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Method of orthogonal projections and approximation of the spectrum of a bounded operator II

by

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Abstract. A necessary and sufficient condition is given in order that a compact subset of the complex plane be a limit of spectra of operators $A_n = P_n A|_{P_n H}$, where A is a given bounded operator on a Hilbert space H and $\{P_n\}$ — a sequence of orthogonal projections converging strongly to the identity operator on H.

This paper is a continuation of the study of asymptotical behaviour of spectra of operators $A_n = P_n A|_{P_n H}$, where P_n is a sequence of orthogonal projections in Hilbert space H converging strongly to the identity operator and A is a bounded operator in H. We use the same notations as in the first authors paper [4] on this subject.

The main result is the following theorem, which completes Theorem 1 in paper [4].

THEOREM. *If A is a bounded operator on a separable Hilbert space H and its essential numerical range $W_e(A)$ contains an interior point, then for any sequence $\{S_n\}$ of finite nonvoid subsets of the interior of $W_e(A)$ there exists a sequence $\{P_n\}$ of orthogonal projections of finite rank such that $P_n \to 1$ strongly and*

$$S_n \subset \Sigma(A_n) \subset S_n \cup \big(\Sigma_d(A) \setminus W_e(A)\big),$$

where

$$A_n = P_n A|_{P_n H} \in L(P_n H).$$

Before proving this theorem, we need some auxiliary lemmas. The idea of conformal mapping in the proof of Lemma 1 is taken from Herrero's paper [3].

We start with the following simple example.

EXAMPLE. Let H stand for a separable Hilbert space, the operator S satisfying the relations $Se_n = e_{n+1}$ (where $\{e_n\}_{n=-\infty}^{\infty}$ is a fixed orthonormal basis in H) is called the *bilateral shift*. It is known that S is a normal operator and that its spectrum $\Sigma(S)$ is a unit circle $S(0, 1)$. Define a sequence

$\{P_n\}_{n=1}^{\infty}$ of orthogonal projections in H by the formulas

$$P_{2n}x = \sum_{i=-n}^{n-1} \langle x, e_i\rangle e_i, \qquad P_{2n+1}x = \sum_{i=-n}^{n} \langle x, e_i\rangle e_i$$

and the operators $S_n = P_n S|_{P_n H} \in L(P_n H)$. It is easy to verify that $\Sigma(S_n) = \{0\}$, $W(S_n) \subset W(S) = K(0,1)$ (the open disc) and since $P_n \to 1$ strongly, we have $\operatorname{dist}(W(S_n), W(S)) \to 0$ with $n \to \infty$.

Lemma 1. *If s_0 is an interior point of the open, convex and bounded set $\Omega \subset \boldsymbol{C}$, then there exists a normal operator $N \in L(H)$ with $\Sigma(N) = \partial\Omega$ such that for the operators $A_n = P_n N|_{P_n H}$ the following relations hold:*

$$\Sigma(A_n) = \{s_0\}, \qquad W(A_n) \subset \bar{\Omega},$$

$$\operatorname{dist}(W(A_n), \Omega) \to 0 \quad \textit{with} \quad n \to \infty.$$

Proof. By Riemann's theorem there exists a conformal mapping f of the open disc $K(0,1)$ on the set Ω such that $f(0) = s_0$. By the Osgood–Caratheodory theorem, f may be assumed to be a continuous function on the closed disc $K(0,1)$ and $f(S(0,1)) = \partial\Omega$. Theorem 10.14 of [7] implies that $f(z) = \sum_{n=0}^{\infty} s_n z^n$ and the series $\sum_{n=0}^{\infty} |s_n|$ is convergent.

We shall show that the operator $N = f(S)$ satisfies our lemma. By the spectral mapping theorem for normal operators $\Sigma(N) = f(\Sigma(S)) = f(S(0,1)) = \partial\Omega$. Note that S is a unitary power dilation of any operator S_n, i.e. $S_n^k = P_n S^k$, $k = 0, 1, 2, \ldots$ This implies that

$$A_n = P_n f(S)|_{P_n H} = f(P_n S|_{P_n H}) = f(S_n).$$

Therefore

$$\Sigma(A_n) = f(\Sigma(S_n)) = f(\{0\}) = \{s_0\}.$$

As N is a normal operator, $\overline{W(N)} = \operatorname{conv}\Sigma(N) = \bar{\Omega}$, now the rest of the lemma follows from the convergence $P_n \to 1$ strongly. ■

If Ω is a convex polygon with extremal points $\{\mu_j\}_{j=1}^{m}$ indexed in such a way that $\bigcup_{j=1}^{m} [\mu_j, \mu_{j+1}] = \partial\Omega$ $(\mu_{m+1} \overset{\mathrm{df}}{=} \mu_1)$, then we can decompose the operator N into a direct sum $N = \bigoplus_{j=1}^{m} N_j$, where $N_j = N|_{H_j}$, $H_j = E([\mu_j, \mu_{j+1}), N)H$. For each j the operator

$$T_j = \frac{N_j - \mu_j}{\mu_{j+1} - \mu_j} \in L(H_j)$$

is a selfadjoint positive contraction, and so the operator matrix

$$R_j = \begin{bmatrix} T_j & \sqrt{T_j - T_j^2} \\ \sqrt{T_j - T_j^2} & 1 - T_j \end{bmatrix}$$

is a selfadjoint projection in $H_j \times H_j$. This shows that $N = QR|_{Q(H\times H)}$, where

$$R = \bigoplus_{j=1}^{m} \big(\mu_j + (\mu_{j+1} - \mu_j)R_j\big) \in L(H \times H), \qquad Q = \bigoplus_{j=1}^{m} \begin{bmatrix} 1_{H_j} & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that R is a normal operator with $\Sigma_e(R) = \Sigma(R) = \{\mu_j\}_{j=1}^{m}$, and all normal operators which satisfy this relations are unitary equivalent, note also that the operators $P_n Q$ are orthogonal projections of the rank n. In this way we have proved the following lemma.

Lemma 2. *Let s_0 be an interior point of a convex polygon Ω with extremal points $\{\mu_j\}_{j=1}^{m}$. If $R \in L(H)$ is a normal operator with $\Sigma_e(R) = \Sigma(R) = \{\mu_j\}_{j=1}^{m}$, then there exists a sequence $\{P_n\}$ of orthogonal projections in H such that:*

$$\dim P_n = n, \qquad \Sigma(P_n R|_{P_n H}) = \{s_0\},$$

$$\operatorname{dist}(W(P_n R|_{P_n H}), \Omega) \to 0, \quad \textit{with} \quad n \to \infty.$$

Lemma 3. *Let H stand for a Hilbert space with orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Let s_0 be an interior point of the triangle $\{\mu_0, \mu_1, \mu_2\}$. Define the operator $A \in L(H)$ by the relations $Ae_{3n+i} = \mu_i e_{3n+i}$, $i = 0, 1, 2$, $n = 0, 1, 2, \ldots$ Then for any $\varepsilon > 0$ there exists a projection $P \in P_f(H)$ such that*

$$\Sigma(PA|_{PH}) = \{s_0\}, \quad \left\| e_0 - \frac{Pe_0}{\|Pe_0\|} \right\|^2 < \varepsilon.$$

Proof. By Lemma 2 there exists a $Q \in P_f(H)$ such that $\Sigma(QA|_{QH}) = \{s_0\}$ and $\operatorname{dist}(W(QA|_{QH}), W(A)) < \varepsilon^2 h/4$, where $h = d(\mu_0, [\mu_1, \mu_2])$ (> 0), so there exists a unit vector $x \in QH$ such that $|\mu_0 - \langle Ax, x\rangle| < \varepsilon^2 h/4$. Note that $x = \sum_{i=0}^{2} a_i y_i$ where $\|y_i\| = 1$, $Ay_i = \mu_i y_i$, $a_i \geqslant 0$, $i = 0, 1, 2$. Hence $1 = \|x\|^2 = \sum_{i=0}^{2} a_i^2$, $\langle Ax, x\rangle = \sum_{i=0}^{2} a_i^2 \mu_i$, because y_i are orthogonal. Therefore

$$\tfrac{1}{4}\varepsilon^2 h > |\mu_0 - \langle Ax, x\rangle| = (1 - a_0^2)\left|\mu_0 - \frac{a_1^2\mu_1 + a_2^2\mu_2}{a_1^2 + a_2^2}\right| \geqslant (1 - a_0^2)h;$$

so $1 - a_0^2 < \varepsilon^2/4$ and

$$\|x - y_0\|^2 = \|(a_0 - 1)y_0 + a_1 y_1 + a_2 y_2\|^2 = 2\,\frac{1 - a_0^2}{1 + a_0} \leqslant \frac{\varepsilon^2}{2}.$$

Let U stand for the unitary operator satisfying the relations $Uz = z$ for z orthogonal to e_0 and y_0, $Ue_0 = y_0$, $Uy_0 = e_0$. It is obvious that $U^2 = 1$ and $A = UAU$, let $P = UQU \in P_f(H)$. The operators $QA|_{QH}$ and $PA|_{PH}$ are unitary equivalent, and $\Sigma(PA|_{PH}) = \{s_0\}$. Now the lemma follows from the inequality

$$\left\|e_0-\frac{Pe_0}{\|Pe_0\|}\right\|^2 \leqslant 2\|e_0-Pe_0\| = 2\|U(y_0-Qy_0)\| = 2\|y_0-Qy_0\|$$
$$= 2\inf_{x\in QH}\|y_0-x\| \leqslant 2\|y_0-x\| \leqslant \varepsilon.$$

LEMMA 4. *If Ω is a convex subset of C, $K(s,\delta)\subset\Omega$, $d(\mu,\Omega)<\varepsilon\delta$, $0<\varepsilon<1$, then $(1-\varepsilon)\mu+\varepsilon s\in\Omega$.*

Proof. There exists a $\lambda\in\Omega$ such that $|\mu-\lambda|<\varepsilon\delta$. Let

$$z = \frac{1-\varepsilon}{\varepsilon}(\mu-\lambda);$$

then $|z|<\delta$ and $s+z\in\Omega$, this implies $(1-\varepsilon)\mu+\varepsilon s = (1-\varepsilon)\lambda+ +\varepsilon(s+z)\in\Omega$. ∎

LEMMA 5. *If projections P, Q in H are defined by the formulas*

$$Px = \sum_1^n\langle x, x_i\rangle x_i, \qquad Qx = \sum_1^n\langle x, y_i\rangle y_i,$$

where $\{x_i\}_1^n$, $\{y_i\}_1^n$, are two orthonormal sets such that $\langle x_i, y_j\rangle = 0$ for $i\neq j$ and $\|x_j-y_j\|\leqslant\varepsilon$, $j=1,2,\ldots,n$, then $\|P-Q\|\leqslant\varepsilon$.

Proof. Note that there exists an orthonormal set $\{x_j\}_{j=1}^{2n}$ such that $y_j = \alpha_j x_j+\beta_j x_{j+n}$, $j=1,2,\ldots,n$, hence

$$|\beta_j|^2\leqslant\|(1-\alpha_j)x_j-\beta_j x_{j+n}\|^2 = \|x_j-y_j\|^2\leqslant\varepsilon^2.$$

The following identity holds:

$$(P-Q)z = \sum_{j=1}^n\beta_j(\langle z, \bar\beta_j x_j-\bar\alpha_j x_{j+n}\rangle x_j-\langle z, y_j\rangle x_{j+n});$$

so

$$\|(P-Q)z\|^2\leqslant\varepsilon^2\sum_{j=1}^n(|\langle z, \bar\beta_j x_j-\bar\alpha_j x_{j+n}\rangle|^2+|\langle z, y_j\rangle|^2)\leqslant\varepsilon^2\|z\|^2,$$

which follows from the Bessel inequality since the vectors $\{\bar\beta_j x_j-\bar\alpha_j x_{j+n}, y_j\}_{j=1}^n$ form an orthonormal set.

LEMMA 6. *If $P, Q\in L(H)$, $P=P^2$, $Q=Q^2$, $PQ=QP$ and $\|P-Q\|<1$, then $P=Q$.*

Proof. Note that $(P-Q)^4 = (P-Q)^2$ and $\|(P-Q)^2\|<1$. Therefore $(P-Q)^2$ is a projection with a norm strictly less than 1; hence $0 = (P-Q)^2 = P+Q-2PQ$. Multiplying this identity first on Q, and then on P, we obtain $P = PQ = Q$. ∎

Proof of the Theorem. 1. It follows from Lemma 7 of [4] that it is enough to prove the theorem in the case where S_n are one-point sets, therefore in the sequel we assume that $\{s_n\} = S_n$. Then there exists a sequence $\{\delta_n\}_{n=1}^\infty$ of positive numbers such that: $K(s_n, \delta_n)\cap\operatorname{Int}W_e(A)$, $\delta_n\to 0$, $0<\delta_n<1$ and $\partial G_n\cap\Sigma(A) = \emptyset$, where $G_n = W_e(A)+(\delta_n^2)$.

2. Let $Q_n = E(C\setminus G_n, A)$, and $\{\tilde P_n\}_{n=1}^\infty\subset P_f(H)$ be a sequence strongly convergent to the identity operator. Let $P_{m,n}$ stand for the orthogonal projection on the subspace $\tilde P_n H+Q_m H+Q_m^* H$. As $P_{k,n}\to 1$ strongly with $n\to\infty$, it follows from Theorem 1 of [6] that there exists a sequence $\{n_k\}_{k=1}^\infty$ such that

$$\|Q_k-E(C\setminus G_k, P_{k,n_k}A|_{P_{kn_k}H}P_{k,n_k}\|<1, \qquad n_k>k.$$

For simplicity assume that $\tilde P_k = P_{k,n_k}$. Since Q_n commutes with A and $\tilde P_n$, so it commutes also with $\tilde P_n A$ and therefore Q_n commutes also with $E(C\setminus G_n, \tilde P_n A|_{\tilde P_n H})\tilde P_n$. Lemma 6 shows that $Q_n = E(C\setminus G_n, \tilde P_n A|_{\tilde P_n H})\tilde P_n$.

3. Now we fix n. Let $\hat Q_n$ stand for the orthogonal projection with range $Q_n H$; then $\hat Q_n Q_n = Q_n$, $Q_n\hat Q_n = \hat Q_n$ and $\hat Q_n<\tilde P_n$. The following identity holds:

$$\tilde P_n A\tilde P_n = \hat Q_n A\hat Q_n+\hat Q_n A(\tilde P_n-\hat Q_n)+(\tilde P_n-\hat Q_n)A(\tilde P_n-\hat Q_n);$$

so the operator $\tilde P_n A|_{\tilde P_n H}$ may be represented by the operator matrix

$$\begin{bmatrix}\hat Q_n A\hat Q_n & \hat Q_n A(\tilde P_n-\hat Q_n)\\ 0 & (\tilde P_n-\hat Q_n)A(\tilde P_n-\hat Q_n)\end{bmatrix}.$$

By a theorem on a triangular matrix form ([1], p. 107) there exists an orthonormal set $\{\tilde x_{-s},\ldots,\tilde x_0,\tilde x_1,\ldots,\tilde x_r\}$ such that

$$\hat Q_n = \sum_{j=-s}^0\langle\ ,\tilde x_j\rangle\tilde x_j, \qquad \tilde P_n-\hat Q_n = \sum_{j=1}^r\langle\ ,\tilde x_j\rangle\tilde x_j$$

and

$$\langle A\tilde x_i, \tilde x_j\rangle = 0 \text{ for } i<j.$$

Let $\tilde\mu_j = \langle A\tilde x_j, \tilde x_j\rangle$; so $\{\tilde\mu_j\}_{j=-s}^r = \Sigma(\tilde P_n A|_{\tilde P_n H})$. Since $\hat Q_n A\hat Q_n|_{\hat Q_n H} = A|_{Q_n H}$, we have

$$\{\tilde\mu_j\}_{j=-s}^0 = \Sigma(A|_{Q_n H}) = \Sigma(A)\setminus G_n.$$

Since $Q_n = E(C\setminus G_n, \tilde P_n A|_{\tilde P_n H})\tilde P_n$, we have

$$\{\tilde\mu_j\}_{j=1}^r = \Sigma(\tilde P_n A|_{\tilde P_n H})\cap G_n;$$

therefore

(1) $$d(\mu_j, W_e(A))<\delta_n^2 \quad (1\leqslant j\leqslant r).$$

Now let

$$\mu_j = \begin{cases} \tilde{\mu}_j, & -s \leqslant j \leqslant 0, \\ (1-\delta_n)\tilde{\mu}_j + \delta_n s_n, & 1 \leqslant j \leqslant r. \end{cases}$$

It follows from Lemma 4 and (1) that $\mu_j \in \operatorname{Int} W_e(A)$, whence $1 \leqslant j \leqslant r$.

Lemma 7 of [4] implies that there exist vectors $\tilde{x}_j$ $(j = r+1, r+2, \dots, 2r)$ such that $\{\tilde{x}_j\}_{j=-s}^{2r}$ is an orthonormal set in H and

$$\langle A\tilde{x}_j, \tilde{x}_i\rangle = \langle A^*\tilde{x}_j, \tilde{x}_i\rangle = 0, \text{ for } j > r, j \neq i,$$

$$\langle A\tilde{x}_j, \tilde{x}_j\rangle = s_n \quad (r < j \leqslant 2r).$$

Now put

$$x_j = \begin{cases} \tilde{x}_j, & -s \leqslant j \leqslant 0, \\ \sqrt{1-\delta_n}\,\tilde{x}_j + \sqrt{\delta_n}\,\tilde{x}_{j+r}, & 1 \leqslant j \leqslant r \end{cases}$$

and note that:

$$\langle x_i, x_j\rangle = \delta_{ij}, \quad \langle Ax_j, x_j\rangle = \mu_j, \quad \langle Ax_i, x_j\rangle = 0 \quad \text{for} \quad i < j. \tag{2}$$

Let an orthogonal projection $\bar{P}_n$ be defined by the formula $\bar{P}_n = \sum_{j=-s}^{r} \langle\ , x_j\rangle x_j$. It follows from (2) that

$$\Sigma(\bar{P}_n A|_{\bar{P}_n H}) = \{\mu_j\}_{j=-s}^{r} \quad \text{and} \quad \{\mu_j\}_{j=-s}^{0} = \Sigma(A) \setminus G_n,$$

$$\{\mu_j\}_{j=1}^{r} \subset W_e(A).$$

From the definition of x_j we see that $\|x_j - \tilde{x}_j\|^2 \leqslant 2\delta_n$; hence by Lemma 5

$$\|\tilde{P}_n - \bar{P}_n\| \leqslant \sqrt{2\delta_n}. \tag{3}$$

4. There exist complex numbers μ_j $(r < j \leqslant 3r)$ such that: $\mu_j \in \operatorname{Int} W_e(A)$ and s_n is an interior point of the triangle $[\mu_j, \mu_{j+r}, \mu_{j+2r}]$, $1 \leqslant j \leqslant r$ (in the case where $\mu_j = s_n$ we put $\mu_{j+r} = \mu_{j+2r} = s_n$).

Let $\mu_{j+3rk} = \mu_j$, $k = 1, 2, \dots$ By Lemma 7 of [4] there exist unit vectors x_j $(r < j < \infty)$ such that:

$$\{x_j\}_{j=-s}^{\infty} \text{ is an orthonormal set in } H,$$

$$\langle Ax_j, x_j\rangle = \mu_j \quad (-s \leqslant j < \infty),$$

$$\langle Ax_j, x_i\rangle = \langle A^*x_j, x_i\rangle = 0 \quad \text{for} \quad j > r,\ j \neq i.$$

Now define orthogonal projections R_j $(0 \leqslant j \leqslant r)$ by the formulas:

$$R_0 = \hat{Q}_n, \quad R_j = \sum_{k=0}^{\infty} \langle\ , x_{j+rk}\rangle x_{j+rk} \quad (1 \leqslant j \leqslant r).$$

Note that:

(i) $R_i R_j = 0$ for $i \neq j$;

(ii) $R_j A x_{j+rk} = \sum_{i=0}^{\infty} \langle Ax_{j+rk}, x_{j+ri}\rangle x_{j+ri} = \mu_{j+rk} x_{j+rk}$ for $j \geqslant 1$, $k = 0, 1, \dots$, therefore the operator $R_j A|_{R_j H}$ is normal for $j \geqslant 1$ and

$$\Sigma(R_j A|_{R_j H}) = \Sigma_e(R_j A|_{R_j H}) = \{\mu_j, \mu_{j+r}, \mu_{j+2r}\} \quad (1 \leqslant j \leqslant r);$$

(iii) if $i > j \geqslant 1$, then

$$R_i A x_{j+rk} = \sum_{l=0}^{\infty} \langle Ax_{j+rk}, x_{i+rl}\rangle x_{i+rl} = \langle Ax_{j+rk}, x_i\rangle x_i = 0;$$

hence $R_i A R_j = 0$ also $R_j A R_0 = R_j R_0 Q_n A R_0 = 0$, and so

$$R_i A R_j = 0 \quad \text{if} \quad 0 \leqslant j < i \leqslant r.$$

5. By Lemma 3 there exists a projection $S_j \in P_f(H)$ $(1 \leqslant j \leqslant r)$ such that $S_j < R_j$, $\Sigma(S_j A|_{S_j H}) = \{s_n\}$, $\|x_j - S_j x_j / \|S_j x_j\|\| < \delta_n$. Put also $S_0 = R_0$ $(= \hat{Q}_n)$. Note that:

(i) $S_i S_j = S_i R_i R_j S_j = 0$ for $i \neq j$; hence if we define $P_n = \sum_{j=0}^{r} S_j$, then $P_n \in P_f(H)$;

(ii) if $0 \leqslant j < i \leqslant r$, then $S_i A S_j = S_i R_i A R_j S_j = 0$.

This shows that the operator $A_n = P_n A_{P_n H}$ may be represented in an upper triangular matrix form

$$A_n = \begin{bmatrix} S_0AS_0 & S_0AS_1 & \dots & S_0AS_r \\ 0 & S_1AS_1 & \dots & S_1AS_r \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & S_rAS_r \end{bmatrix}.$$

Therefore

$$\Sigma(A_n) = \bigcup_{j=0}^{r} \Sigma(S_j A|_{S_j H}) = \big(\Sigma_d(A) \setminus G_n\big) \cup \{s_n\}.$$

It remains to prove that $P_n \to 1$ strongly. Let

$$\hat{P}_n = \sum_{j=1}^{r} \left\langle\ , \frac{S_j x_j}{\|S_j x_j\|}\right\rangle \frac{S_j x_j}{\|S_j x_j\|} + \sum_{j=-s}^{0} \langle\ , x_j\rangle x_j.$$

By Lemma 5 $\|\bar{P}_n - \hat{P}_n\| \leqslant \delta_n$ this, with (3), gives $\|\tilde{P}_n - \hat{P}_n\| \to 0$. Since $\tilde{P}_n \to 1$ strongly, $\hat{P}_n \to 1$ strongly; since $P_n > \hat{P}_n$, also $P_n \to 1$ strongly. This ends the proof.

Remark 1. If $A \in L(H)$ is a selfadjoint operator, then the operators $A_n = P_n A|_{P_n H}$ are also selfadjoint. Using the identity

$$d\big(\lambda, \Sigma(A)\big) = \inf\{\|(A-\lambda)x\|:\ \|x\| = 1\},$$

which is valid for any normal operator, we see that if $P_n \to 1$ strongly, $P_n \in P_f(H)$, then for any $\varepsilon > 0$ and n large enough $\Sigma(A) \subset \Sigma(A_n) + (\varepsilon)$. In such a case the assumption of our Theorem that $W_e(A)$ contains an interior point is not satisfied; $W_e(A)$ is an interval.

However, the asymptotical behaviour of spectra of the operators A_n is not clear in the case where A differs from a selfadjoint operator by a compact one.

Remark 2. If $W_e(A)$ is a one-point set, then ([6]) A is of the form $\lambda+K$ where K is compact. Then for any sequence $\{P_n\}$ of projections in H (not necessarily orthogonal) converging strongly to identity $\operatorname{dist}\big(\Sigma(A), \Sigma(P_nA|_{P_nH})\big) \to 0$.

The case of not necessarily orthogonal projections is easy to settle. Remark 2 and the corollary of Lemma 7 give a characterization of the asymptotical behaviour of spectra of the operators $P_nA|_{P_nH}$.

Lemma 7. *If $A \in L(H)$ and $W_e(A)$ is not a one-point set, then for any bounded set $\Omega \subset \boldsymbol{C}$ there exists an operator $C \in L(H)$ with $C^{-1} \in L(H)$ such that $\Omega \subset W_e(C^{-1}AC)$.*

Proof. If $a, b \in W_e(A)$, $a \neq b$, then there exists a number $r > 0$ such that

$$\Omega \subset K\left(\frac{a+b}{2}, r\frac{|a-b|}{2}\right);$$

it follows from Lemma 7 of [4] that there exists an orthonormal set $\{x_n, y_n\}_{n=1}^{\infty}$ such that: $\langle Ax_n, x_n\rangle \to a$, $\langle Ay_n, y_n\rangle \to b$, $\langle Ax_n, y_n\rangle = \langle Ay_n, x_n\rangle = 0$. Let Q stand for the orthogonal projection on the subspace spanned by $\{x_n, y_n\}_{n=1}^{\infty}$. Define operator $C \in L(H)$ by the formula

$$Cz = (1-Q)z + \sum_{n=1}^{\infty} (\langle z, x_n + ry_n\rangle x_n + \langle z, y_n\rangle y_n);$$

then

$$C^{-1} \in L(H) \quad \text{and} \quad (C^{-1})^* z = (1-Q)z + \sum_{n=1}^{\infty} (\langle z, x_n\rangle x_n + \langle z, y_n - rx_n\rangle y_n).$$

Let $v_n = v_n(t) = \frac{1}{\sqrt{2}}(x_n + e^{it}y_n)$ where $t \in \boldsymbol{R}$. $\{v_n(t)\}_{n=1}^{\infty}$ is an orthonormal sequence in H and

$$\langle C^{-1}ACv_n, v_n\rangle = \langle ACv_n, (C^{-1})^*v_n\rangle$$

$$= \tfrac{1}{2}[(1+re^{it})\langle Ax_n, x_n\rangle + (1-re^{it})\langle Ay_n, y_n\rangle] \to \frac{a+b}{2} + re^{it}\frac{(a-b)}{2} \stackrel{\text{df}}{=} \lambda(t).$$

This shows that $\lambda(t) \in W_e(C^{-1}AC)$. Since $W_e(C^{-1}AC)$ is a convex set and

$$\bigcup_t \{\lambda(t)\} = S\left(\frac{a+b}{2}, r\frac{|a-b|}{2}\right),$$

we see that

$$\Omega \subset K\left(\frac{a+b}{2}, r\left|\frac{a-b}{2}\right|\right) \subset W_e(C^{-1}AC). \ \blacksquare$$

Corollary. *Suppose $A \in L(H)$ is not of the form $s+K$ where K is a compact operator on H, then for each bounded set $\Omega \subset \boldsymbol{C}$ there exists a sequence $\{P_n\}$ of projections in H of finite rank (we do not assume $P_n = P_n^*$) such that:*

$$P_n \to 1 \ \textit{strongly}, \quad P_n^* \to 1 \ \textit{strongly}$$

and

$$\operatorname{dist}\big(\Sigma(A_n), \Omega\big) \to 0, \quad \textit{where} \quad A_n = P_nA|_{P_nH}.$$

Proof. By the previous lemma there exists a $C \in L(H)$ with $C^{-1} \in L(H)$ such that $\Omega \cup \Sigma(A) \subset W_e(CAC^{-1})$. Theorem implies that there exists a sequence $\{Q_n\}_{n=1}^{\infty} \subset P_f(H)$ such that $Q_n \to 1$ strongly and

$$\operatorname{dist}\big(\Sigma((Q_n \cdot CAC^{-1})|_{Q_nH}), \Omega\big) \to 0. \tag{4}$$

Let $P_n = C^{-1}Q_nC$, so $P_n = P_n^2$ and it is easy to verity that $P_n \to 1$ strongly, $P_n^* \to 1$ strongly.

Moreover, $A_n = P_nA|_{P_nH} = C^{-1}(Q_nCAC^{-1}|_{Q_nH})\,(C|P_nH)$, i.e. A_n is similar to $Q_nCAC^{-1}|_{Q_nH}$. Therefore $\Sigma(A_n) = \Sigma(Q_nCAC^{-1}|_{Q_nH})$, this and (4) end the proof.

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