection suffices. In either case we have that x/y is continuous at k.

Now let D be the clopen set $\operatorname{int}\{k|\ k \in \operatorname{supp} y,\ x/y(k) = \infty\}$. Then for each natural number n, $\{E|\ \|E\chi_D x\| \geqslant n\|E\chi_D y\|\}$ is co-final in $\mathfrak{A}$ since $\|Ex\|/\|Ey\| \to \infty$ along each ultrafilter U_k with $k \in D$. Since $\mathfrak{A}$ is uniformly decomposing, it follows that $\|\chi_D x\| \geqslant n\|\chi_D y\|$ for all n and thus that $\chi_D y = 0$. Since $D \leqslant \operatorname{supp} y$ it follows that $D = \emptyset$. Thus x/y is finite almost everywhere.

That the mapping $x \mapsto x/y$ is sub-linear follows immediately from the sub-linearity of the norm. To check absolute homogeneity let x be an element in X and f a function in $C(K)$. We must show that for all $k \in \operatorname{supp} y$ $(fx/y)(k) = |f(k)|(x/y)(k)$ whenever $x/y(k)$ is finite. Let k be a point in $\operatorname{supp} y$ and $\alpha = f(k)$.

$$\big|\|Efx\| - \|\alpha Ex\|\big| \leqslant \|Efx - \alpha Ex\| = \|E(f-\alpha I)Ex\| \leqslant \|E(f-\alpha I)\| \cdot \|Ex\|$$

since the operator f lies in $\overline{\operatorname{lin}}\,\mathfrak{A}$ and therefore commutes with E. If D is the clopen set in K corresponding to E, we have

$$\|E(f-\alpha I)\| = \|\chi_D(f-\alpha I)\| = \sup_{l\in D} |f(l)-\alpha|.$$

Since $\alpha = f(k)$ and f is continuous at k, we have $\|E(f-\alpha I)\| \underset{U_k}{\to} 0$. It follows that

$$\lim_{U_k}\left|\frac{\|Efx\|}{\|Ey\|} - \frac{\|\alpha Ex\|}{\|Ey\|}\right| \leqslant \lim_{U_k}\|E(f-\alpha I)\|\frac{\|Ex\|}{\|Ey\|} = 0$$

at each point k where $x/y(k)$ $(= \lim_{U_k}\|Ex\|/\|Ey\|)$ is finite. Since

$$\lim_{U_k}\frac{\|Efx\|}{\|Ey\|} = (fx/y)(k) \quad \text{and} \quad \lim_{U_k}\frac{\|\alpha Ex\|}{\|Ey\|} = |\alpha| x/y(k) = |f(k)|(x/y)(k)$$

this is the required identity.

The mappings $x \mapsto x/y$ for each y thus have the properties of a norm resolution except for the norm preserving property since the range space $C_{\overline{\mathbf{R}}}(\operatorname{supp} y)$ is not normed and in general not even normable (as a lattice). We can however construct a Banach function lattice in $C_{\overline{\mathbf{R}}}(\operatorname{supp} y)$ which is large enough to contain the functions x/y. This is the purpose of the following definitions and results.

3.4. Definitions. Let X be a Banach space and $\mathfrak{A}$ a Boolean algebra of projections on X. We define the ordering $\geqslant_{\mathfrak{A}}$ (or $\geqslant$ if $\mathfrak{A}$ is clear) by

$$x \geqslant_{\mathfrak{A}} y \Leftrightarrow \text{there is an } E \text{ in } \mathfrak{A} \text{ with } Ex = y,$$

i.e. x is larger than y if and only if it is an extension of it. This is clearly a partial order on X.

Let X be a Banach space and $\mathfrak{A}$ a complete uniformly decomposing Boolean algebra of projections on X with Stonean space K. Suppose $\Gamma \subseteq X$ is a subset which is directed by $\geqslant_{\mathfrak{A}}$. The *support of* Γ is the set $\operatorname{supp}\Gamma := (\bigcup_{y\in\Gamma} \operatorname{supp} y)^-$ a clopen subset of K. We define the mapping m_Γ from $C(\operatorname{supp}\Gamma)$ into $[0, \infty]$ by virtue of

$$m_\Gamma(f) := \sup_{y\in\Gamma}\|fy\|$$

and $C_\Gamma \subseteq C(\operatorname{supp}\Gamma)$ by

$$C_\Gamma := \{f \mid f \in C(\operatorname{supp}\Gamma),\ \{fy\}_{y\in\Gamma} \text{ is Cauchy}\}.$$

Finally we define $M_\Gamma \subseteq C_{\overline{\mathbf{R}}}(\operatorname{supp}\Gamma)$ as the set

$$\{f \mid f \in C_{\overline{\mathbf{R}}}(\operatorname{supp}\Gamma), \text{ the increasing net of positive } C_\Gamma\text{-functions which are majorised by } |f| \text{ is } m_\Gamma\text{-Cauchy}\}$$

and extend m_Γ to M_Γ by means of

$$m_\Gamma(f) := \sup\{m_\Gamma(g) \mid 0 \leqslant g \leqslant |f|,\ g \in C_\Gamma\}.$$

If Γ consists solely of one element y, we shall write M_y and m_y instead of M_Γ and m_Γ.

3.5. Proposition. *M_Γ is an order ideal in $C_{\overline{\mathbf{R}}}(\operatorname{supp}\Gamma)$, m_Γ is an order-continuous lattice norm for M_Γ and M_Γ is complete in m_Γ. Thus M_Γ with m_Γ is a Banach lattice with order-continuous norm.*

Proof. It is clear from the definitions that M_Γ is an order ideal in $C_{\overline{\mathbf{R}}}(\operatorname{supp}\Gamma)$ and that m_Γ is a lattice norm. It remains to show that m_Γ is order-continuous and that M_Γ is complete. Let $\{f_\alpha\}$ be a downwards directed net in M_Γ whose infimum is 0. We may suppose without loss of generality that $\{f_\alpha\}$ has a largest element, say f. Since f lies in M_Γ, there is an f_0 in C_Γ with $0 \leqslant f_0 \leqslant f$ and $m_\Gamma(f-f_0) \leqslant \varepsilon/5$ for a given $\varepsilon > 0$. For each α we define $D_\alpha := \{k \mid f_\alpha(k) < \varepsilon f_0(k)/5m_\Gamma(f_0)\}^-$ and $D := (\bigcup D_\alpha)^-$. We then have $\varepsilon(f_0 - \chi_D f_0)/5m_\Gamma(f_0) \leqslant f_\alpha$ for all α. Since the net converges in order to 0 and $f_0 - \chi_D f_0$ is positive, we have $f_0 - \chi_D f_0 = 0$. As f_0 lies in C_Γ, there is an $y_0 \in \Gamma$ such that $\|f_0(y-y_0)\| \leqslant \varepsilon/5$ for $y_0 \leqslant y \in \Gamma$. $(1-\chi_D)f_0 y_0$ is 0, therefore by the completeness of $\mathfrak{A}$ there is an α_0 with

$$\alpha \geqslant \alpha_0 \Rightarrow \|(1-\chi_{D_\alpha})f_0 y_0\| \leqslant \varepsilon/5$$

and then for $y \geqslant y_0$ we have

$$\begin{aligned}\|(1-\chi_{D_\alpha})f_0 y\| &\leqslant \|(1-\chi_{D_\alpha})f_0(y-y_0)\| + \|(1-\chi_{D_\alpha})f_0 y_0\| \\ &\leqslant \|f_0(y-y_0)\| + \|(1-\chi_{D_\alpha})f_0 y_0\| \leqslant \varepsilon/5 + \varepsilon/5 = 2\varepsilon/5.\end{aligned}$$

We thus have $m_\Gamma((1-\chi_{D_\alpha})f_0) \leqslant 2\varepsilon/5$. But then

$$\begin{aligned}m_\Gamma(f_\alpha) &\leqslant m_\Gamma((1-\chi_{D_\alpha})f_\alpha) + m_\Gamma(\chi_{D_\alpha}f_\alpha) \\ &\leqslant m_\Gamma((1-\chi_{D_\alpha})(f_\alpha - f_0)) + m_\Gamma((1-\chi_{D_\alpha})f_0) + \varepsilon m_\Gamma(\chi_{D_\alpha}f_0)/5m_\Gamma(f_0) \\ &\leqslant m_\Gamma(f-f_0) + 2\varepsilon/5 + \varepsilon/5 \leqslant 4\varepsilon/5 < \varepsilon \quad \text{for} \quad \alpha \geqslant \alpha_0.\end{aligned}$$

Thus $\{m_\Gamma(f_\alpha)\} \to 0$. This shows that m_Γ is order-continuous.

In order to show that M_Γ is complete, let (f_n) be a monotone increasing Cauchy sequence of positive functions in M_Γ. Let f be the supremum of (f_n) in $C_{\overline{\mathbf{R}}}(\operatorname{supp}\Gamma)$; if we show that f is in M_Γ, we shall be finished since the order-continuity of m_Γ implies that (f_n) converges to f. Let $\varepsilon > 0$ be given. Since (f_n) is Cauchy, there is an n such that $m_\Gamma(f_m - f_n) < \varepsilon/2$ for $m \geqslant n$. Since f_n is in M_Γ, there is a g_n in C_Γ with $m_\Gamma(f_n - g_n) < \varepsilon/2$. But then for g in C_Γ, $g_n \leqslant g \leqslant f$ we have

$$\begin{aligned}m_\Gamma(g-g_n) &\leqslant m_\Gamma(g - g\wedge f_m) + m_\Gamma(g\wedge f_m - g_n) \\ &\leqslant m_\Gamma(g - g\wedge f_m) + m_\Gamma(f_m - g_m) < m_\Gamma(g - g\wedge f_m) + \varepsilon.\end{aligned}$$

As $g \leqslant f$, we have $(g - g\wedge f_m) \to 0$ in order and thus also in norm. So $m_\Gamma(g - g_n) \leqslant \varepsilon$ which implies (since ε was arbitrary) that f is in M_Γ.

Since the M_Γ's are Banach lattices with order-continuous norm, an M_Γ with $\operatorname{supp}\Gamma = K$ is clearly a prime candidate for the range space of a norm resolution for X. However, we must first check that the M_Γ's are large enough to contain the functions x/y.

3.6. DEFINITION. If $\Gamma \subseteq X$ is directed and $\operatorname{supp}\Gamma \neq \emptyset$, then for non-zero $y_1, y_2 \in \Gamma$, $y_1 \geqslant y_2$ the functions x/y_1, x/y_2 for x in X are clearly equal on their common domain of definition. Thus we can define $x/\Gamma\colon \operatorname{supp}\Gamma \to [0, \infty]$ as the unique continuous extension to $\operatorname{supp}\Gamma$ of the functions x/y on $\operatorname{supp} y$, $y \in \Gamma$.

Note. If Γ contains a maximal element y, then $x/\Gamma = x/y$; this is in particular the case when Γ only contains one y.

3.7. LEMMA. *With $X, \mathfrak{A}, K, \Gamma$ as above we have:*

(i) *$x+z/\Gamma \leqslant x/\Gamma + z/\Gamma$ for all x, z in X;*

(ii) *$fx/\Gamma = |f|(x/\Gamma)$ for all x in X, f in $C(K)$.*

Furthermore, for every x in X, x/Γ lies in M_Γ and $m_\Gamma(x/\Gamma) = \|Ex\|$ whereby E is the projection in $\mathfrak{A}$ corresponding to $\operatorname{supp}\Gamma$.

Proof. (i) and (ii) follow obviously from the corresponding relations for x/y.

Let x be an element in X for which x/Γ is finite. We write $f = x/\Gamma \in C(\operatorname{supp}\Gamma)$. Then $fy/y = f|_{\operatorname{supp} y} = x/y$ for y in Γ. If E_y denotes the carrier projection of an element y, then we have $y_2 = E_{y_2} y_1$ for y_1, y_2 in Γ, $y_1 \geqslant y_2$. Thus

$$\|fy_1 - fy_2\| = \|E_{y_1} f y_1 - E_{y_2} f y_1\| = \|(E_{y_1} - E_{y_2}) f y_1\|.$$

Since $E_{y_2} \leqslant E_{y_1}$, we have $\|(E_{y_1} - E_{y_2}) f y_1\| = \|(E_{y_1} - E_{y_2}) x\|$ as a consequence of the equality $fy_1/y_1 = x/y_1$. That $\{E_y x\}_{y \in \Gamma}$ is Cauchy follows from the completeness of $\mathfrak{A}$, thus $\{fy\}_{y\in\Gamma}$ is also Cauchy and f therefore lies in C_Γ. $m_\Gamma(f) = \sup\|fy\| = \sup\|E_y x\| = \|Ex\|$ by the completeness of $\mathfrak{A}$. Now suppose x is an arbitrary element in X. We write $f = x/\Gamma$, $D_n := \{k \mid k \in \operatorname{supp}\Gamma, f(k) < n\}^-$ clopen in K, E_n the corresponding projection and $f_n = \chi_{D_n} f$. Then $\chi_{D_n} f = E_n x/\Gamma$ so that $\chi_{D_n} f$ lies in C_Γ for all n. Since x/Γ is clearly the supremum of the f_n's, it suffices to show that (f_n) is Cauchy in M_Γ. But $f_n - f_m = (\chi_{D_n} - \chi_{D_m}) f = (E_n - E_m) x/\Gamma$ and thus $m_\Gamma(f_n - f_m) = \|(E_n - E_m) Ex\|$. The sequence $(E_n Ex)$ is Cauchy by the completeness of $\mathfrak{A}$. So x/Γ is in M_Γ and

$$m_\Gamma(x/\Gamma) = \sup m_\Gamma(f_n) = \sup\|E_n Ex\| = \|Ex\|.$$

We are now ready to prove our main theorem except that we need the existence of Γ's with $\operatorname{supp}\Gamma = K$.

3.8. LEMMA. *With $X, \mathfrak{A}, K$ as above let Γ_0 be any finite subset of X with $\operatorname{supp} x \cap \operatorname{supp} y = \emptyset$ for $x, y \in \Gamma_0$, $x \neq y$. Then there is a directed set $\Gamma \subseteq X$ containing Γ_0 with $\operatorname{supp}\Gamma = K$.*

In particular, there are directed sets Γ with $\operatorname{supp}\Gamma = K$ (simply set $\Gamma_0 = \{x\}$ for some non-zero x).

Proof. By Zorn's Lemma there is a maximal set $\Gamma_1 \subseteq X$ containing Γ_0 such that the supports of distinct elements of Γ_1 are disjoint. Let $D = \big(\bigcup_{\Gamma_1} \operatorname{supp} y\big)^-$. If $D \neq K$, then there is a non-zero element x in X with $\operatorname{supp} x \subseteq K \setminus D$. This would contradict the maximality of Γ_1. Thus $\big(\bigcup_{\Gamma_1} \operatorname{supp} y\big)^- = K$. Let Γ be the set of all finite sums of elements in Γ_1. Since the sum of two elements with disjoint support majorises both elements in the order $\geqslant_{\mathfrak{A}}$, Γ is a directed set containing Γ_0 for which $\operatorname{supp}\Gamma = K$.

3.9. THEOREM. *Let X be a Banach space and $\mathfrak{A}$ a complete uniformly decomposing projection algebra on X with Stonean space K. Then there is a norm resolution for X with respect to $\overline{\operatorname{lin}}\,\mathfrak{A} \cong C(K)$ taking values in a Banach lattice of continuous numerical functions on K with order-continuous norm.*

Proof. Let Γ be a directed set in X with $\operatorname{supp}\Gamma = K$. Then by Lemma 3.7, the mapping $x \mapsto x/\Gamma$ is a norm resolution (since $E = I$ in this case) taking values in the Banach lattice M_Γ which is a lattice of continuous numerical functions on K with the order-continuous norm m_Γ.

4. Cycles and ideals. With the help of the norm resolutions defined in the last section we can show that the $\mathfrak{A}$-cycles and ideals have several nice properties. The reader is reminded of the following definitions.

4.1. DEFINITION. Let X be a Banach space and A a commutative subset of $B(X)$. A closed subspace J of X is called an *A-cycle* if J is an invariant subspace for every operator in A and an *A-ideal* if it is invariant for every operator in $[A]_{\text{comm}}$.

Since A itself is commutative, an A-ideal is an A-cycle. Also if J_α is a family of A-cycles (resp. A-ideals), then $\bigcap J_\alpha$ and $\big(\sum J_\alpha\big)^-$ are also A-cycles (resp. A-ideals). In particular, we can define the A-cycle (resp. A-ideal) generated by a subset of X as the intersection of all A-cycles (A-ideals) containing it. In our context we are naturally interested in the $\mathfrak{A}$-cycles and ideals where $\mathfrak{A}$ is a complete uniformly decomposing projection algebra on X. The cycles generated by directed sets turn out to have a very simple form.

4.2. PROPOSITION. *Let $\mathfrak{A}$ be a complete uniformly decomposing projection algebra on a Banach space X and Γ a directed subset of X. $S(\mathfrak{A}:\Gamma)$, the $\mathfrak{A}$-cycle generated by Γ, is isometrically isomorphic to the Banach space M_Γ.*

Proof. A simple calculation shows that $S(\mathfrak{A}:\Gamma) = \{fy \mid f \in C(K), y \in \Gamma\}^-$. Consider the mapping $fy \mapsto f\chi_{\operatorname{supp} y}$ for f in $C(K)$, y in Γ. This is well-defined and since for $y_1 \geqslant y$ in Γ we have $f\chi_{\operatorname{supp} y} y_1 = fy$, $f\chi_{\operatorname{supp} y}$ lies in C_Γ and has norm $\|fy\|$. Furthermore, since $fy_1 + gy_2 = (f\chi_{\operatorname{supp} y_1} + g\chi_{\operatorname{supp} y_2})\, y$ for any $y \geqslant y_1, y_2$, the mapping is linear. This mapping then extends to an isometry between $S(\mathfrak{A}:\Gamma)$ and the closure in M_Γ of the functions of the form $f\chi_{\operatorname{supp} y}$, $f \in C(K)$, $y \in \Gamma$. Inspection of the definitions of C_Γ and M_Γ shows that this closure is all of M_Γ. Observe that this mapping maps an element x onto a function whose absolute value is x/Γ.

One of the most interesting problems in the general theory of cycles is the question of the existence of a projection in $[A]_{\mathrm{comm}}$ projecting onto an A-cycle J. The positive answer for cycles of the form $S(\mathfrak{A}:\Gamma)$ is a simple corollary of the preceding proposition.

4.3. Corollary. *Let $\mathfrak{A}$ be a complete uniformly decomposing projection algebra on a Banach space X and Γ a directed subset of X. There is a contractive projection from X onto $S(\mathfrak{A}:\Gamma)$ which commutes with $\mathfrak{A}$.*

Proof. Let j be the isometry between $S(\mathfrak{A}:\Gamma)$ and M_Γ which was constructed in the preceding proposition. We have $j(x) \leqslant |j(x)| = x/\Gamma$ for all x in $S(\mathfrak{A}:\Gamma)$. Since M_Γ is order-complete, we can apply the Hahn–Banach theorem to obtain a linear mapping $T: X \to M_\Gamma$ which extends j and for which $Tx \leqslant x/\Gamma$ for all x in X. Since also $-Tx = T(-x) \leqslant (-x)/\Gamma$ $= x/\Gamma$, we have $|Tx| \leqslant x/\Gamma$. Consider the mapping $j^{-1}T$. This clearly maps X into $S(\mathfrak{A}:\Gamma)$ and for x in $S(\mathfrak{A}:\Gamma)$ we have $j^{-1}T(x) = j^{-1}j(x) = x$. This is thus a projection onto $S(\mathfrak{A}:\Gamma)$. Since

$$\|j^{-1}T(x)\| = m_\Gamma(Tx) = m_\Gamma(|Tx|) \leqslant m_\Gamma(x/\Gamma) = \|Ex\|,$$

it is also contractive (E as in 3.7). Let F be a projection in $\mathfrak{A}$. Then for all x in X we have

$$\begin{aligned} Fj^{-1}T(I-F)x/\Gamma &= \chi_D\big(j^{-1}T(I-F)x/\Gamma\big) \leqslant \chi_D\big((I-F)x/\Gamma\big) \\ &= \chi_D(1-\chi_D)(x/\Gamma) = 0, \end{aligned}$$

where D is the clopen set in K corresponding to F. Since $Fj^{-1}T(I-F)x \in S(\mathfrak{A}:\Gamma)$, this implies that $Fj^{-1}T(I-F)x = 0$. Since this holds for all projections in $\mathfrak{A}$ and

$$Fj^{-1}Tx - j^{-1}TFx = Fj^{-1}T(I-F)x - (I-F)j^{-1}TFx,$$

the projection $j^{-1}T$ commutes with $\mathfrak{A}$.

In our concrete case the other subspaces, the $\mathfrak{A}$-ideals, have an even simpler form.

4.4. Proposition. *Let $\mathfrak{A}$ be a complete uniformly decomposing projection algebra on a Banach space X. A closed subspace J of X is an $\mathfrak{A}$-ideal if and only if it is the range of a projection in $\mathfrak{A}$.*

Proof. Clearly the range of a projection in $\mathfrak{A}$ is an $\mathfrak{A}$-ideal. Now suppose J is an $\mathfrak{A}$-ideal and let E be the supremum in $\mathfrak{A}$ of the carrier projections E_y of elements y in J. Clearly $J \subseteq EX$; we shall show the reverse inclusion. Take x in EX and $\varepsilon > 0$. Since $E = \bigvee E_y$, there is a y in J with $\|x - E_y x\| < \varepsilon/2$. For each n in N let $D_n := \{k|\ x/y(k) < n\}^-$ clopen in K, the Stonean space of $\mathfrak{A}$, and E_n the corresponding projection in $\mathfrak{A}$. Since $E_y = \bigvee E_n$, there is an E_n with $\|x - E_n x\| < \varepsilon$. Set $g = E_n x/y$. Consider the mapping $fy \mapsto fE_n x$ for $f \in C(K)$. Since $E_n \leqslant E_y$ it is well-defined, is clearly linear and commutes with $C(K)$ and thus with $\mathfrak{A}$. Also $fE_n x/y = fg = fgy/y$ so that $\|fE_n x\| = \|fgy\| \leqslant \|g\|\|fy\|$. The mapping therefore extends to a continuous linear mapping from $S(\mathfrak{A}: y)$ into $S(\mathfrak{A}: x)$. Let F be a projection in $[\mathfrak{A}]_{\mathrm{comm}}$ mapping X onto $S(\mathfrak{A}: y)$, then TF lies in $[\mathfrak{A}]_{\mathrm{comm}}$ and $TFy = Ty = E_n x$. Since y is in J and J is an $\mathfrak{A}$-ideal, $E_n x$ is also in J. But $\|x - E_n x\| < \varepsilon$ and ε was arbitrary, so x itself lies in J.

The results of this section generalize 2.10–2.12 of [5] which also form 4.2, 4.4 and 4.5 of [1].

5. A characterization and a representation theorem. This section is devoted to the proof of two theorems. The first is an application of the results of the last section to obtain a Banach space characterization of Banach lattices with order-continuous norm. The second is a representation theorem analogous to the function module representation of Cunningham [4] and our own integral module representation [5], [1].

5.1. Theorem. *Let X be a Banach space. Then the following properties of a complete uniformly decomposing projection algebra $\mathfrak{A}$ on X are equivalent:*

(i) $[\mathfrak{A}]_{\mathrm{comm}} = \overline{\operatorname{lin}}\,\mathfrak{A}$;

(ii) *$\mathfrak{A}$ is a maximal Boolean algebra of bounded projections;*

(iii) *Every $\mathfrak{A}$-cycle is an $\mathfrak{A}$-ideal.*

Moreover, there is such an algebra on X if and only if X is isometrically isomorphic to a Banach lattice with order-continuous norm.

Proof. Clearly (i) $\Rightarrow$ (ii) and by 4.3, (ii) $\Rightarrow$ (iii).

Assume (iii) and let Γ be a directed set in X with $\operatorname{supp}\Gamma = K$, the Stonean space of $\mathfrak{A}$. By 4.2 $S(\mathfrak{A}:\Gamma)$ is isometrically isomorphic to M_Γ a Banach lattice with order-continuous norm. However, $S(\mathfrak{A}:\Gamma)$ is an $\mathfrak{A}$-cycle and therefore also an $\mathfrak{A}$-ideal. By 4.4 $S(\mathfrak{A}:\Gamma)$ is the range of a projection in $\mathfrak{A}$. Since $\operatorname{supp}\Gamma = K$, this must be the identity, i.e. $S(\mathfrak{A}:\Gamma)$ $= X$. Thus X is isometrically isomorphic to a Banach lattice with order-continuous norm. Note that the projections in $\mathfrak{A}$ correspond to the projections $f \mapsto f\chi_D$ for clopen D and these are exactly the band projections in M_Γ.

Now suppose that X is isometrically isomorphic to a Banach lattice M with order-continuous norm. Then X is itself a Banach lattice with order-continuous norm in the induced ordering. Let $\mathfrak{A}$ be the Boolean algebra of band projections on X. Since X has order-continuous norm, $\mathfrak{A}$ is a complete projection algebra. Also a simple calculation shows that $\overline{\mathrm{lin}\,\mathfrak{A}}$ is the centre of X (i.e. all operators T for which $-aI \leqslant T \leqslant aI$ for some $a \in \boldsymbol{R}$). It follows from Lemma 2.3 that $\mathfrak{A}$ is uniformly decomposing since the band projection onto the band generated by $(|x|-|y|)^+$ clearly satisfies (b) of the lemma. Let T be an operator in $[\mathfrak{A}]_{\mathrm{comm}}$ and x an element of X. Set $z := \big(|Tx|-(\|T\|+1)|x|\big)^+$ and let E be the band projection onto the principal band generated by z. Then

$$\|TEx\| = \|E|Tx|\| \geqslant \big\|E(\|T\|+1)|x|\big\| = (\|T\|+1)\|Ex\|.$$

It follows that $Ex = 0$ and so also $z = 0$. Thus $|Tx| \leqslant (\|T\|+1)|x|$ for all x and T belonging to the centre of X which is $\overline{\mathrm{lin}\,\mathfrak{A}}$. Thus $\mathfrak{A}$ satisfies (i) and this completes the proof of the theorem.

The representation theorem is a direct generalisation of the author's integral module representation and shows that a Banach space X can be considered as a space of vector-valued functions on the Stonean space of any complete uniformly decomposing projection algebra on X. The following definition defines the appropriate type of vector-valued function space.

5.2. Definition. Let M be a Banach lattice with order-continuous norm consisting of continuous numerical functions on the extremally disconnected compact Hausdorff space K. Let $\{X_k\}$ be a family of Banach spaces indexed by the points of K.

By $M(K;\{X_k\})$ we denote the set of all functions z from K into the disjoint union of the Banach spaces X_k with each other and with a distinct element ∞, such that:

(i) $z(k) \in X_k \cup \{\infty\}$ for each k and

(ii) $|z| \in M$ whereby $|z|(k) := \|z(k)\|_{X_k}$ (with $\|\infty\|_{X_k} := +\infty$).

A *lattice module* in $M(K;\{X_k\})$ is a subset Y for which:

(i) x, y in $Y \Rightarrow \exists z$ in Y with $z(k) = x(k)+y(k)$ where $x(k), y(k) \neq \infty$;

(ii) x in Y, f in $C(K) \Rightarrow \exists z$ in Y with $z(k) = f(k)x(k)$ where $x(k) \neq \infty$;

(iii) $X_k = \{x(k) \mid x \text{ in } Y,\ x(k) \neq \infty\}^-$ for each k.

Since the equations in (i) and (ii) determine the element z uniquely, a lattice module carries the structure of a $C(K)$-module and $\|z\| := m(|z|)$ is a norm for the lattice module whereby m is the norm of the lattice M. We shall say that a normed $C(K)$-module *has a representation in* $M(K;\{X_k\})$ if it is isometrically isomorphic (as a $C(K)$-module) to a lattice module in $M(K;\{X_k\})$.

Note. In the same way as in [5] or [1] one can show that if Y is a lattice module in $M(K;\{X_k\})$ which is complete in the norm, then Y is maximal amongst those subsets of $M(K;\{X_k\})$ for which (i) holds, and that we have equality in (iii) without taking the closure, i.e. if x_k is in X_k, then there is an x in Y with $x(k) = x_k$.

5.3. Theorem. *Let X be a Banach space and $\mathfrak{A}$ a complete uniformly decomposing projection algebra on X with Stonean space K. Then there is a Banach function lattice M on K and Banach spaces X_k indexed by the points of K such that X has a representation in $M(K;\{X_k\})$.*

Furthermore, if X also has a representation in $N(K;\{Y_k\})$, then $M \cong N$ as Banach lattices and $X_k \cong Y_k$ whenever both spaces are non-trivial.

Note. X is a $C(K)$-module by virtue of $C(K) = \overline{\mathrm{lin}\,\mathfrak{A}}$.

Proof. By 3.9 X has a norm resolution with respect to $C(K) = \overline{\mathrm{lin}\,\mathfrak{A}}$ taking values in a lattice M of continuous numerical functions on K whose norm m is order-continuous. Let $x \mapsto [x]$ be such a norm resolution. For each k, $x \mapsto [x](k)$ is a semi-norm on the subspace of X where it takes finite values. Let X_k be the Banach space obtained by factorising out the kernel of the semi-norm and completing the space in the resulting norm. The mapping $x \mapsto \langle x\rangle$ from X into $M(K;\{X_k\})$ is defined as follows:

$\langle x\rangle(k) := \infty$ if $[x](k) = +\infty$.

$\langle x\rangle(k) :=$ the equivalence class of x in X_k if $[x](k)$ is finite.

Since $\|\langle x\rangle(k)\|_{X_k}$ is simply $[x](k)$, the functions $\langle x\rangle$ all lie in $M(K;\{X_k\})$. It is easily verified that $\{\langle x\rangle \mid x \text{ in } X\}$ is a lattice module in $M(K;\{X_k\})$ and that $x \mapsto \langle x\rangle$ is an isometric isomorphism.

Suppose now that X has a representation in $M(K;\{X_k\})$ and also in $N(K:\{Y_k\})$. Each x in X is represented as $\langle x\rangle_M$ in $M(K;\{X_k\})$ and as $\langle x\rangle_N$ in $N(K;\{Y_k\})$. Writing $[x]_M := \|\langle x\rangle_M(k)\|_{X_k}$ and similarly $[x]_N := \|\langle x\rangle_N(k)\|_{Y_k}$, we have two norm resolutions for X taking values in M and N, respectively. Suppose that for a point k there is an x in X with $[x]_M(k)$ and $[x]_N(k)$ both finite and non-zero. Then the identity

$$[y]_M(k)/[x]_M(k) = \lim_{U_k} \|Ey\|/\|Ex\| = [y]_N(k)/[x]_N(k)$$

implies that the quotient $[x]_M(k)/[x]_N(k)$ is independent of the choice of x. Set λ_k equal to this quotient. Then the λ_k's are defined on a dense open subset of K and the mapping $k \mapsto \lambda_k$ is continuous since we can retain the same element x in a neighbourhood of k and the two norm resolutions are continuous. Let f be the unique continuous numerical function with $f(k) = \lambda_k$ wherever this is defined. Then $[x]_M = f[x]_N$ for all x in X. Note also that f is invertible since the λ_k's are all non-zero. We claim that $g \mapsto fg$ is a Banach lattice isometry of N onto M. Clearly the mapping is linear and positive. Also, if m and n are the norms of M and N, respectively, then $n([x]_N) = \|x\| = m([x]_M) = m(f[x]_N)$ so that the mapping is an isometry for elements of the form $[x]_N$. However, these elements are order-dense in N_+ and thus, by the order-continuity of n, also norm-dense. Since the norm in a lattice is determined by the norm on the positive cone, $g \mapsto fg$ is an isometry. That it is surjective follows from the invertibility of f.

To show that X_k and Y_k are isometric if they are non-trivial let k be a point with $f(k)$ finite and $X_k \neq \{0\}$. Since either f or f^{-1} is always finite and the situation is symmetrical, this does not involve any loss of generality. Let D be a clopen neighbourhood of k on which f is finite, then $f\chi_D$ is in $C(K)$. For each x_k in X_k there is an x in X with $\langle x\rangle_M(k) = x_k$. Set $Tx_k = \langle f\chi_D x\rangle_N(k)$. T is well-defined since $\langle x\rangle_M(k) = 0$ implies that $[x]_M(k) = 0$ and thus that $[f\chi_D x]_N(k) = f(k)[x]_N(k) = [x]_M(k) = 0$. T is clearly linear and the identity $[x]_M(k) = f(k)[x]_N(k)$ shows that it is an isometry. It remains to show that T is surjective. Let y_k be in Y_k with $\|y_k\|_{Y_k} = 1$. Choose an x in X with $\operatorname{supp} x \subseteq D$ and $[x]_M(k) = 2$ (since X_k is non-trivial there is such an element), then $[f\chi_D x]_N(k) = 2$. Let y be in X with $\langle y\rangle_N(k) = y_k$ and $[y]_N \leqslant [f\chi_D x]_N$. Let $\mathcal{S}$ be the directed set of those clopen subsets of D on which f^{-1} is finite, ordered by inclusion. For B in $\mathcal{S}$ we have $[f^{-1}\chi_B y]_M = f[f^{-1}\chi_B y]_N = \chi_B[y]_N \leqslant \chi_B[f\chi_D x]_N = \chi_B[x]_M$. Thus $\{f^{-1}\chi_B y\}_{B\in\mathcal{S}}$ is a Cauchy net in X. Let z be the limit of this net. Then $\chi_B z = f^{-1}\chi_B y$ for B in $\mathcal{S}$. In particular, $\langle f\chi_D z\rangle_N(l) = \langle f\chi_D f^{-1}\chi_B y\rangle_N(l) = \langle y\rangle_N(l)$ for l in B, B in $\mathcal{S}$. Since $\big(\bigcup_{\mathcal{S}} B\big)^- = D$, we have by continuity $\langle f\chi_D z\rangle_N(k) = \langle y\rangle_N(k)$, i.e. $T\big(\langle z\rangle_N(k)\big) = \langle y\rangle_N(k) = y_k$. Thus T is surjective.

It is straightforward to show that the points of K for which the component spaces are trivial in all representations are those points of K any neighbourhood of which contains uncountably many pairwise disjoint open sets (the so-called intrinsic null points, see [1], 3.11). Let us denote the set of these points by K_0. Note that K_0 is determined by the topology of K and thus by the Boolean algebra structure of $\mathfrak{A}$ and not by its action on X. Theorem 5.3 now shows that X and $\mathfrak{A}$ together define an unique non-trivial Banach space X_k for each point k in $K \setminus K_0$.

6. Conclusion. In this paper we have generalised the integral module representation of [5] to apply to a fairly large class of projection algebras. In particular, given a Banach space X and an embedding $C(K) \hookrightarrow B(X)$ we have a criterion for deciding whether X has a norm resolution over K taking values in a lattice with order-continuous norm and a method for constructing such a resolution and the associated representation. The key to this was the definition of uniform decomposition. A re-formulation of this property in terms of $C(K)$ rather than the projection algebra $\mathfrak{A}$ would seem like a sound starting point for a general criterion to decide whether a space X has a norm resolution with respect to a given embedding $C(K) \hookrightarrow B(X)$. This general criterion would of course have to contain both

the present theory and the M-structure of Cunningham as special cases.

Theorem 5.3 raises another interesting point. Namely, suppose that X and Y are two Banach spaces with isomorphic complete uniformly decomposing projection algebras $\mathfrak{A}_X$ and $\mathfrak{A}_Y$, respectively. Then we have two uniquely determined Banach function lattices M_X and M_Y on the common Stonean space K and two families X_k and Y_k of non-trivial Banach spaces indexed by the points of $K \setminus K_0$. This raises the following question:

If $X_k \cong Y_k$ for each k and $M_X \cong M_Y$ (with respect to the common representation on K), does it follow that $X \cong Y$, i.e. do the function lattice and the component spaces determine the space itself?

This problem and other allied ones, such as the sense in which the mapping $k \mapsto X_k$ is 'continuous' seem almost insoluble even in fairly simple cases.

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