

$\mu$  is replaced by  $2\mu$ . Specifically we expect to have that

$$(*) \quad f^*(t) = 2g^*(2t) \quad \text{implies that} \quad w(f) = w(g).$$

But it follows from (\*) that the unit ball of  $W$  with respect to the dual norm has no extreme points, which is a contradiction. To see this we first note that, if  $f \in \text{Weak } L^1$  has the form  $f = \sum_{k=1}^{\infty} \lambda_k \chi_{E_k}$  where the sets  $E_k$  are disjoint, then  $f$  cannot be an extreme point of the dual norm unit ball of  $W$ , since  $f = f_0 + f_1$  where  $w(f_0) = w(f_1) = \frac{1}{2}w(f)$  and  $f_0 \neq f_1$ . To construct  $f_0$  and  $f_1$  divide each set  $E_k$  into two disjoint sets of equal measure  $E_k = E_{k0} \cup E_{k1}$  and let  $f_j = \sum_{k=1}^{\infty} \lambda_k \chi_{E_{kj}}$ ,  $j = 0, 1$ . Clearly  $f_j^*(t) = f^*(2t)$ , so that  $w(f_j) = \frac{1}{2}w(f)$ . Finally it suffices to observe that for any function  $g \in \text{Weak } L^1$  there exists a function  $f = \sum_{k=1}^{\infty} \lambda_k \chi_{E_k}$  of the above form which represents the same element of  $W$ . To construct  $f$  we let  $G_n = \{x \mid 2^n \leq |g(x)| < 2^{n+1}\}$  for each integer  $n$  and let  $f$  restricted to  $G_n$  be a simple function such that  $\|(f-g)\chi_{G_n}\|_{L^1} \leq 2^{-|n|}$ . Clearly  $f$  has the required form. For each positive integer  $N$  let  $V_N = \bigcup_{|n| \leq N} G_n$ . Then  $I((f-g)\chi_{V_N}) = 0$  for each  $N$  and

$$I(f-g) = I((f-g)\chi_{X \setminus V_N}) \leq \|(f-g)\chi_{X \setminus V_N}\|_{L^1} \leq \sum_{|n| > N} 2^{-|n|}.$$

Since  $N$  may be arbitrarily large, we deduce that  $f$  and  $g$  are equivalent.

**Acknowledgement.** We thank Dr. Yoav Benyamini for a helpful remark.

# References

- [1] M. Cwikel, *On the conjugates of some function spaces*, Studia Math. 45 (1973), pp. 49–55.
- [2] — *The dual of Weak  $L^p$* , Ann. Inst. Fourier, Grenoble 25, 2 (1975), pp. 81–126. (Correction note to appear.)
- [3] M. Cwikel and Y. Sagher,  $L(p, \infty)^*$ , Indiana Univ. Math. J. 21 (1972), pp. 781–786.
- [4] A. Haaker, *On the conjugate space of Lorentz space*, Technical report, Lund 1970.

DEPARTMENT OF MATHEMATICS  
TECHNION-I.I.T., HAIFA  
and  
DEPARTMENT OF MATHEMATICS  
PRINCETON UNIVERSITY, PRINCETON, N.J.

Received June 20, 1978

(1440)

## On the strong maximal function and Zygmund's class $L(\log^+ L)^n$

by

N. A. FAVA, E. A. GATTO and C. GUTIÉRREZ (Buenos Aires)

**Abstract.** We show that the basic inequality of a weak type estimate may in a sense be reversed, thus allowing us to give a characterization of the class  $L(\log^+ L)^n$  in terms of the integrability of certain functions associated with the strong differentiability of integrals.

**1. Introduction and statement of results.** In this paper  $m$  is used to denote Lebesgue measure in any number of dimensions; this number being clear in each case from the context.

Let  $f(x) = f(x_1, \dots, x_n)$  be an integrable function with support in the unit cube of  $\mathbf{R}^n$ , defined by the inequalities  $0 \leq x_i \leq 1$  ( $i = 1, \dots, n$ ). We consider the *partial* maximal operators  $M_i$  defined by

$$(1) \quad M_i f(x) = \sup_{a < x_i < b} \frac{1}{b-a} \int_a^b |f(x_1, \dots, x_{i-1}, u, x_{i+1}, \dots, x_n)| du$$

at each point  $x$  in  $\mathbf{R}^n$ . We also consider the *strong maximal function*

$$(2) \quad f^*(x) = \sup_{I \ni x} \frac{1}{m(I)} \int_I |f(y)| dy,$$

where the supremum is taken over the set of all intervals  $I$  (cells with sides parallel to the axes) containing the point  $x$ . As usual, we denote by  $L(\log^+ L)^k$  the class of all functions  $f$  such that the integral

$$\int |f(x)| (\log^+ |f(x)|)^k dx$$

is finite.

One of the authors proved in [1] the following theorem:

**THEOREM A.** *The operation  $M_n \dots M_1 f$  is well defined for any function  $f$  in  $L(\log^+ L)^{n-1}$  and there exists a constant  $C$  depending only on  $n$ , such that for each  $t > 0$ ,*

$$(3) \quad m\{x: M_n \dots M_1 f(x) > 4t\} \leq C \int \frac{|f|}{t} \left( \log^+ \frac{|f|}{t} \right)^{n-1} dx.$$

Moreover, if  $f \in L(\log^+ L)^n$ , then  $M_n \dots M_1 f$  is integrable over every set of finite measure.

Since  $f^* \leq M_n \dots M_1 f$ , a similar statement holds with  $f^*$  in place of  $M_n \dots M_1 f$ .

The preceding theorem was used in [1] to obtain an easy proof of the theorem of Jessen, Marcinkiewicz and Zygmund [3]. Essentially the same inequality was found some time later by de Guzmán [2], p. 192, who made a similar application. The purpose of this note, which follows the lines of [4], is to show that in a certain sense our inequality (3) may be reversed, enabling us to prove the following theorem:

**THEOREM B.** Suppose that  $f \in L(\log^+ L)^{n-1}$  so that  $M_n \dots M_1 f$  is well defined and finite almost everywhere. Then this function is integrable over every set of finite measure if and only if  $f \in L(\log^+ L)^n$ .

At the end of the paper we consider a conjecture concerning the strong maximal function. Although we do not solve this conjecture, we explain how the present contribution may help to give an answer by reducing it to an equivalent—hopefully easier to prove—statement.

**2. Proof of the results.** The proofs will be based on three simple lemmas:

**LEMMA 1.** Suppose that  $f(x) \in L^1(\mathbf{R})$  and that

$$Mf(x) = \sup_{a < x < b} \frac{1}{b-a} \int_a^b |f(y)| dy$$

is the maximal function on the real line. Then for every  $t > 0$ , we have

$$(4) \quad m\{x: Mf(x) > t\} \geq \frac{2^{-1}}{t} \int_{|f|>t} |f(x)| dx.$$

This is a particular case of an inequality found by E. Stein [4] by making use of Calderón-Zygmund's decomposition. We only remark that it holds true even if  $f$  is not integrable. For if  $f$  is non negative, then we select an increasing sequence of non negative integrable functions  $f_k$  converging pointwise to  $f$ . Since for each  $k$ ,

$$m\{x: Mf_k(x) > t\} \geq \frac{2^{-1}}{t} \int_{f_k>t} f_k(x) dx,$$

we only have to let  $k \rightarrow \infty$ .

Going back to  $\mathbf{R}^n$ , we state

**LEMMA 2.** For each of the partial maximal operators (1) and for each  $t > 0$ , we have

$$(5) \quad m\{x: M_i f(x) > t\} \geq \frac{2^{-1}}{t} \int_{|f|>t} |f(x)| dx.$$

Here is the proof for  $i = 1$ . Starting from the inequality

$$m\{x_1: M_1 f(x_1, x_2, \dots, x_n) > t\} \geq \frac{2^{-1}}{t} \int_{\{x_1: |f(x_1, x_2, \dots, x_n)| > t\}} |f(x_1, x_2, \dots, x_n)| dx_1,$$

we integrate both members with respect to  $x_2, \dots, x_n$  to obtain (5) by the virtue of Fubini's theorem.

**LEMMA 3.** For each integer  $k$  such that  $1 \leq k \leq n$  and for each  $t > 0$ , we have

$$(6) \quad m\{x: M_k \dots M_1 f(x) > t\} \geq \frac{2^{-k}}{(k-1)!} \int_{|f|>t} \frac{|f|}{t} \left( \log \frac{|f|}{t} \right)^{k-1} dx.$$

This inequality reduces to (5) if  $k = 1$  and follows for  $k > 1$  by induction on  $k$ .

Let us assume that (6) holds as stated for some positive integer  $k < n$ . Then

$$m\{x: M_{k+1} M_k \dots M_1 f(x) > t\} \geq \frac{2^{-1}}{t} \int_{\{M_k \dots M_1 f > t\}} M_k \dots M_1 f(x) dx \geq 2^{-1} \int_{\{M_k \dots M_1 g > 1\}} M_k \dots M_1 g(x) dx,$$

where  $g = f^t/t$  with  $f^t = f$  if  $|f| > t$  and  $f^t = 0$  otherwise. On the other hand, the last integral may be evaluated as follows:

$$\begin{aligned} & \int_0^\infty m(\{M_k \dots M_1 g > 1\} \cap \{M_k \dots M_1 g > u\}) du \\ & \geq \int_1^\infty m\{M_k \dots M_1 g > u\} du \\ & \geq \int_1^\infty du \frac{2^{-k}}{(k-1)!} \int_{|g|>u} \frac{|g|}{u} \left( \log \frac{|g|}{u} \right)^{k-1} dx \\ & = \frac{2^{-k}}{(k-1)!} \int dx |g| \int_1^{|g|} \left( \log \frac{|g|}{u} \right)^{k-1} \frac{du}{u} \\ & = \frac{2^{-k}}{k!} \int |g| (\log |g|)^k dx = \frac{2^{-k}}{k!} \int_{|f|>t} \frac{|f|}{t} \left( \log \frac{|f|}{t} \right)^k dx. \end{aligned}$$

**Note.** The proof above is a good example of the extremely simple technique used in [1].

Now we come to the proof of Theorem B. Suppose that  $f \in L(\log^+ L)^{n-1}$  so that the set where  $M_n \dots M_1 f > 1$  has finite measure in virtue of Theorem A. If we assume that  $M_n \dots M_1 f$  is integrable over every set of finite measure, then

$$\begin{aligned} \infty &> \int_{\{M_n \dots M_1 f > 1\}} M_n \dots M_1 f(x) dx \geq \int_1^\infty m \{M_n \dots M_1 f > t\} dt \\ &\geq \frac{2^{-n}}{(n-1)!} \int_1^\infty dt \int_{|f| > t} \left( \log \frac{|f|}{t} \right)^{n-1} dx \\ &= \frac{2^{-n}}{(n-1)!} \int dx |f| \int_1^{|f|} \left( \log \frac{|f|}{t} \right)^{n-1} \frac{dt}{t} = \frac{2^{-n}}{n!} \int |f| (\log^+ |f|)^n dx. \end{aligned}$$

**3. An open problem.** Suppose that  $f \in L(\log^+ L)^{n-1}$  so that  $f^*$  is finite almost everywhere. We conjecture that  $f^*$  is integrable over every set of finite measure if and only if  $f \in L(\log^+ L)^n$ .

By virtue of Theorems A and B this conjecture is reduced to proving (or disproving) that if  $f^*$  is integrable over every set of finite measure, then so is  $M_n \dots M_1 f$ ; but this remains, as far as we know, an open question.

#### References

- [1] N. A. Fava, *Weak type inequalities for product operators*, Studia Math. 42 (1972), pp. 271–288.
- [2] M. de Guzmán, *An inequality for the Hardy–Littlewood maximal operator with respect to a product of differentiation bases*, ibid. 49 (1974), pp. 185–194.
- [3] B. Jessen, J. Marcinkiewicz and A. Zygmund, *Note on the differentiability of multiple integrals*, Fund. Math. 25 (1935), pp. 217–234.
- [4] E. Stein, *Note on the class  $L(\log L)$* , Studia Math. 32 (1969), pp. 305–310.

Received June 20, 1978

(1442)

#### Zero-one laws for Gaussian measures on metric abelian groups

by

T. BYCZKOWSKI (Wrocław)

**Abstract.** We prove zero-one laws for Gaussian measures on metric abelian groups. As a consequence, we derive the zero-one law for Gaussian processes with values in an LCA group  $G$ . The latter result contains, in particular, all the zero-one laws of Kallianpur and Jain as well as the zero-one law for the  $G$ -valued Wiener process.

In 1951 Cameron and Graves proved that every measurable rational subspace of  $C[0, 1]$  has Wiener measure zero or one [7]. This result has been generalized by Kallianpur [16] to a large class of Gaussian processes and by Baker [1], Rajput [18] and others to Gaussian measures on Banach spaces or on Fréchet spaces.

The proofs presented by these authors depend heavily on methods of linear spaces as well as on the stability of Gaussian measures.

A completely different approach has been proposed in the proof of Theorem 2.2 in [4]. This proof, suggested by an algebraic definition of Gaussian measure, seems to be more natural and relatively simple. The arguments, being of group-theoretic nature, enable us to extend this result to Gaussian measures on measurable groups, as pointed out in [6].

The aim of this paper is to generalize the result of Theorem 2.2 in [4] in two directions. First, we consider Gaussian measures on arbitrary Hausdorff abelian groups; secondly, we prove the zero-one law for measurable subgroups (instead of rational subspaces, as in [4]).

Section 1 is preliminary. In Section 2 we prove that every symmetric Gaussian measure without idempotent factors on a Hausdorff abelian group  $G$  can be embedded in a unique continuous semigroup of symmetric Gaussian measures under the assumption that  $x \rightarrow 2x$  is a Borel automorphism of  $G$ . From this result it follows, in particular, that every symmetric Gaussian measure on  $G$  is infinitely divisible.

In Section 3 we prove that every Gaussian measure on a metric abelian group  $G$  is a translation of a symmetric Gaussian measure (under the assumption that  $G$  has no non-zero elements of order 2). This property of Gaussian measures is well known if  $G$  is a Banach space (or, more