in l^1. That is, $\{S^n x_0\}$ is a basis for l^1 which, as our proof shows, is similar to $\{e_n\}$.

Using essentially the same ideas we can prove:

THEOREM 4.2. *If* $x_0 = (1, a_1, a_2, \ldots) \in l^1$ *and* $f(z) = 1 + a_1 z + a_2 z^2 + \ldots$ *with* $f(z) \neq 0$ *for all* z *in the unit disc* $|z| \leqslant 1$, *then* $\{S^n x_0\}$ *is a basis for* l^1 *which is similar to* $\{e_n\}$.

Proof. As in the proof of Theorem 4.2, our assumptions imply that $f^{-1}(z) = 1 + c_1 z + \ldots$ for $\{c_i\}_{i=0}^{\infty}$ in l^1. Therefore the operator $Q = I + c_1 S + c_2 S^2 + \ldots$ is a bounded linear operator on l^1 and clearly $QT = TQ = I$. That is, T is invertible on l^1 and since $Te_n = S^n x_0$, the theorem is proved.

COROLLARY 4.3. *I* $x_0 \in l^1$, *the following are equivalent:*

(i) $\{S^n x_0\}$ *is a basis for* l^1 *similar to* $\{e_n\}$.

(ii) $\{S^n x_0\}$ *is a basis for* l^1.

(iii) *The function* $f(z) = 1 + a_1 z + a_2 z^2 + \ldots$ *has no zero on the disc* $|z| \leqslant 1$.

Finally we state without proof a more general version of Theorem 3.1 (for the case $X = l^1$) which settles the question of similarity of $\{S^n x_0\}$ and $\{e_n\}$ in l^1.

THEOREM 4.4. Let $x_0 = (1, a_1, a_2, \ldots) \in l^1$ *and* $f(z) = 1 + a_1 z + a_2 z^2 + \ldots$ *Then* $\{S^n x_0\}$ *is a basic sequence in* l^1 *which is similar to* $\{e_n\}$ *if and only if* $f(z) \neq 0$ *for all* z *on the unit circle* $|z| = 1$.

The proof uses essentially the same ideas outlined above along with certain estimates on the norms of linear combinations of $\{S^n x_0\}$ and will be given in a subsequent paper devoted to more general problems in this area.

References

[1] I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Trans. Math. Monographs, Vol. 18, Amer. Math. Soc., Providence, R. I. 1969.

[2] H. Helson and G. Szegö, *A problem in prediction theory*, Ann. Mat. Pura Appl. 51 (1960), 107–138.

[3] K. Hoffman, *Banach spaces of analytic functions*, Prentice Hall, Englewood Cliffs, N. J. 1962.

[4] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Springer-Verlag, Berlin 1973.

[5] I. Singer, *Bases in Banach spaces I*, Springer-Verlag, Berlin 1970.

[6] A. Szankowski and M. Zippin, *A stability property of the unit vector basis of* l_p, Israel J. Math., 20 (1975), 216–227.

VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY
BLACKSBURG, VIRGINIA 24061

Received September 6, 1978 (1465)

STUDIA MATHEMATICA, T. LXXI. (1981)

A refinement of the Helson–Szegö theorem and the determination of the extremal measures

by

RODRIGO AROCENA (Caracas)

Abstract. Let R_M be the set of measures on the unit circle which satisfy the M. Riesz inequality for the Hilbert transform with constant M. R_M is determined by an associated class H_M of analytic functions. We give a geometric characterization of the elements of H_M and derive a refinement of the Helson–Szegö theorem. The extremal measures in the cone R_M are determined. Our basic result is the construction of a subset of extremal measures by means of which every element in R_M can be naturally obtained.

I. Introduction. Let T denote the unit circle, $\hat{f}$ the Fourier transform of $f \in L_1(T)$ and $\tilde{f}$ its conjugate function. If $M > 1$ is a fixed constant and $\mu \geqslant 0$ a measure on T, the Hilbert transform, we shall write $\mu \in R_M$ if:

$$\int_T |\tilde{f}|^2 d\mu \leqslant M \int_T |f|^2 d\mu, \qquad \forall f \in L^2(\mu) \cap L^2. \tag{I.1}$$

Set $R = \bigcup_{M>1} R_M$. Helson and Szegö proved [3] that $\mu \in R$ iff μ is absolutely continuous with respect to Lebesgue measure, $d\mu = w(x)\,dx$, and

$$w = \exp(u + \tilde{v}), \qquad u,\ v \in L^\infty, \qquad \|v\|_\infty < \pi/2. \tag{I.2}$$

Consequently, from now on we shall write $w \in R_M$ iff $\mu \in R_M$. Cotlar and Sadosky proved [2] that $w \in R_M$ iff there exists $h \in H^1(T)$ such that:

$$-4Mw^2 - 2(M+1)\operatorname{Re}(h)w - |h|^2 \geqslant 0, \text{ a.e.} \tag{I.3}$$

In this paper we study first those functions h that, by (I.3), characterize R_M. Then we state a version of Helson–Szegö theorem for each R_M, and, in particular, a simple proof of (I.2), deduced from (I.3). Finally we determine the extremal rays of the cone R_M. Our basic result is:

THEOREM 1. *Let* $g = g_1 + ig_2 \neq 0$ *be a function of* $H^1(T)$ *with* $\hat{g}(0) = 0$ *and* $\tau(g) = (v_0, u_0)$, *where*

$$\begin{aligned} v_0 &= \operatorname{artg}\{(M-1)g_1/[(M+1)^2 g_2^2 + 4Mg_1^2]^{1/2}\}, \\ u_0 &= \operatorname{arch}\left(\frac{M+1}{2\sqrt{M}}\cos v_0\right)\cdot(\chi_{\{g_2>0\}} - \chi_{\{g_2<0\}}). \end{aligned} \tag{I.4}$$

Then $w = C\exp(u_0+\tilde{v}_0)$, *where* C *is any positive constant, belongs to an extremal ray in* R_M. *Let* E_M *be the set of rays obtained in this way: if* $w \in R_M$, $\log w$ *is a weak star limit, in the dual of*

$$H_{1,0} = \{f \in L^1_r: \tilde{f} \in L^1, \hat{f}(0) = 0\},$$

of convex combinations of logarithms of elements of E_M.

II. The class H_M of analytic functions. Let H_M be the set of all functions $h \in H^1(T)$ that verify (I.3), that is,

$$H_M = \{h \in H^1(T): \exists w \geqslant 0,\ w \in L^1(T) \text{ and } h,\ w \text{ verify (I.3)}\}.$$

(II.a) LEMMA. *We have*

$$H_M = \{h = h_1+ih_2 \in H^1(T): h_1 \leqslant 0,\ |h_2| \leqslant (M-1)|h_1|/2M^{1/2},\ \text{a.e.}\}.$$

Proof. If $h \in H_M$ and $h \not\equiv 0$, (I.3) says that for almost every x there exists a positive solution of

$$-4Mw^2-2(M+1)wh_1(x)-|h(x)|^2 = 0,$$

so $h_1(x) \leqslant 0$ and

$$(M-1)^2h_1^2(x)/4M \geqslant h_2^2(x).$$

Reciprocally, if $h = h_1+ih_2 \in H^1(T)$ and $h_1 \leqslant 0$, $(M-1)^2h_1^2/4M \geqslant h_2^2$ setting $w = -(M+1)h_1/4M$, it is clear that $h \in H_M$, thus proving (II.a) and consequently that H_M is a convex cone.

Given $h \in H_M$, we set:

(II.1) $$w_j = (4M)^{-1}\{-(M+1)h_1+(-1)^j[(M-1)^2h_1^2-4Mh_2^2]^{1/2}\},\quad j = 1, 2.$$

w_1 and w_2 are the solutions of the second degree equation considered above, so we have

(II.b) COROLLARY. *Given* $h \in H_M$, w *and* h *verify* (I.3) *iff* $w_1 \leqslant w \leqslant w_2$. *In particular* $(-\operatorname{Re} H_M) \subset R_M$.

So each $h \in H_M$ defines a "band" of functions belonging to R_M. We denote this band with $W_{M,h} = \{w,\ \text{measurable function},\ w_1 \leqslant w \leqslant w_2\}$, where w_1, w_2 are given by (II.1). Therefore $w \in R_M \Leftrightarrow \exists h \in H_M$ such that $w \in W_{M,h}$. Now we consider the angle:

(II.2) $$S_M = \{(x, y) \in R^2: x < 0,\ |y| < (M-1)|x|/2M^{1/2}\},\quad \text{so}$$
$$H_M = \{h \in H^1(T): h(T) \subset \bar{S}_M\}.$$

(II.c) PROPOSITION. $h \in H_M$ *and* $h \not\equiv 0$ *iff* h *is given by the boundary values of* f, *an analytic function on the unit disc* D, *such that* $f(D) \subset S_M$.

Proof. If $h \in H_M$, $h \in H^1(T)$. Set

$$f(z) = (2\pi)^{-1}\int_0^{2\pi} P(r, u-t)h(t)\,dt,\qquad z = re^{iu},\qquad r < 1,$$

where P is the Poisson kernel; then

$$h(e^{iu}) = \lim_{r\to 1} f(re^{iu})\ \text{a.e.}$$

Being $P > 0$, (II.a) implies that $f(D) \subset S_M$.

Reciprocally, let f be analytic on D and $f(D) \subset S_M$. Set:

(II.3) $$a(M) = \operatorname{artg}[(M-1)/2M^{1/2}],$$

so the measure of the angle S_M equals $2a(M)$, and $f \in H^p(D)$, $\forall p \in \big(0, \pi/2a(M)\big)$. (See [1].) Since $2a(M) < \pi$, $f \in H^1(D)$; then $h(e^{iu}) = \lim_{r\to 1} f(re^{iu})$ exists a.e. and $h \in H^1(T)$. Clearly $h \in H_M$.

In the above proof we have shown that $h \in H_M$ implies $h \in H^p$, $\forall p \in \big(0, \pi/2a(M)\big)$.

Since $w_2 \leqslant |h|/2$, (II.b) says that

(II.d) COROLLARY. *If* $w \in R_M$, *then* $w \in L^p(T)$, $\forall p \in \big(0, \pi/2a(M)\big)$, $a(M)$ *given by* (II.3).

Also, since $|h_1| \leqslant 2Mw_1$, it is evident that

$$W_{M,h} \subset L^p(T) \Leftrightarrow h_1 \in L^p(T)$$
$$\Leftrightarrow \exists w \in W_{M,h} \text{ such that } w \in L^p(T).$$

If $w \in R_M$, there exists $\varphi \in H^p(T)$, $\forall p \in \big(0, \pi/2a(M)\big)$, such that $w = |\varphi|$; in fact, if $w \in R_M$ $\exists h \in H_M$ such that $w \in W_{M,h} \Rightarrow w \geqslant C|h|$, C a constant, and $\log|h| \in L^1$ because $h \in H^1 \Rightarrow \log w \in L^1$; then $\exists \varphi \in H^1$ such that $w = |\varphi|$; since $w \in L^p$, $\varphi \in H^p$, $\forall p \in \big(0, \pi/2a(M)\big)$.

(II.e) COROLLARY. *If* $h \in H_M$, $h^{-1} \in H_M$ *and* $(W_{M,h})^{-1} = 4M\ W_{M,h^{-1}}$. *If* $w \in R_M$, $w^{-1} \in R_M$.

Proof. If $h(D) \subset S_M$, then $h^{-1}(D) \subset S_M$. It is easy to see that, with obvious notation, $w_{1,h}^{-1} = 4Mw_{2,h}/|h|^2 = 4Mw_{2,h^{-1}}$ and $w_{2,h}^{-1} = 4Mw_{1,h^{-1}}$, so $(W_{M,h})^{-1} = \{w^{-1}: w \in W_{M,h}\} = 4M\ W_{M,h^{-1}}$.

EXAMPLES. (i) Let $k \in [0, 2a(M)/\pi]$, $f(z) = -\dfrac{(1+z)^k}{(1-z)^k}$. Then $f(D) \subset S_M$ and $w_j(u) = |\operatorname{cotg} u/2|^k \cos(k\pi/2)\{(M+1)+(-1)^j[(M-1)^2-4M\operatorname{tg}^2(k\pi/2)]^{1/2}\}/4M$, $j = 1, 2$.

(ii) If $h_1 = -2$, $h_2 = 0$, then $w_1 = M^{-1}$, $w_2 = 1$, so if w is a measurable function such that $0 < a \leqslant w \leqslant b$, with $b/a \leqslant M$, then $w \in R_M$.

If $g: D \to D$ is analytic and

$$F(z) = -a\{|1+g(z)|/|1-g(z)|\}^k,$$

$a > 0$, $0 \leqslant k \leqslant 2a(M)/\pi$, then $f(D) \subset S_M$. By subordination, this is the general form of the functions in H_M.

We shall say that $h \in H_M$ is a "boundary function" in H_M if $h(T) \subset \partial S_M \Leftrightarrow (M-1)^2h_1^2/4M = h_2^2$ a.e.

If $F\colon D\to\{\operatorname{Re} z<0\}$ is analytic, then

$$F(z) = -|F(0)|\exp\left(\frac{i}{4}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}f(t)\,dt\right) \tag{II.4}$$

with $|f(t)|\leqslant 1$, $t\in[0,2\pi]$, so $\arg F(e^{it}) = \pi f(t)/2$. Then every $F\in H_M$ has a representation (II.4) with

$$f\in L_M = \{g\colon T\to R,\ |g(t)|\leqslant 2a(M)/\pi\},$$

convex and compact with the weak star topology of L^∞. The Krein–Milman theorem states that every $g\in L_M$ can be approximated, in that topology, by convex combinations of extremal points of L_M. Now, f is extremal in L_M iff $|f(t)| = 2a(M)/\pi$ a.e., that is, iff the F associated to f by (II.4) is a "boundary function" in H_M. In this sense, going from $F\in H_M$ to the associated $f\in L_M$, we can say that the "boundary functions" determine H_M.

III. Propositions related to the Helson–Szegö theorem.

(III.a) LEMMA. *Let $h\in H_M$ and $w\in W_{M,h}$. Then $w = C\exp(u+\tilde v)$, with*

(i) $v = \pi-\arg h$, *so* $\|v\|_\infty\leqslant a(M)$;

(ii) $C = |h(0)|/2M^{1/2}$;

(iii) $u = \log(2M^{1/2}w/|h|)$.

Let $U_M(v) = \operatorname{arch}[(M+1)\cos v/2M^{1/2}]$. *Then* $|u|\leqslant U_M(v)$, $U_M(v)+u = \log(w/w_1)$, $U_M(v)-u = \log(w_2/w)$.

Proof. Since $h(D)\subset S_M$, $\log h = \log|h|+i\arg h$ can be defined, with $|\pi-\arg h|\leqslant a(M)$. Set (in T) $v = \pi-\arg h \Rightarrow \|v\|_\infty\leqslant a(M)$ and $\tilde v = \log|h/\hat h(0)|$. From the definition (II.1) of w_1, w_2 get

$$\log(w_1/|h|) = \log\left\{\frac{M+1}{2\sqrt{M}}\cos v-\left[\frac{(M+1)^2}{4M}\cos^2 v-1\right]^{1/2}\right\}-\log 2M^{1/2},$$

$$\log(w_2/|h|) = \log\left\{\frac{M+1}{2\sqrt{M}}\cos v+\left[\frac{(M+1)^2}{4M}\cos^2 v-1\right]^{1/2}\right\}-\log 2M^{1/2},$$

so

$$\log(2M^{1/2}w_1/|h|) = -U_M(v),\qquad \log(2M^{1/2}w_2/|h|) = U_M(v).$$

Then, setting $u = \log(2M^{1/2}w/|h|)$, the result follows.

Note. Helson–Szegö's proof of (I.2) uses special properties of analytic outer functions. The characterization (I.3) has been proved in a direct and elementary way; moreover, it is immediate that (I.2) implies (I.3) for some M [2]. (III.a) states the reciprocal. So (III.a) with (I.3) gives a new proof of Helson–Szegö theorem. (See also (III.e) below.) As (III.a) rests only on (II.b), this proof is a simpler and more elementary one, because it does not use refined properties of H^1.

(III.b) LEMMA. *Let $w = C\exp(u+\tilde v)$, C a positive constant, $|u|\leqslant U_M(v)$, $\|v\|_\infty\leqslant a(M)$. Set*

$$h(z) = 2M^{1/2}C\exp\{i[\pi-\hat v(0)]+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}v(t)\,dt.\}.$$

Then

(i) $h\in H_M$ *and (on T)* $\arg h = \pi-v$;

(ii) $w/w_1 = \exp(U_M(v)+u)$, $w_2/w = \exp(U_M(v)-u)$;

(iii) $w\in W_{M,h}$.

Proof. (i) From the definition of h it follows that

$$\arg h = \operatorname{Im}\{i[\pi-\hat v(0)]+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}v\,dt\}$$

so, on T, $\arg h = \pi-\hat v(0)+\tilde{\tilde v} = \pi-v \Rightarrow h\in H_M$.

(ii) On T, $|h| = 2M^{1/2}Ce^{\tilde v}$ so $w = e^u|h|/2M^{1/2}$. In the last proof we saw that $w_j = e^{(-1)^j U_M(v)}|h|/2M^{1/2}$, $j = 1, 2$.

(iii) Since $U_M(v)\pm u\geqslant 0$, (ii) says that $w_1\leqslant w\leqslant w_2$ so $w\in W_{M,h}$.

(III.c) PROPOSITION. *$w\in R_M$ iff $w = C\exp(u+\tilde v)$ with C a positive constant, $|u|\leqslant U_M(v)$ and $\|v\|_\infty\leqslant a(M)$.*

Proof. If $w\in R_M$, $\exists h\in H_M$ such that $w\in W_{M,h}$; (III.a) says that w is as stated. (III.b) proves the reciprocal.

(III.d) COROLLARY (Helson–Szegö theorem for R_M). *$w\in R_M$ iff $w = \exp(u+\tilde v)$, with $\|v\|_\infty\leqslant a(M)$, $|u+C|\leqslant U_M(v)$ for some real constant C.*

Proof. (III.c), with u, C instead of $(u+\log C), -\log C$.

(III.e) COROLLARY (Helson–Szegö theorem). *We have*

$$w\in R = \bigcup\{R_M\colon M>1\}\Leftrightarrow w = \exp(u+\tilde v),\ u, v\in L^\infty,$$

$\|v\|_\infty<\pi/2$.

Proof. If $w = \exp(u+\tilde v)$, $\|v\|_\infty<\pi/2$, $u\in L^\infty$, then $\exists M$ such that $\cos v\geqslant 2M^{3/4}/(M+1)$, and $|u|\leqslant\operatorname{arch}(M^{1/4})$, so $\|v\|_\infty<\arccos(2M^{1/2}/(M+1)) = a(M)$ and $|u|\leqslant U_M(v)$; the result follows by (III.d), which also states the reciprocal.

IV. Characterization of the extremal rays of R_M in terms of H_M. $w\in R_M$ defines an extremal ray of R_M iff $w = (w'+w'')/2$, w' and $w''\in R_M$ imply $w' = C'w$, $w'' = C''w$, C' and C'' positive constants. Set

$$\operatorname{Ext}R_M = \{w\in R_M,\ w \text{ belongs to an extremal ray in } R_M\}.$$

(IV.a) PROPOSITION. *Let* $h \in H_M$, $w \in W_{M,h}$. *If there exists* $A \subset T$, $|A| > 0$, *such that, in* A, $w_1 < w < w_2$, *then* $w \notin \operatorname{Ext} R_M$.

Proof. $\exists \varepsilon > 0$ such that in a set $A_1 \subset T$, $|A_1| > 0$, $w_1 \leqslant w - \varepsilon$, $w + \varepsilon \leqslant w_2$.

Seting $w' = w\chi_{T-A_1} + (w-\varepsilon)\chi_{A_1}$, $w'' = w\chi_{T-A_1} + (w+\varepsilon)\chi_{A_1}$, the result follows.

(IV.b) COROLLARY. *If* $w \in \operatorname{Ext} R_M$, *then* $\exists h = h_1 + ih_2 \in H_M$ *and* $A \subset T$ *such that*

$$\begin{aligned} w &= w_2\chi_A + w_1\chi_{T-A} \\ &= \{-(M+1)h_1 - (\chi_A - \chi_{T-A})[(M-1)^2h_1^2 - 4Mh_2^2]^{1/2}\}/4M. \end{aligned}$$

(IV.c) PROPOSITION. *If* $h \in H_M$ *and* $M' > M$, *then* $w_{M',1} \leqslant w_{M,1}$, $w_{M,2} \leqslant w_{M',2}$ *and the inequalities are strict in the set (of positive measure)* $\{h_2 \neq 0\}$, $\{h_1 \neq 0\}$, *respectively. In particular* $W_{M',h} \supsetneqq W_{h,M}$.

Proof. Let $-f_M(w)$ be the first member of (I.3), that is, $f_M(w) = 4Mw^2 + 2(M+1)h_1w + |h|^2$. Then $f_{M'}(w) - f_M(w) = 2(M'-M)w(2w + h_1)$. Now, $2w_{M,2} + h_1 \leqslant 0$; moreover, $2w_{M,2} + h_1 < 0$ iff $h_2 \neq 0$. Consequently, $f_{M'}(w_{M,2}) \leqslant 0$, the inequality being strict iff $h_2 \neq 0$. Since $f_{M'}(w)$ is a second degree polynomial with two real roots, $w_{M',1} < w_{M',2}$, it follows that $w_{M,2} \leqslant w_{M',2}$, with strict inequality iff $h_2 \neq 0$. In the same way we can prove the result concerning $w_{M',1}$ and $w_{M,1}$.

(IV.d) COROLLARY. $\operatorname{Ext} R_{M'} \subset R_{M'} - \bigcup_{1<M<M'} R_M$.

(IV.e) PROPOSITION. *If* h', $h'' \in H_M$, $0 < t < 1$, *and* $h = th' + (1-t)h''$, *then*

$$w_1 \leqslant tw_1' + (1-t)w_2'', \qquad tw_2' + (1-t)w_2'' \leqslant w_2;$$

equalities hold iff h'/h'' *is real. So* $W_{M,h} \supset tw_{M,h'} + (1-t)W_{M,h''}$.

Proof. We want to compare $W_{M,h}$ with

$tW_{M,h'} + (1-t)W_{M,h''} = \{tw' + (1-t)w'' \colon w' \in W_{M,h'} w'' \in W_{M,h''}\}$.

Set

$$\begin{aligned} A &= \{t^2[(M-1)^2h_1'^2 - 4Mh_2'^2] + (1-t)^2[(M-h_2''^2-1)^2 4Mh_2''^2] + \\ &\quad + 2t(1-t)[(M-1)^2h_1'h_1'' - 4Mh_2'h_2'']\}^{1/2}, \\ B &= t[(M-1)^2h_1'^2 - 4Mh_2'^2]^{1/2} + (1-t)[(M-1)^2h_1''^2 - 4Mh_2''^2]^{1/2}. \end{aligned}$$

Then

$$w_2 - [tw_2' + (1-t)w_2''] = (A-B)/4M,$$

with A, $B \geqslant 0$. Set

$$C = (M-1)^2h_1'h_1'' - 4Mh_2'h_2'' \geqslant 0,$$

$$D = [(M-1)^2h_1'^2 - 4Mh_2'^2]^{1/2}[(M-1)^2h_2''^2 - 4Mh_2''^2]^{1/2} \geqslant 0.$$

Then

$$A^2 - B^2 = 2t(1-t)(C-D)$$

and

$$C^2 - D^2 = 4M(M-1)^2(h_1'h_2'' - h_1''h_2')^2.$$

Consequently $w_2 \geqslant tw_2' + (1-t)w_2''$ and equality holds only when $h_1'h_2'' - h_1''h_2' = 0$. Analogously,

$$w_1 - [tw_1' + (1-t)w_1''] = -(A-B)/4M.$$

Now suppose h'/h'' is real a.e. (on T). Consider the analytic function (on D) $f = h'/h''$; then $f(D) \subset S = \{z \colon |\arg z| < b\}$ for some $b < \pi$ because h', $h'' \in H_M$. Set $F(z) = \left(\frac{1+z}{1-z}\right)^t t$ such that $F(D) = S$; by subordination, $f = F \cdot g$, $g \colon D \to D$, analytic. Now $f(e^{iu}) \in R$ a.e. on $T \Leftrightarrow g(e^{iu}) \in R$ a.e. on $T \Rightarrow g$ is constant. So we have the following

(IV.f) LEMMA *If* h', $h'' \in H_M$ *and* h'/h'' *is real* a.e., *then* h'/h'' *is constant (and positive).*

Note. The above lemma implies that if h is a "boundary function" of H_M, then it belongs to an extremal ray of the cone H_M. In fact, if $h = th' + (1-t)h''$, h', $h'' \in H_M$, it is clear that, on T, $\arg h = \arg h' = \arg h''$, so h'/h'' is real (a.e.). (IV.e) and (IV.f) state:

(IV.g) PROPOSITION. *Let* h', $h'' \in H_M$, $0 < t < 1$ *and* $h = th' + (1-t)h''$. *If* $h'' \neq Ch'$, C *a positive constant, then*

(i) $W_{M,h} \neq tW_{M,h'} + (1-t)W_{M,h''}$;

(ii) $w_1 < tw_1' + (1-t)w_1''$, $tw_2' + (1-t)w_2'' < w_2$ *in a set of positive measure in* T.

(IV.h) COROLLARY. *Under the hypothesis of* (IV.g), *if* $w = w_2\chi_A + w_1\chi_{T-A}$, *then* $w \notin tW_{M,h'} + (1-t)W_{M,h''}$.

(IV.i) LEMMA. *If* $w \in R_M - \operatorname{Ext} R_M$, $\exists h \in H_M$ *and* $A \subset T$ *such that* $w \in W_{M,h'}|A| > 0$ *and* $w_1 < w < w_2$ *on* A.

Proof. Since $w \notin \operatorname{Ext} R_M$, $\exists w'$, $w'' \in R_M$, not belonging to the same ray that w and such that $w = (w' + w'')/2$.

Let h', $h'' \in H_M$ be such that $w' \in W_{M,h'}$, $w'' \in W_{M,h''}$; then $w \in \frac{1}{2}(W_{M,h'} + W_{M,h''})$. Set $h = (h' + h'')/2$; (IV.e) says that $w \in W_{M,h}$. Suppose $h'/h'' = C$, a positive constant; (IV.e) implies that $w_2 = (w_2' + w_2'')/2$, $w_1 = (w_1' + w_1'')/2$. If $w = w_2\chi_B + w_1\chi_{T-B}$ for some $B \subset T$, then $w' = w_2'\chi_B + w_1'\chi_{T-B}$, $w'' = w_2''\chi_B + w_1''\chi_{T-B}$; since $w_2'/w_2'' = w_1'/w_1'' = C$, then $w'/w'' = C$, a contradiction. But if h'/h'' is not a real constant, (IV.g) states that $w = w_2\chi_B + w_1\chi_{T-B}$ is impossible. The result follows.

The last lemma and (IV.a) prove the following

(IV.j) PROPOSITION. *Let* $w \in R_M$. *Then* $w \in \operatorname{Ext} R_M$ *iff* $w \in W_{M,h}$ *implies* $w = w_1\chi_A + w_2\chi_{T-A}$.

That is, w belongs to an extremal ray if and only if it belongs to a "band", it equals the "upper" function of the band in some subset of T and, in its complement, it equals the "lower" function.

DEFINITION. For every $w \in R_M$ let

(IV.1) $$H_{M,w} = \{h \in H_M \colon w \in W_{M,h}\}.$$

If $0 < t < 1$, then

$$|th' + (1-t)h''|^2 = t(h_1'^2 + h_2'^2) + (1-t)(h_1''^2 + h_2''^2) + \\ +2t(1-t)[h_1'h_1'' - (h_1'^2 + h_2''^2)/2 + h_2'h_2'' - (h_2'^2 + h_2''^2)/2],$$

so:

$$|th' + (1-t)h''|^2 \leqslant t|h'|^2 + (1-t)|h''|^2, \qquad 0 < t < 1,$$

and equality holds only when $h' = h''$, (IV.2). This relation shows that:

(IV.k) PROPOSITION. *The set $H_{M,w}$ is convex.*

(IV.l) PROPOSITION. *Let $h \in H_M$ and $w = w_1\chi_A + w_2\chi_{T-A}$. Then h is an extremal point of the convex set $H_{M,w}$.*

Proof. Suppose $h = (h' + h'')/2$, $h', h'' \in H_{M,w}$, $h' \neq h''$. The hypothesis on w says that $0 = -4Mw^2 - 2(M+1)h_1 w - |h|^2$, a.e. (IV.2) ensures that $\exists A \subset T$, $|A| > 0$, such that in A,

$$-4Mw^2 - 2(M+1)h_1 w - |h|^2 > -4Mw^2 - 2(M+1)h_1 w - (|h'|^2 + |h''|^2)/2 \\ \Rightarrow 0 > -4Mw^2 - 2(M+1)(h_1' + h_2'')/2 - (|h'|^2 + |h''|^2)/2,$$

which contradicts $h', h'' \in H_{M,w}$.

(IV.m) COROLLARY. *If $w \in \operatorname{Ext} R_M$, then $H_{M,w}$ contains only one element.*

Proof. (IV.j) and (IV.l) state that every element in $H_{M,w}$ is an extremal point. In this way we get a first characterization of the extremal rays of R_M.

(IV.n) THEOREM 2. *The following conditions are equivalent:*

(i) $w \in \operatorname{Ext} R_M$;

(ii) *there exists one and only one $h \in H_M$ such that $w \in W_{M,h}$ and*

$$w = w_1\chi_A + w_2\chi_{T-A} \\ = \{-(M+1)h_1 + (\chi_A - \chi_{T-A})[(M-1)h_1^2 - 4Mh_2^2]^{1/2}\}/4M, \text{ a.e.}$$

Proof. (IV.m) and (IV.j) say that (i) implies (ii); (IV.j) proves the reciprocal.

V. Characterization of the extremal rays of R_M by means of Helson–Szegö theorem.

(V.a) LEMMA. *Let $w \in \operatorname{Ext} R_M$. Then there exists only one v such that*

(i) $\|v\|_\infty \leqslant \alpha(M)$;

(ii) $w = Ce^{u+\tilde{v}}$, *with C a positive constant and* $|u| \leqslant U_M(v)$.

Proof. There exists at least one such v because of (III.c). Lemma (III.b) says that $w \in W_{M,h}$, with $v = \pi - \arg h$. Since $w \in \operatorname{Ext} R_M$, (IV.n) shows that h, and consequently v, is well determined.

(V.b) LEMMA. *Under the same hypothesis and with the same notation that in* (V.a), *it must be* $|u| = U_M(v)$.

Proof. (IV.n) shows that $w = w_1$ or $w = w_2$ a.e., so (III.b.ii) implies, respectively, $U_M(v) + u = 0$ or $U_M(v) - u = 0$.

(V.c) LEMMA. *Under the same hypothesis and with the same notation that in* (V.a), *u is also unique.*

Proof. If $w = C_1\exp(u_1 + \tilde{v}) = C_2\exp(u_2 + \tilde{v})$, then $u_2 = u_1 + k$, k a real constant. Considering (V.b), we can see that $|u_2| = |u_1| \Rightarrow u_1$ is constant $\Rightarrow |v|$ equals a.e. a constant t.

Suppose $t < \alpha(M) = \operatorname{artg}[(M-1)/2M^{1/2}] \Rightarrow \exists M'$, $1 < M' < M$, such that $t \leqslant \alpha(M')$; then $w = C_1 \exp(u_1 + \tilde{v}) = C\exp(u + \tilde{v})$ with $u = 0$, $C = C_1\exp u_1$ and $\|v\|_\infty \leqslant \alpha(M')$. (III.c) says that $w \in R_{M'}$ and (IV.d) that w would not belong to $\operatorname{Ext} R_M$. Then, it must be $t = \alpha(M) \Rightarrow |v| = \alpha(M)$ a.e. $\Rightarrow U_M(v) = 0$ a.e. $\Rightarrow u_1 = u_2$.

(V.d) DEFINITION. A (measurable) function *has property* (E_M) if the following conditions are satisfied.

$(E_M)_1$ There exists one and only one pair (v, u) such that

(i) $\|v\|_\infty \leqslant \alpha(M)$;

(ii) $|u| \leqslant U_M(v)$;

(iii) $w = Ce^{u+\tilde{v}}$, for a (well determined) positive constant C.

$(E_M)_2$ $|u| = U_M(v)$.

What we have proved up to now is that $w \in \operatorname{Ext} R_M \Rightarrow w$ *has property* (E_M).

Reciprocally:

(V.e) LEMMA. *If w has property (E_M), then $w \in \operatorname{Ext} R_M$.*

Proof. $w \in R_M$ because of (III.c). Lemma (III.b) says that $w \in W_{M,h}$, with h defined there. Suppose $h' \in H_M$ is such that $w \in W_{M,h'}$; then (III.a) states that $w = C'e^{u'+\tilde{v}'}$, with $v' = \pi - \arg h'$, $C' = |h'(0)|/2M^{1/2}$, and $\|v'\|_\infty \leqslant \alpha(M)$, $|u'| \leqslant U_M(v')$. So, by $(E_M)_1$, $v = v'$, $u = u' \Rightarrow C = C'$. From $\arg h = \pi - v = \pi - v' = \arg h'$ we see that h'/h is real a.e. on T; then (IV.f) shows that $h' = kh$, k a positive constant. Consequently $k = |h'(0)/h(0)| = C'/C = 1 \Rightarrow h' \equiv h$. So there is only one $h \in H_M$ such that $w \in W_{M,h}$.

$(E_M)_2$ says that $|u| = U_M(v)$; by (III.a) $w = w_1$ or $w = w_2$ a.e. Considering (IV.n), the result follows.

Summing up we have

(V.f) THEOREM 3. $w \in \operatorname{Ext} R_M \Leftrightarrow w$ *has property* (E_M).

VI. The set P_M.

(VI.a) Definitions. Set $G(v,u) = u+\tilde{v}$,

$$P_M = \{(v,u) \in L_r^\infty - L_r^\infty \colon \ \|v\|_\infty \leqslant \alpha(M),\ |u| \leqslant U_M(v)\}, \qquad L_M = G(P_M).$$

(VI.b) Lemma. *U_M is a strictly concave function.*

From the lemma, considering that U_M is a decreasing function in $[0, \alpha(M)]$, the following result follows.

(VI. c) Proposition. *P_M is convex, symmetric and absorbent. In the norm topology of $L_r^\infty \times L_r^\infty$, P_M is the same as the closure of its interior.*

(VI. d) Definitions. Let $\tilde{P}_M$ and $\tilde{L}_M$ be the "cylinders" given by

$$\tilde{P}_M = \{[(v,u)+(0,c)] \in L_r^\infty \times L_r^\infty \colon \ (v,u) \in P_M,\ c \in R\},$$

$$\tilde{L}_M = G(\tilde{P}_M) = L_M + R.$$

Let us consider the sets $\operatorname{Ext}\tilde{P}_M$ and $\operatorname{Ext}\tilde{L}_M$ of extremal generators of these cylinders and the space $L = \{c+u+\tilde{v}\colon\ c \in R,\ u, v \in L_r^\infty\}$, where functions that differ in a constant are identified. Then $\operatorname{Ext}\tilde{L}_M$ is just the set of extreme points of $\tilde{L}_M$, a convex subset of L.

(VI.e) Proposition. *$\tilde{L}_M$ is a weak star compact and convex subset of L.*

Proof. L is the dual of $H_{1,0}$ with a norm equivalent to the one given by

$$\|\varphi\|_L = \inf\{\|u\|_\infty + \|v\|_\infty\colon\ \varphi = c+u+\tilde{v},\ c \in R,\ u, v \in \mathrm{L}_r^\infty\}.$$

In this norm $\tilde{L}_M$ is bounded, because $\varphi \in \tilde{L}_M$ implies $\|\varphi\|_L \leqslant U_M(0)+\alpha(M)$.

Let us prove now that L_M is closed. If φ belongs to the norm closure of $\tilde{L}_M$, for each natural number $n\varphi$ may be written as $\varphi = c_n + u_n + \tilde{v}_n + u_n' + \tilde{v}_n'$, with $c_n \in R$, $(v_n, u_n) \in P_M$, $\|u_n'\|_\infty, \|v_n'\|_\infty \to 0$. Considering eventually subsequences, we may assume that v_n, u_n converge — in the weak star topology of L_r — to v, u, respectively. Set $\varphi_1 = u+\tilde{v}$; clearly, $\hat{\varphi}(n) = \hat{\varphi}_1(n)$, $\forall\ n \neq 0$, so φ and φ_1 represent the same element of L.

Obviously, $\|v\|_\infty \leqslant \alpha(M)$; moreover, in a set $A \subset T$ such that $|T-A| = 0$ the following inequalities hold: $|v| \leqslant \overline{\lim}\,|v_n|$, $|u| \leqslant \overline{\lim}\,|u_n|$. Now, since U_M is a decreasing function with a continuous inverse on $[0, \alpha(M)]$, in every point of A we have: $U_M(v) \geqslant U_M(\overline{\lim}\,|v_n|) = \underline{\lim}\, U_M(v_n) \geqslant \overline{\lim}\,|u_n| \geqslant |u|$. Consequently $(v,u) \in P_M$, and so $\varphi \in \tilde{L}_M$. Considering the theorem of Bourbaki–Alaoglu, the proof is over.

(VI.f) Lemma. *Let $(v,u) \in P_M$; then (v,u) is an extremal point of the convex set P_M iff $|u| = U_M(v)$, a.e.*

Proof. If $|u| < U_M(v)$ in a set of positive measure it is easy to construct u_1, u_2 such that $u = (u_1+u_2)/2$, $(u-u_1)$ and $(u-u_2)$ are not constant — and a fortiori not zero — and $|u_1|, |u_2| \leqslant U_M(v)$. Then $(v,u) = \frac{1}{2}(v,u_1)+\frac{1}{2}(v,u_2)$ in P_M, so (v,u) is not an extremal point of P_M. This proves also that $(v,u) \notin \operatorname{Ext}\tilde{P}_M$.

Reciprocally, suppose $|u| = U_M(v)$ a.e. and $(v,u) = \frac{1}{2}(v',u')+\frac{1}{2}(v'',u'')$ in P_M. Then $|u| = U_M(v) \geqslant \frac{1}{2}U_M(v')+\frac{1}{2}U_M(v'') \geqslant (|u'|+|u''|)/2 \geqslant |u'+u''|/2 = |u|$ a.e. Since U_M is strictly concave, we must have $v = v' = v''$ to get equality in the first inequality, hence $|u|$ a.e., $|u''| \leqslant U_M(v) = |u|$ a.e. and we must have $u = u' = u''$ a.e. to get equality in all the inequalities.

(VI.g) Proposition. *Let $(v,u) \in P_M$; then the following conditions* (a) *and* (b) *are equivalent:*

(a) $(v,u) \in \operatorname{Ext}\tilde{P}_M$.

(b) $|u| = U_M(v)$ *and at least one of the following conditions is satisfied:* (b_1) *u changes its* sign, (b_2) $\|v\|_\infty = \alpha(M)$.

Proof. (a) $\Rightarrow$ (b): We saw at the beginning of the proof of (VI.f) that (a) implies $|u| = U_M(v)$. Suppose that (b_1) and (b_2) are both false; then we may assume that $u \geqslant 0$ and that $\exists M'$, $1 < M' < M$, such that

$$\|v\|_\infty \leqslant \alpha(M') < \alpha(M) \Rightarrow u = U_M(v) \geqslant U_M(\|v\|_\infty) \geqslant U_M[\alpha(M')] > 0.$$

Set $u_1 = u - U_M[\alpha(M')]$; then $u_1 \geqslant 0$ and $u_1 < u$, so $(v,u_1) \in P_M$ and (v,u_1) is not an extremal point of $P_M \Rightarrow (v,u_1) \notin \operatorname{Ext}\tilde{P}_M$. The result follows.

(b) $\Rightarrow$ (a): Let (v',u'), $(v'',u'') \in P_M$ and C a constant such that $v = (v'+v'')/2$, $u = (u'+u'')/2+C$. In the set $A = \{u > 0\}$ we have

$$u = U_M(v) \geqslant \tfrac{1}{2}U_M(v')+\tfrac{1}{2}U_M(v'') \geqslant \tfrac{1}{2}|u'|+\tfrac{1}{2}|u''| \Rightarrow u \geqslant (u'+u'')/2;$$

so, if $|A| > 0$, $C \geqslant 0$. Analogously, if $|\{u<0\}| > 0$, $C \leqslant 0$. Then, if (b_1) holds, $C = 0$ and $v = v' = v''$, $u = u' = u''$, because $|u| = U_M(v)$ ensures that (v,u) is an extremal point of P_M. If (b_1) does not hold, then $\|v\|_\infty = \alpha(M)$; for each natural $n\ \exists B_n \subset T$ such that $|B_n| > 0$ and, in B_n,

$$|v| \geqslant \alpha(M)-1/n \Rightarrow |v'|,\ |v''| \geqslant \alpha(M)-2/n$$
$$\Rightarrow |u-(u'+u'')/2| \leqslant 2U_M[\alpha(M)-2/n];$$

consequently $|C| \leqslant 2U_M[\alpha(M)-2/n] \to 0$, and the result follows.

(VI.h) Corollary. *Let $(v,u) \in P_M$ be such that* $\exp(u+\tilde{v}) \in \operatorname{Ext}R_M$. *Then* $(v,u) \in \operatorname{Exp}\tilde{P}_M$.

Proof. Suppose the statement is false. Since (V.b) says that $|u| = U_M(v)$, both (b_1) and (b_2), in (VI.g), must be false. So we may assume that $u \geqslant 0$ and $\|v\|_\infty < \alpha(M) \Rightarrow \exists M' \in (1, M)$ such that $\|v\|_\infty \leqslant \alpha(M')$. Set $u_1 = u - U_M[\alpha(M')] \Rightarrow u_1 \geqslant 0$; we shall see that $|u_1| \leqslant U_{M'}(v)$ which is equivalent to

$$U_M(v) - U_M[\alpha(M')] \leqslant U_{M'}(v) \Leftrightarrow \operatorname{arch}\,[(M+1)\cos v/2M^{1/2}] -$$
$$-\operatorname{arch}[(M'+1)\cos v/2M'^{1/2}] \leqslant \operatorname{arch}\left(\frac{M+1}{2\sqrt{M}}\cdot\frac{2\sqrt{M'}}{M'+1}\right);$$

setting

$$a = (M+1)2M'^{1/2}/2M^{1/2}(M'+1) > 1$$

the last inequality is true because $\operatorname{arch}(at) - \operatorname{arch} t \leqslant \operatorname{arch} a$, $\forall t \geqslant 1$. So $|u_1| \leqslant U_{M'}(v)$ and

(IV.d) $$\|v\|_\infty \leqslant a(M') \Rightarrow (v, u_1) \in P_{M'} \Rightarrow e^{u+\tilde{v}} = Ce^{u_1+\tilde{v}} \in R_{M'} \Rightarrow e^{u+\tilde{v}} \notin \operatorname{Ext} R_M.$$

Now we can state the relation between the extremal generators of these cylinders and the extremal rays of R_M.

(VI.i) Proposition. *The following conditions are equivalent*:

(a) $f \in \operatorname{Ext} \tilde{L}_M$;

(b) $e^f \in \operatorname{Ext} R_M$;

(c) $\varphi^{-1}(f) \cap \tilde{P}_M$ *contains only one element that belongs to* $\operatorname{Ext} \tilde{P}_M$.

Proof. (a) $\Rightarrow$ (b): Let $f = \varphi(v_1, u_1 + C)$, $(v_1, u_1) \in P_M$, C a constant; (III.b) says that $w = e^f \in R_M$. Suppose $w \notin \operatorname{Ext} R_M$; then (IV.j) ensures that $\exists h \in H_M$ and $A \subset T$ such that $w \in W_{M,h}$, $|A| > 0$ and, in A, $w_1 < w < w_2$; so $\exists B \subset T$ and $a > 1$ such that $|B| > 0$ and, in B, $aw_1 \leqslant w \leqslant w_2/a$. Set

$$w' = w\chi_{T-B} + \frac{1}{a} w\chi_B, \quad w'' = w\chi_{T-B} + aw\chi_B;$$

then $w', w'' \in W_{M,h}$ and $w^2 = w'w''$. Then (III.a) shows that

$$w = C\exp(u + \tilde{v}), \quad w' = C\exp(u' + \tilde{v}), \quad w'' = C\exp(u'' + \tilde{v}),$$

(v, u), (v, u'), $(v, u'') \in P_M$. From $w^2 = w'w''$ we get $u + \tilde{v} = (u' + \tilde{v})/2 + (u'' + \tilde{v})/2$; since w'/w is not a constant, $(u + \tilde{v}) - (u' + \tilde{v})$ is not one either, so $(u + \tilde{v}) \notin \operatorname{Ext} L_M \Rightarrow f \notin \operatorname{Ext} \tilde{L}_M$.

(b) $\Rightarrow$ (c): Suppose (v, u), $(v', u') \in P_M$ and $f = k + u + v = k' + u' + v'$, k and k' real constants; then $w = e^f = Ce^{u+\tilde{v}} = C'e^{u'+\tilde{v}'}$, C and C' positive constants.

Let h be defined in terms of v and C as in (III.b) and, in the same way, construct h' by means of v' and C'; then $w \in W_{M,h} \cap W_{M',h'}$. Since $w \in \operatorname{Ext} R_M$, (IV.n) states that $h = h'$, so $v = v'$, $C = C'$ and, consequently, $u = u'$. Thus $\varphi^{-1}(f) \cap \tilde{P}_M$ contains only one element, $(v, u + k)$; since $e^{u+\tilde{v}} = e^f/C \in \operatorname{Ext} R_M$, (VI.h) shows that $(v, u + k) \in \operatorname{Ext} \tilde{P}_M$.

(c) $\Rightarrow$ (a): Let $f = (f' + f'')/2 = k + (u' + u'')/2 + (v' + v'')/2$; since $((u' + u'')/2, (v' + v'')/2) \in P_M$, $(k + (u' + u'')/2, (v' + v'')/2)$ is the only element in $\varphi^{-1}(f) \cap \tilde{P}_M$; in order that it belong to an extremal generator, it is necessary that $v' = v''$, $u' = u''$, so $f \in \operatorname{Ext} \tilde{L}_M$, and the proof is over.

Let N be the kernel of φ. Then following is the basic result of this section.

(VI.j) Theorem 4. *Let* $w \in R_M$; *then* (a) *and* (b) *are equivalent.*

(a) $w \in \operatorname{Ext} R_M$;

(b) $w = C\exp(u_0 + \tilde{v}_0)$, C *a positive constant*, $(v_0, u_0) \in P_M$, $|u_0| = U_M(v_0)$, $\{(v_0, u_0) + N\} \cap \tilde{P}_M = \{(v_0, u_0)\}$ *and at least one of the following conditions is satisfied*: (b$_1$) u *changes its sign*, (b$_2$) $\|v\|_\infty = a(M)$.

Proof. (a) $\Rightarrow$ (b): (III.c) says that $w = C\exp(u_0 + \tilde{v}_0)$, with $(v_0, u_0) \in P_M$. Let $f = u_0 + \tilde{v}_0$; (VI.i) states that $\{(v_0, u_0) + N\} \cap \tilde{P}_M$ contains only one element, evidently (v_0, u_0). Since $(v_0, u_0) \in \operatorname{Ext} P_M$, (VI.g) finishes this part of the proof.

(b) $\Rightarrow$ (a): (VI.g) Shows that $(v_0, u_0) \in \operatorname{Ext} \tilde{P}_M$, so the result follows from (VI.i).

VII. Proof of the main theorem. The preceeding characterizations of the extremal rays are not constructive; now we shall construct explicitely a subset $E_M \subset \operatorname{Ext} R_M$ such that, in the sense specified in Theorem 1, every $w \in R_M$ can be obtained by means of elements of E_M.

Let K_1 be the interior of P_M in the norm topology of $L_r^\infty \times L_r^\infty$ and $K_2 = (v_0, u_0) + N$, where (v_0, u_0) is as in (VI.j) and $N = \{(v, -\tilde{v})$: $v, \tilde{v} \in L_r^\infty\}$. Considering (VI.c), a well known corollary of the Hahn–Banach theorem ensures the existence of an hyperplane H that separates K_1 and K_2. Since K_1 is open, $K_1 \cap H = \emptyset$, so H is not dense; consequently, it is not difficult to prove the following result.

(VII.a) Proposition. *Let* (v_0, u_0) *be as in* (VI.j). *Then there exists* μ, ν, *belonging to the topological dual of* L_r^∞ *such that*:

(i) $$\int_T e_n(d\mu + i\,d\nu) = 0, \quad \forall\ n \geqslant 0,$$

(ii) $$1 = \int_T (v_0\,d\mu + u_0\,d\nu) \geqslant \int_T (v\,d\mu + u\,d\nu), \quad \forall (v, u) \in P_M.$$

It seems reasonable to suppose that (v_0, u_0) will have some special properties when the functional can be represented by functions. So we set the following.

(VII.b) Definition. (v_0, u_0) as in (VI.j) belongs to the set EP_M if there exists $f_1, f_2 \in L_r^1$ such that $d\mu = f_1 dt$, $d\nu = f_2 dt$ satisfy the assertion of Proposition (VII.a)

In the way to prove the first part of the main theorem, it will be shown that $(v, u) \in EP_M$ iff $(v, u) \in \tau(\{g \in H_{1,0}\colon \|g\|_1 > 0\})$, where τ is as defined in Section I. (VII.a) and (VII.b) say that:

(VII.c) Proposition. *Let* (v_0, u_0) *be as in* (VI.j). *Then* $\exists f = (f_1, f_2)$ *such that* $(-f_1 + if_2) \in H^1$, $\hat{f}_1(0) = \hat{f}_2(0) = 0$,

(VII.1) $$\int_T f_1 v + \int_T f_2 u \leqslant \int_T f_1 v_0 + \int_T f_2 u_0, \quad \forall (v, u) \in P_M.$$

Proof. Since P_M is convex and absorbent, N a closed subspace and—by (VI.j)—$\{(v_0, u_0) + N\} \cap \tilde{P}_M = (v_0, u_0)$, the Hahn–Banach theorem states that $\exists f \in (L^\infty \times L^\infty)'$ such that $f(N) = 0$ and $f(v_0, u_0) \geqslant f(s)$,

$\forall s \in \tilde{P}_M$. If $(v, u) \in P_M$ and C is any real constant $(v, u+C) \in \tilde{P}_M$, so it is necessary that $\hat{f}_2(0) = 0$; the result follows. From now on (v_0, u_0) and $f = (f_1, f_2)$ shall be as in (VII.c). Consequently $|u_0| = U_M(v_0)$, so $u_0 = U_M(v_0)(\chi_{A_0} - \chi_{T-A_0})$.

(VII.d) PROPOSITION. *With the above notation, the following relations are true, except for sets of measure* 0:

(VII.2)
$$\{f_2 > 0\} \cap \{|v_0| < \alpha(M)\} \subset A_0,$$
$$\{f_2 < 0\} \cap \{|v_0| < \alpha(M)\} \subset T - A_0.$$

Proof. Set $v = v_0$, $u = U_M(v)\cdot(\chi_A - \chi_{T-A})$; then $(v, u) \in P_M$ and $\int_T f_1 v = \int_T f_1 v_0$ so (VII.1) says that

(#)
$$\int_A f_2 U_M(v_0) - \int_{T-A} f_2 U_M(v_0) \leqslant \int_{A_0} f_2 U_M(v_0) - \int_{T-A_0} f_2 U_M(v_0).$$

If the first statement of (VII.2) is false, setting
$$A = \{f_2 > 0\} \cap \{|v_0| < \alpha(M)\} \equiv \{f_2 > 0\} \cap \{U_M(v_0) > 0\}$$
it follows that $|A - A_0| > 0$ and $A_0 - A \subset \{f_2 U_M(v_0) \leqslant 0\}$. Then
$$\int_A f_2 U_M(v_0) - \int_{T-A} f_2 U_M(v_0) - \int_{A_0} f_2 U_M(v_0) + \int_{T-A_0} f_2 U_M(v_0)$$
$$= 2\int_{A-A_0} f_2 U_M(v_0) - 2\int_{A_0-A} f_2 U_M(v_0) \geqslant \int_{A-A_0} f_2 U_M(v_0) > 0,$$
which contradicts (#).

The second statement of (VII.2) can be proved in the same way.

(VII.e) PROPOSITION. *With the same notation, the following relation is true, except for a set of measure zero*:

(VII.3)
$$\operatorname{sg} v_0 = \operatorname{sg} f_1.$$

Proof. Let $|v| = |v_0|$, $u = u_0$, so $(v, u) \in P_M$ and (VII.1) says that $\int_T f_1 \leqslant \int_T f_1 v_0$. Set $B = \{f_1 < 0\}$, $v = v_0\chi_{T-B} - v_0\chi_B$; then it is clear that:

(i)
$$|\{f_1 v_0 < 0\}| = 0.$$

Now set $C = \{f_1 = 0\}$, $v = v_0\chi_{T-C}$, $u = u_0\chi_{T-C} + U_M(0)(\operatorname{sg} f_2)\chi_C$; then (VII.1) shows that
$$\int_C |f_2| U_M(0) \leqslant \int_C f_2 u_0; \qquad |u_0| \leqslant U_M(0)$$
and $|\{f_2 = 0\} \cap C| = 0$ because $(-f_1 + if_2) \in H^1$; so $|u_0| = U_M(0)$ a.e. in $C \Rightarrow v_0 = 0$ a.e. in C. Consequently

(ii)
$$|\{f_1 = 0\} - \{v_0 = 0\}| = 0.$$

Let $D = \{v_0 = 0\} - \{f_1 = 0\}$; suppose $|D| > 0$; then $\exists D_1 \subset D$, k, a, positive numbers such that $|D_1| > 0$ and, in D_1, $|f_1| \geqslant a$, $|f_2| \leqslant k$. Let d be such that

(iii)
$$U_M(0) - U_M(d) \leqslant ad/2k;$$

such a d exists because $U'_M(0) = 0$.

Now set $v = v_0\chi_{T-D_1} + d(\operatorname{sg} f_1)\chi_{D_1}$, $u = u_0\chi_{T-D_1} + U_M(d)(\operatorname{sg} f_2)\chi_{D_1}$. Then (VII.1) says that
$$\int_{D_1} |f_1| d + \int_{D_1} |f_2| U_M(d) \leqslant \int_{D_1} f_2 U_M(0) \Rightarrow ad|D_1| \leqslant k[U_M(0) - U_M(d)]\,|D_1|$$
which contradicts (iii), so

(iv)
$$|\{v_0 = 0\} - \{f_1 = 0\}| = 0.$$

From (i), (ii) and (iv) the result follows.

DEFINITION. Let $J_M(v) = -U'_M(v)$, $|v| < \alpha(M)$. Then
$$J_M(v) = \frac{M+1}{2M^{1/2}} \sin v \left[\frac{(M+1)^2}{4M}\cos^2 v - 1\right]^{-1/2},$$
so J_M is odd, $J_M(0) = 0$, J_M is positive and increasing in $(0, \alpha(M))$, and $J_M(v) \to \infty$ when $v \to \alpha(M)$.

(VII.f) PROPOSITION. *Let* (v_0, u_0), $f = (f_1, f_2)$ *be as in* (VII.c). *Then*

(VII.4)
$$|f_1/f_2| = J_M(|v_0|) \text{ a.e.}$$

Proof. Let v be such that $\operatorname{sg} v = \operatorname{sg} v_0$ and $\|v\|_\infty \leqslant \alpha(M)$; set $u = U_M(v)(\chi_{A_0} - \chi_{T-A_0})$; then $(v, u) \in P_M$. (VII.1) shows that
$$\int_T f_1 v + \int_{A_0} f_2 U_M(v) - \int_{T-A_0} f_2 U_M(v)$$
$$\leqslant \int_T f_1 v_0 + \int_T f_1 v_0 + \int_{A_0} f_2 U_M(v_0) - \int_{T-A_0} f_2 U_M(v_0).$$

Since $\operatorname{sg} f_1 = \operatorname{sg} v_0 = \operatorname{sg} v$, it follows that

(i)
$$\int_T |f_1|\,|v| + \int_{A_0} f_2 U_M(v) - \int_{T-A_0} f_2 U_M(v)$$
$$\leqslant \int_T |f_1|\,|v_0| + \int_{A_0} f_2 U_M(v_0) - \int_{T-A_0} f_2 U_M(v_0).$$

Assume that $|v_0| < \alpha(M) \Leftrightarrow |v| < \alpha(M)$. Then (i) and (VII.2) show that

(ii)
$$\int_T [|f_1|\,|v| + |f_2| U_M(v)] \leqslant \int_T [|f_1|\,|v_0| + |f_2| U_M(v_0)]$$

for every v such that

(iii) $\operatorname{sg} v = \operatorname{sg} v_0$, $|v| \leqslant \alpha(M)$ and $|v| = \alpha(M)$ iff $|v_0| = \alpha(M)$.

Set $v_n = v_0$ if $v_0 = 0$ or $|v_0| = \alpha(M)$, and $v_n = v_0(1 - 1/n)$ if $0 < |v_0| < \alpha(M)$.

Then, for every natural n, v_n verifies (iii) and $|v_n|<|v_0|$ in $\{0<|v_0|<\alpha(M)\}$. Given any interval J, set $v=v_0\chi_{T-J}+v_n\chi_J$, so v verifies (iii). Then (ii) states that

$$\int_J [|f_1|\,|v_n|+|f_2|U_M(v_n)]\leqslant\int_J [|f_1|\,|v_0|+|f_2|U_M(v_0)],\qquad \forall J$$

$$\Rightarrow |f_1|\,|v_n|+|f_2|U_M(v_n)\leqslant |f_1|\,|v_0|+|f_2|U_M(v_0)\ \text{a.e.}$$

Consequently,

$$\frac{U_M(|v_n|)-U_M(|v_0|)}{|v_0|-|v_n|}\leqslant\left|\frac{f_1}{f_2}\right|$$

holds a.e. in $\{0<|v_0|<\alpha(M)\}$. So we have

(iv) $\qquad J_M(|v_0|)\leqslant|f_1/f_2|$ a.e. in $\{0<|v_0|<\alpha(M)\}$.

Now set $v_n'=v_0$ if $v_0=0$ or $|v_0|=\alpha(M)$,

$$v_n'=\left[|v_0|+\frac{\alpha(M)-|v_0|}{n}\right]\operatorname{sg}v_0\quad\text{in}\quad\{0<|v_0|<\alpha(M)\}.$$

Then v_n' verifies (iii) and $|v_n'|>|v_0|$ in $\{0<|v_0|<\alpha(M)\}$ so, in the same way as above, we see that

(v) $\qquad |f_1/f_2|\leqslant J_M(|v_0|)$ a.e. in $\{0<|v_0|<\alpha(M)\}$.

Since $(-f_1+f_2)\in H^1$, (VII.3) shows that, except for a set of measure 0, $\{J_M(|v_0|)=0\}=\{v_0=0\}=\{f_1=0\}=\{|f_1/f_2|=0\}$, so (VII.4) is proved in $\{v_0=0\}$. In order to finish the proof it is thus enough to show that $f_2=0$ a.e. in $B=\{|v_0|=\alpha(M)\}$. Set

$$v_n=v_0[\chi_{T-B}+(1-1/n)\chi_B],\qquad u_n=u_0\chi_{T-B}+U_M(v_n)\cdot(\operatorname{sg}f_2)\chi_B;$$

then $(v_n,u_n)\in P_M$ and $\operatorname{sg}v_n=\operatorname{sg}v_0$, so (VII.1) and (VII.3) show that

$$\int_B|f_1|\,|v_n|+\int_B|f_2|U_M(v_n)\leqslant\int_B|f_1|\,|v_0|$$

$$\Rightarrow\frac{\int_B|f_1|}{\int_B|f_2|}=\frac{U_M[\alpha(M)(1-1/n)]}{(1/n)\alpha(M)}\to\infty\quad\text{when}\quad n\to\infty.$$

The result follows.

(VII.g) Corollary. *The following relations hold except for sets of measure* 0:

$$\text{(VII.5)}\qquad \begin{aligned}\{f_2>0\}&=A_0\cap\{|v_0|<\alpha(M)\},\\ \{f_2<0\}&=(T-A_0)\cap\{|v_0|<\alpha(M)\}.\end{aligned}$$

Proof. (VII.f) says that $|\{f_2\neq0\}\Delta\{|v_0|<\alpha(M)\}|=0$. So, except for sets of measure 0, (VII.2) shows that

$$\{f_2>0\}=\{f_2>0\}\cap\{|v_0|<\alpha(M)\}\subset A_0\cap\{|v_0|<\alpha(M)\},$$
$$\{f_2<0\}\cap\{|v_0|<\alpha(M)\}\subset(T-A_0)\cap\{|v_0|<\alpha(M)\}.$$

Since $\{f_2>0\}\cup\{f_2<0\}=\{|v_0|<\alpha(M)\}$, the result follows.

Now we easily see that f determines (v_0,u_0); let J_M^{-1} be the inverse function to J_M, and remember that $u_0=U_M(|v_0|)\cdot(\chi_{A_0}-\chi_{T-A_0})$. Then (VII.3), (VII.4) and (VII.5) state that:

$$\text{(VII.6)}\qquad (v_0,u_0)=\big((\operatorname{sg}f_1)J_M^{-1}(|f_1/f_2|),\ U_M[J_M^{-1}(|f_1/f_2|)]\cdot(\chi_{\{f_2>0\}}-\chi_{\{f_2>0\}})\big).$$

Set $b=(M+1)/2\sqrt{M}$; then $U_M(x)=\operatorname{arch}(b\cos x)$; if

$$y=J_M(x),\qquad x=J_M^{-1}(y)=\operatorname{artg}\left[\frac{y(b^2-1)^{1/2}}{(b^2+y^2)^{1/2}}\right]$$

and

$$\cos x=[(b^2+y^2)/(b^2+b^2y^2)]^{1/2}.$$

Consequently, (VII.6) can be written in the following way:

$$\text{(VII.7)}\qquad \begin{aligned}v_0&=\operatorname{artg}\big((M-1)f_1/[(M+1)^2f_2^2+4Mf_1^2]^{1/2}\big),\\ u_0&=\operatorname{arch}\big([(M+1)^2f_1^2+4Mf_1^2]^{1/2}/[4M(f_2^2+f_1^2)]^{1/2}\big)\cdot\\ &\qquad\cdot(\chi_{\{f_2>0\}}-\chi_{\{f_2<0\}}).\end{aligned}$$

So we have proved that

(VII.h) Proposition. *Let* (v_0,u_0) *be as in* (VI.j). *Then* $\exists f=(f_1,f_2)\neq0$ *such that* $(-f_1+if_2)\in H^1$, $\hat f_1(0)=\hat f_2(0)=0$, *and* (v_0,u_0) *are given by* (VII.7).

(VII.i) Lemma. *If* $f_1,f_2\in L^1(T)$ *and* (v_0,u_0) *are defined by* (VII.7), *then* $(v_0,u_0)\in P_M$.

Proof. Clearly $|v_0|\leqslant\operatorname{artg}[(M-1)/\sqrt{4M}]=\alpha(M)$; also

$$\cos v_0=\frac{2M^{1/2}}{M+1}\left[\frac{(M+1)^2f_2^2+4Mf_1^2}{4M(f_1^2+f_2^2)}\right]^{1/2},$$

so

$$|u_0|=\operatorname{arch}\left(\frac{M+1}{2M^{1/2}}\cos v_0\right)=U_M(v_0).$$

(VII.j) Lemma. *In the same hypothesis of* (VII.i), $f(v,u)<f(v_0,u_0)$ *holds for every* $(v,u)\in P_M-\{(v_0u_0)\}$.

Proof. Since $|u| \leqslant U_M(v)$, $u = hU_M(v)$, with $|h| \leqslant 1$. Then

(i) $$f(v, u) \leqslant \int_T [|f_1|\,|v| + |f_2| U_M(v)].$$

(VII.7) shows that

(ii) $$\operatorname{sg} v_0 = \operatorname{sg} f_1,$$

and also that

(iii) $\operatorname{tg}(|v_0|) = (M-1)|f_1|/[(M+1)^2 f_2^2 + 4Mf_1^2]^{1/2} \Rightarrow |f_1/f_2| = J_M(|v_0|)$.

Also

(iv) $$u_0 = U_M(v_0) . (\chi_{\{f_2>0\}} - \chi_{\{f_2<0\}}).$$

(ii) and (iv) show

(v) $$f(v_0, u_0) = \int_T [|f_1|\,|v_0| + |f_2| U_M(v_0)].$$

Suppose first that $|v| = |v_0|$ a.e. If $\operatorname{sg} f_1 = \operatorname{sg} v$ does not hold a.e., (v) states that $f(v, u) < f(v_0, u_0)$. If $\operatorname{sg} f_1 = \operatorname{sg} v$ a.e., (ii) shows that $v = v_0$, so

$$f(v_0, u_0) - f(v, u) = f(v_0, u_0) - f\big(v_0, hU_M(v_0)\big)$$
$$= \int_T |f_2| U_M(v_0) - \int_T f_2 h U_M(v_0) > 0$$

except $h = \operatorname{sg} f_2$ a.e. in $\{|v_0| < a(M)\}$, which would imply $u = u_0$. Suppose that $A = \{|v| \neq |v_0|\}$ has positive measure. Consider, in A, $C = -[U_M(v) - U_M(v_0)]/[|v| - |v_0|]$; if $|v| > |v_0|$, $C > -U_M'(|v_0|) = J_M(|v_0|)$, and, if $|v| < |v_0|$, $C < -U_M'(|v_0|)$, so, in A, $U_M(v_0) - U_M(v) > J_M(|v_0|).(|v| - |v_0|)$.

Then (iii) shows that, in A, $|f_2| U_M(v_0) + |f_1|\,|v_0| > |f_2| U_M(v) + |f_1|\,|v|$. Since $|A| > 0$,

$$\int_T [|f_1|\,|v_0| + |f_2| U_M(v_0)] > \int_T [|f_1|\,|v| + |f_2| U_M(v_0)];$$

considering (i) and (v), the result follows.

(VII.k) PROPOSITION. *In the same hypothesis of* (VII.i), $\exp[G \circ \tau(g)] = \exp(u_0 + \tilde v_0) \in \operatorname{Ext} R_M$ *and* $\tau(g) \in EP_M$.

Proof. (VII. j) shows that $(v_0, u_0) \in \operatorname{Ext} \tilde P_M$. Suppose

$$(v, u+c) \in \{(v_0, u_0) + N\} \cap \tilde P_M;$$

since $N \subset \operatorname{Ker} f$, it is clear that

$$G^{-1}(u_0 + \tilde v_0) \cap \tilde P_M = \{(v_0, u_0) + N\} \cap \tilde P_M$$

contains only one element and that it belongs to $\operatorname{Ext} \tilde P_M$. So $\exp(u_0 + \tilde v_0) \in \operatorname{Ext} R_M$ because of (VI.i). Consequently (v_0, u_0) is as in (VI.j); set $d\mu = g\,dt$, $dv = \tilde g\,dt$. Then, the result follows from (VII.j) and Definition (VII.b).

Remark that the first part of Theorem 1 has been proved. In order to prove the second we shall give a geometrical interpretation of E_M. We say that $\varphi \in \tilde L_M$ *belongs to the set* $\operatorname{Exp} \tilde L_M$ of exposed points of $\tilde L_M$ if there exists a linear functional F, continuous in the weak star topology of L, such that $F(\varphi) > F(\psi)$, $\forall \psi \in \tilde L_M - \{\varphi\}$.

Proposition (VI.i) shows that there exists a bijection between $\operatorname{Ext} R_M$ and $\operatorname{Ext} \tilde L_M$, given by $f \to e^f$. We shall show that the same relation holds between E_M and $\operatorname{Exp} \tilde L_M$.

(VII.k) PROPOSITION. $\varphi \in \operatorname{Exp} \tilde L_M \Leftrightarrow e^\varphi \in E_M$.

Proof. If $\varphi \in \operatorname{Exp} \tilde L_M$, clearly $\varphi \in \operatorname{Ext} \tilde L_M$, so $e^\varphi \in \operatorname{Ext} R_M$. Then $\varphi = c + u_0 + \tilde v_0$, with (v_0, u_0) as in (VI.j). Moreover, there exists $g \in H_{1,0}$ such that

$$\int_T (g v_0 + \tilde g u_0) \geqslant \int_T (g v + \tilde g u) \quad \text{for every } (v, u) \in P_M.$$

Thus $(v_0, u_0) \in EP_M$. Proposition (VII.h) and the definition of E_M finish this part of the proof. Reciprocally, if $e^\varphi \in E_M$, $\varphi = c + u_0 + \tilde v_0$, with (v_0, u_0) as in (VII.i). By (VII.j), $\varphi \in \operatorname{Exp} \tilde L_M$.

We know that $\tilde L_M$ is a weak star compact and convex subset of L. So we may refer to the following theorem of V. Klee ([4]): Let E be a separable Banach space and E^* its topological dual. Let C be a weak star compact and convex subset of E^*. Then C is the weak star closure of the convex hull of the set of exposed points of C.

This theorem, with $E = H_{1,0}$, $C = \tilde L_M$, says that the proof of Theorem 1 is over.

References

[1] C. T. Cargo, J. Math. Anal. Appl. 7 (1963), 471–474.
[2] M. Cotlar and C. Sadosky, Proc. Symp. Pure Math., vol. 35, AMS.
[3] H. Helson and G. Szegö, Ann. Mat. Pura Appl. 51 (1960), 107–138.
[4] V. Klee, Math. Z. 69 (1958), 90–104.

Received April 21, 1979 (1536)