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Series expansions for Fourier transforms and Lebesgue functions

by

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Dedicated to Professor Jan Mikusiński

Abstract. A series expansion for Fourier transforms of integrable functions in terms of simple invertible functions is obtained. Essential use of Mikusiński's series expansion of Lebesgue functions is made and a type of nonorthogonal "Fourier series" of such functions is obtained.

Mikusiński [2] has defined Lebesgue integrable functions as limits of series of brick functions (i.e., characteristic functions of bounded intervals) with a special type of convergence. This definition is equivalent to the original one and the integral of an integrable function is obtained simply upon adding the term by term integrals of the series. Specifically, if $f \in \mathscr{L}$ (the class of Lebesgue integrable functions of one variable), then there exist numbers a_n and characteristic functions of bounded intervals χ_n such that

$$\sum |a_n| \int \chi_n < \infty \tag{i}$$

and

$$f = \sum a_n \chi_n \text{ (almost everywhere, a.e.).} \tag{ii}$$

Because of (i) and the dominated convergence theorem, the convergence in (ii) can be stipulated as being absolute and

$$\int f = \sum a_n \int \chi_n . \tag{iii}$$

Mikusiński uses (i) and (ii) to define the $\mathscr{L}$ class, and (iii) as the definition of the integral for this class; he calls the χ_n *brick functions*, a term suggested by their rectangularly shaped graphs.

We shall employ an analogous expansion using, however, functions whose graphs are isoceles triangles. These will be called *tent functions* or simply *tents*. They turn out to be somewhat more appropriate for purposes of the Fourier transform than are the bricks. Actually, bricks can be expanded in terms of tents, as the following theorem shows, hence they are equivalent collections for purposes of integration since, of course, tents can be expanded in terms of bricks by Mikusiński's theorem.

THEOREM 1. *Let χ be a brick function. Then there exist nonnegative tent functions τ_n such that*

$$\chi = \sum \tau_n \ (\textit{absolutely a.e.}).$$

Proof. Choose the tent τ_1 so that its graph is the largest isoceles triangle (with horizontal base) lying within the brick χ. The area of τ_1 is then one-half that of the brick. Next choose the two tents τ_2 and τ_3 so that their graphs are the largest isoceles triangles which when translated vertically lie within the two disjoint triangular regions between the brick and the tent τ_1. The combined area of these two tents is then equal to one-quarter of that of the two disjoint regions. Next choose the tents $\tau_4, \ldots, \tau_{10}$ so that their graphs are the largest isoceles triangles which when translated vertically lie within the six disjoint triangular regions between the brick and the three (translated) triangles obtained from the tents τ_1, τ_2, τ_3. The combined area of these six tents is again equal to one-quarter of that of the six disjoint regions. In a similar manner, choose the tents successively so that at each step, suitable vertical translates of their triangular graphs cover as much of the brick area remaining as possible. This process leads to a disjoint covering by isoceles triangles of the region inside the brick with a combined area *equal* to that of the brick. Indeed, the area of each τ_n $(n > 1)$ is equal to exactly one-quarter of the area of the corresponding triangular region in which its vertical translation lies, and exactly three-quarters of the area of this triangular area remains to be covered by subsequently chosen isoceles triangles. Thus the total area of the covering is given as the fraction

$$\tfrac{1}{2}+\tfrac{1}{2}\big(\tfrac{1}{4}+\tfrac{3}{4}\big(\tfrac{1}{4}+\tfrac{3}{4}\big(\tfrac{1}{4}+\ldots\big)\big)\big)$$

of that of the brick. But this fraction is equal to

$$\tfrac{1}{2}+\tfrac{1}{6}\big(\tfrac{3}{4}+\tfrac{3^2}{4^2}+\ldots\big) = 1.$$

This proves the theorem since χ and the τ_n are nonnegative functions and $\int |\chi + \sum \tau_n| = 0$.

Using this theorem and Mikusiński's expansion we obtain the desired tent expansion theorem.

THEOREM 2. *Let $f \in \mathscr{L}$. Then there exist tent functions τ_n such that*

$$\text{(iv)} \qquad \sum \int |\tau_n| < \infty$$

and

$$\text{(v)} \qquad f = \sum \tau_n \ (\textit{absolutely a.e.}).$$

Proof. First use the Mikusiński expansion theorem to obtain $f = \sum a_n \chi_n$ (absolutely a.e.), with $\sum |a_n| \int \chi_n < \infty$. Then use Theorem 1 to obtain for each n, $\chi_n = \sum_k \tau_{nk}$ (absolutely a.e.), with τ_{nk} nonnegative tents. It follows that $f = \sum \sum a_n \tau_{nk} = \sum \tau_m$ (absolutely a.e.), with $\tau_m = a_n \tau_{nk}$ some linear reordering of the square array. Moreover, $\sum \int |\tau_m| = \sum \sum |a_n| \int \tau_{nk} = \sum |a_n| \int \chi_n < \infty$, which proves the theorem.

Hereafter if $f \in \mathscr{L}$, then $\mathscr{F}f$ will denote the Fourier transform of f given by

$$\mathscr{F}f(y) = \int_{-\infty}^{\infty} e^{ixy} f(x)\, dx.$$

Also $\check{f}$ will denote the reflected function given by $\check{f}(x) = f(-x)$. The following Fourier inversion result for tents is well known [1]. The proof is omitted; the Fourier transforms of tents will be displayed subsequently.

THEOREM 3. *Let τ be a tent function. Then its Fourier transform $\mathscr{F}\tau$ is continuous, is bounded (by $\int |\tau|$) and is integrable. Moreover, it tends to zero at $\pm\infty$ and satisfies the Fourier inversion formula*

$$\check{\tau} = \frac{1}{2\pi} \mathscr{F}(\mathscr{F}\tau) \ (\textit{everywhere}).$$

In the next theorem we give new proofs of some standard results

THEOREM 4. *Let $f \in \mathscr{L}$. Then its Fourier transform $\mathscr{F}f$ is continuous, is bounded and it tends to zero at $\pm\infty$. If $\mathscr{F}f \in \mathscr{L}$, then f satisfies the Fourier inversion formula*

$$\check{f} = \frac{1}{2\pi} \mathscr{F}(\mathscr{F}f) \ (\textit{a.e.}).$$

In particular, if $\mathscr{F}f = 0$ (everywhere), then $f = 0$ (a.e.).

Proof. Using Theorem 2, let $f = \sum \tau_n$ (absolutely a.e.) be a tent expansion of f, where $\sum \int |\tau_n| < \infty$. By the dominated convergence theorem it follows that

$$\text{(vi)} \qquad \mathscr{F}f = \sum \mathscr{F}\tau_n,$$

$|\mathscr{F}f| \leqslant \int |f|$ and, because of Theorem 3, that the series converges absolutely and uniformly on the real line. Hence the Fourier transform $\mathscr{F}f$ is continuous, is bounded by $\int |f|$ and it tends to zero at $\pm\infty$. Again by Theorem 3, it follows that the series of integrals

$$\text{(vii)} \qquad \sum \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixy} \mathscr{F}\tau_n(y)\, dy = \sum \frac{1}{2\pi} \mathscr{F}(\mathscr{F}\tau_n)(x) = \sum \check{\tau}_n(x)$$

is convergent absolutely almost everywhere and that the sum is $f(-x)$. If in addition $\mathscr{F}f \in \mathscr{L}$, then the integral of the sum is the sum of the integrals in (vii) so that $\check{f} = \frac{1}{2\pi}\mathscr{F}(\mathscr{F}f)$ (a.e.), which completes the proof.

The above proof embodies a new characterization of Fourier transforms of Lebesgue functions.

THEOREM 5. *A function f is integrable if and only if there exist real sequences b_n and $d_n > 0$, and a complex sequence c_n with $\sum |c_n| < \infty$, such that*

$$\mathscr{F}f(y) = \sum 2c_n e^{ib_n y}\cdot\frac{(1-\cos d_n y)}{(d_n y)^2} \tag{viii}$$

holds for all real y.

Proof. The "only if" proof consists of computing the Fourier transforms of tent functions τ_n, which turn out to be terms of the type displayed in (viii). Specifically, if

$$\tau_n(x) = \frac{c_n}{d_n}\left(1-\frac{|x-b_n|}{d_n}\right) \tag{ix}$$

for $|x-b_n| \leqslant d_n$ and zero otherwise, then $\int |\tau_n| = |c_n|$ and the Fourier transform of τ_n is given in (viii). Thus (viii) is merely the expansion (vi) encountered in the proof of Theorem 4. For the "if" proof we simply note that under the conditions on the b_n, d_n and c_n stated in the present theorem, the right-hand side of (viii) defines a function which is the Fourier transform of a Lebesgue integrable function f determined almost everywhere by the termwise Fourier inversion of the series. (See Theorems 2, 3, and 4.)

Because of the expansion (viii) of $\mathscr{F}f$ in Theorem 5, the original tent expansion (v) of f in Theorem 2 takes on a more significant meaning; it is a type of nonorthogonal Fourier series representation of f, where the τ_n are the Fourier inverses of the expansion terms of the series in (viii). Moreover, the expansion (vii) in the proof of Theorem 4 yields a new Fourier inversion theorem which we now state explicitly for greater emphasis.

THEOREM 6. *Let $f \in \mathscr{L}$. Then its Fourier transform $\mathscr{F}f$ can be expressed in the form of an absolutely and uniformly convergent series $\sum \mathscr{F}\tau_n$ of functions each term of which is continuous, is bounded and tends to zero at $\pm\infty$ (see (viii)) and which has a classical Fourier inverse*

$$\tau_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixy}\mathscr{F}\tau_n(y)\,dy.$$

The latter in turn is continuous and has compact support (see (ix)) and is such that the equation

$$f = \sum \tau_n$$

holds (with absolute convergence) almost everywhere. This last equation can be regarded as a "Fourier series" representation of the Lebesgue integrable function f.

We conclude with a few observations and a final theorem.

(a) The convolution of two brick functions can be expressed as the difference of two tent functions. This leads to an elementary proof of the convolution theorem, a proof which avoids the use of Fubini's theorem, but uses instead Mikusiński's expansion (ii) as well as the Fourier expansion (v) and its transform (vi).

(b) Since the Fourier transforms of tent functions and their Fourier inverse transforms can be computed as Riemann type integrals, if the series expansions (v) and (viii) themselves are used to identify the classes of functions involved, the Lebesgue theory can effectively be avoided altogether, simply by operating termwise on the series.

(c) Normalizations of tent functions (ix) converge to delta-functions as $d_n \to 0$ and to constant functions as $d_n \to \infty$. Exactly the opposite holds true for the nonperiodic parts $g_n(y) = (1-\cos d_n y)/(d_n y)^2$ of their Fourier transforms. Since each collection consists of integrable functions, their roles can be, and often are interchanged. (See [1].) We now know that one collection generates $\mathscr{L}$ while the other generates the $\mathscr{L}$-transforms.

(d) Since $\int |g_n| = \int g_n = \pi/d_n$, it follows that if $\sum |c_n|/d_n < \infty$ in Theorem 5, then $\mathscr{F}f \in \mathscr{L}$. This holds iff the "altitudes" of the tents (ix) are summable. In particular, this holds if the bases $2d_n$ of the tents employed are not arbitrarily small. The condition $\sum |c_n|/d_n < \infty$, of course, guarantees the uniform convergence of the tent expansion (v) and is the most obvious condition insuring the (uniform) continuity a.e. of f.

This last observation is, perhaps, of sufficient interest to state as a theorem, in an equivalent form.

THEOREM 7. *Let $f \in \mathscr{L}$, and let $f = \sum \tau_n$ be a tent expansion of f according to Theorem 2, where $\sum \int |\tau_n| < \infty$ and where l_n is the length of the support of the tent τ_n for each n. Then the quantities $\bar{\tau}_n = \frac{1}{l_n}\int |\tau_n|$ are "average" values of the tent functions, and if these averages are summable, i.e. if $\sum \bar{\tau}_n < \infty$, then it follows that the Fourier transform $\mathscr{F}f$ is integrable and f satisfies the Fourier inversion formula $\check{f} = \frac{1}{2\pi}\mathscr{F}(\mathscr{F}f)$ (a.e.).*

Since brick functions of several variables can be expressed as pointwise products of brick functions of one variable at a time, all the results given in this paper may be extended to functions of several variables in a straightforward manner, upon replacing the various series by corresponding series of products representing each variable one at a time. Many quantities merely require a vector interpretation of the indices of summation.

References

[1] R. R. Goldberg, *Fourier Transforms*, Cambridge University Press, Cambridge 1965, p. 21.

[2] J. Mikusiński, *The Bochner Integral*, Academic Press, New York–San Francisco 1978, p. 3.

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Some Fourier transform inversion theorem

by

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Dedicated to Professor Jan Mikusiński on his 70th birthday

Abstract. It is shown that if $x^{\mu}\frac{\partial^{|\nu|}}{\partial x^{\nu}}f \in L^2(R^q)$ for $|\mu+\nu| \leqslant k$, $k > q/2$ then $(\mathscr{F}^{-1}\circ\mathscr{F})f(x) = f(x)$ for $x \in R^q$, where $\mathscr{F}f$ and $\mathscr{F}^{-1}f$ denote the Fourier integral and the inversion Fourier integral of f.

It is well known that the operation of differentiation of functions corresponds to the operation of multiplication by the argument of their Fourier transforms. This makes it possible to look for solutions of differential equations by means of the Fourier transformation. However, this method of solving differential equations is useful provided the inversion formula for the Fourier transform can be applied.

The Fourier transform of a function of the class $\mathscr{S}(R^q)$ of rapidly decreasing smooth functions is defined by means of the formula

$$\mathscr{F}f(\xi) = (2\pi)^{-q/2}\int_{R^q} e^{i\langle x,\xi\rangle}f(x)\,dx, \qquad \langle x, \xi\rangle = x_1\xi_1+\ldots+x_q\xi_q; \quad x, \xi \in R^q. \tag{1}$$

Then, the inversion formula is expressed in the form:

$$\mathscr{F}^{-1}f(\xi) = (2\pi)^{-q/2}\int_{R^q} e^{-i\langle x,\xi\rangle}f(x)\,dx. \tag{2}$$

For f in $\mathscr{S}(R^q)$, we have

$$(\mathscr{F}^{-1}\circ\mathscr{F})f(x) = (\mathscr{F}\circ\mathscr{F}^{-1})f(x) \quad \text{for each } x \in R^q. \tag{3}$$

We can ask for what other functions formula (3) holds, where the Fourier transform and the inversion Fourier transform are given by Fourier integrals (1) and (2), respectively. The author knows only the following