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INSTITUTE OF MATHEMATICS  
WROCLAW TECHNICAL UNIVERSITY  
50-370, WROCLAW, POLAND

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## Convexity, type and the three space problem

by

N. J. KALTON (Columbia, Mo.)\*

**Abstract.** A twisted sum of two quasi-Banach spaces  $X$  and  $Y$  is a quasi-Banach space  $Z$  with a closed subspace  $X_0 \cong X$  such that  $Z/X_0 \cong Y$ .

We show that if  $X$  is  $p$ -convex and  $Y$  is  $q$ -convex where  $p \neq q$ , then  $Z$  is  $\min(p, q)$  convex. Similarly, if  $X$  is a type  $p$  Banach space and  $Y$  is a type  $q$  Banach space where  $p \neq q$  then  $Z$  is type  $\min(p, q)$ .

If  $X$  and  $Y$  are Banach spaces, we show that  $Z$  is *log convex*, i.e., for some  $C < \infty$

$$\|z_1 + \dots + z_n\| \leq C \left( \sum_{k=1}^n \|z_k\| \left( 1 + \log \frac{1}{\|z_k\|} \right) \right)$$

where  $\|z_1\| + \dots + \|z_n\| = 1$ . Conversely, every log convex space is the quotient of a subspace of a twisted sum of two Banach spaces.

If  $X$  and  $Y$  are type  $p$  Banach spaces ( $1 < p < 2$ ) and one is the quotient of a subspace of some  $L_p$ -space, then  $Z$  is *log type  $p$* , i.e.,

$$\left\{ \int_0^1 \|e_1(t)z_1 + \dots + e_n(t)z_n\|^p dt \right\}^{1/p} \leq C \left\{ \sum \|z_k\|^p \left( 1 + \left( \log \frac{1}{\|z_k\|} \right)^p \right) \right\}^{1/p}$$

where  $\|z_1\|^p + \dots + \|z_n\|^p = 1$ . This result is best possible in a certain sense.

We also show that if  $p < 1$  type  $p$  implies  $p$ -convexity, but if  $p = 1$  a type 1 space need not be convex.

We investigate which Orlicz sequence spaces and Köthe sequence spaces are  $X$ -spaces, i.e., such that every twisted sum with  $X$  is a direct sum.

**1. Introduction.** A quasi-Banach space  $Z$  is a twisted sum of  $X$  and  $Y$  if it has a subspace  $X_0 \cong X$  such that  $Z/X_0 \cong Y$ . The so-called *three space problem* is to study the properties of  $Z$  in terms of those of  $X$  and  $Y$ .

In [1], Enflo, Lindenstrauss and Pisier showed that a Banach space which is a twisted sum of two Hilbert spaces need not be a Hilbert space. Independently, the author [6], Ribe [15] and Roberts [16] showed that a twisted sum of a line and a Banach space need not be locally convex. In [9] the author and Peck showed that these results are related by describing a general construction which shows that for every  $p$ ,  $0 < p < \infty$ , there is a twisted sum of  $l_p$  with  $l_p$  which is not a direct sum. In particular, for  $0 < p < 1$ , there is a non  $p$ -convex space which is a twisted sum of two  $p$ -convex spaces.

\* University of Missouri. Research done while the author held visiting positions at Michigan State University and the University of Illinois.

In contrast to these negative results there are a number of theorems which say that a twisted sum cannot be too bad. In [1] the twisted sum of two type 2 Banach spaces is shown to be type  $p$  for all  $p < 2$ . In [6] it is shown that if  $X$  is a Banach space and  $Y$  is a type  $p$  Banach space for some  $p > 1$ , then a twisted sum of  $X$  and  $Y$  (in that order) is convex (i.e., a Banach space). Also if  $X$  is  $p$ -convex ( $0 < p \leq 1$ ) and  $Y$  is  $q$ -convex ( $0 < q < p$ ), then any twisted sum of  $X$  and  $Y$  (in that order) is again  $q$ -convex.

These suggest a general principle, if we regard  $p$ -convexity ( $0 < p \leq 1$ ) or type  $p$  ( $0 < p \leq 2$ ) as an index of "roundness". The twisted sum of two spaces of differing degrees of roundness will retain the properties of the less round space; the twisted sum of two spaces of equal roundness may however be less round than either. The main aim of this paper is to establish such a pattern, and to examine more precisely the case of equal roundness.

First in Section 3, we introduce a new class of quasi-Banach spaces which we name *logconvex*. A space  $X$  is logconvex if either of the following two equivalent conditions holds for some  $C$ ,  $C^* < \infty$

$$(1.0.1) \quad \|x_1 + \dots + x_n\| \leq C \sum_{i=1}^n \|x_i\| \left(1 + \log \frac{1}{\|x_i\|}\right)$$

wherever  $\|x_1\| + \dots + \|x_n\| = 1$ ,  $x_1, \dots, x_n \in X$  or

$$(1.0.2) \quad \|x_1 + \dots + x_n\| \leq C^* \sum_{j=1}^n \|x_j\| (1 + \log j).$$

Logconvex spaces play an important role in this paper; they are, in a sense, the next best thing to being Banach spaces. An example is the space  $L(1, \infty)$  (i.e., weak  $L_1$ ).

In Section 4 we show that if  $p < 1$ , type  $p$  is equivalent to  $p$ -convexity, so that we reduce the study of type to the case  $1 \leq p \leq 2$ .

Section 5 contains some initial technical results in twisted sums which contain very little that is new. Lemma 5.2 essentially reproduces a result of [1] in rather more generality.

In Section 6 we show that if  $X$  and  $Y$  are  $p$ -convex and  $q$ -convex, respectively, where  $p < q$ , then any twisted sum of  $X$  and  $Y$  or of  $Y$  and  $X$  is  $p$ -convex. (One half of this result is in [6], see above). In a similar vein, if  $X$  and  $Y$  are type  $p$  and type  $q$ , respectively, where  $1 \leq p < q \leq 2$ , then any twisted sum  $Z$  of  $X$  and  $Y$  or  $Y$  and  $X$  is type  $p$ . Let us remark here that the methods of [1] (cf. also [13]) show that in either case if  $z_1, \dots, z_n \in Z$

$$(1.0.3) \quad \left\{ \int_0^1 \left\| \sum \varepsilon_i z_i \right\|^2 dt \right\}^{1/2} \leq C n^{1/p-1/2} \left( \sum_{i=1}^n \|z_i\|^2 \right)^{1/2}.$$

The main step in the argument here is to pass from this inequality to establishing type  $p$  (but only for twisted sums where the other space is type  $q > p$ ). As shown in [13] (1.0.3) implies type  $r$  for  $r < p$ . We note that a type 1 space need not be convex.

In Section 7 we study the case  $p = q$ . We show that any twisted sum of two Banach spaces is logconvex, and this result is best possible. In fact, a space is logconvex if and only if it is a quotient of some subspace of a twisted sum of two Banach spaces. The corresponding results for type  $p$  are right if we assume that one of the spaces  $X$  and  $Y$  is a quotient of a subspace of a space  $L_p(\mu)$ . In that case any twisted sum  $Z$  is *log type  $p$* , i.e.,

$$(1.0.4) \quad \left\{ \int_0^1 \left\| \sum \varepsilon_i z_i \right\|^p dt \right\}^{1/p} \leq C \left( 1 + \sum \|z_i\|^p \left( \log \frac{1}{\|z_i\|} \right)^p \right)^{1/p}$$

whenever  $\|z_1\|^p + \dots + \|z_n\|^p = 1$ , or equivalently

$$(1.0.5) \quad \left\{ \int_0^1 \left\| \sum_{i=1}^n \varepsilon_i z_i \right\|^p dt \right\}^{1/p} \leq C^* \left( \sum_{k=1}^n \|z_k\|^p (1 + \log k)^p \right)^{1/p}$$

for  $z_1, \dots, z_n \in Z$ . Furthermore this is best possible for the twisted sum of  $l_p$  and  $l_p$  ( $1 \leq p \leq 2$ ) constructed in [9] contains a copy of the Orlicz space  $l_p$  where

$$\psi(t) = t^p \left[ 1 + \left( \log \frac{1}{t} \right)^p \right]$$

near zero, and for this space (1.0.4) cannot be improved.

In Section 8 we examine twisted sums of  $\mathbf{R}$  and  $l_1$  more closely, showing in particular that the examples in [6] and [15] are non-equivalent but both are best possible in a certain sense.

In Section 9 we classify those non-locally convex Orlicz spaces  $l_\psi \subset l_1$  which are  $\mathcal{X}$ -spaces, i.e., for which every twisted sum of  $\mathbf{R}$  and  $l_\psi$  is a direct sum. In particular, this applied to examining "galb" conditions of the type  $(\sum f(\|x_i\|) < \infty \Rightarrow \sum x_i \text{ converges})$  which are preserved under twisted sums with  $\mathbf{R}$ . It is shown that if  $f$  is submultiplicative, this condition will be preserved if and only if  $f(x) \geq cx^p$  for some  $p < 1$ .

In Section 10, we examine those locally convex  $F$ -spaces  $X$  which are not locally bounded, but are  $\mathcal{X}$ -spaces, so that they have the property that if  $Y$  is locally convex any twisted sum of  $Y$  and  $X$  is locally convex. It is shown that every nuclear space is a  $\mathcal{X}$ -space, and Köthe spaces which are  $\mathcal{X}$ -spaces are characterized exactly.

**2. Quasi-Banach spaces.** Throughout this paper all vector spaces will be real, although most arguments may be modified without difficulty to the complex case.

A quasi-norm on a real vector space  $X$  is a map  $x \rightarrow \|x\|$  ( $X \rightarrow \mathbf{R}$ ) such that for some  $K < \infty$ ,

$$(2.0.1) \quad \|x\| > 0, \quad x \neq 0, \quad x \in X,$$

$$(2.0.2) \quad \|tx\| = |t| \|x\|, \quad t \in \mathbf{R}, \quad x \in X,$$

$$(2.0.3) \quad \|x+y\| \leq K(\|x\| + \|y\|), \quad x, y \in X.$$

A quasi-norm induces a locally bounded topology on  $X$  and conversely any locally bounded topology is given by a quasi-norm. A complete quasi-normed space is called a *quasi-Banach space*. If in addition we have for some  $0 < p \leq 1$

$$(2.0.4) \quad \|x+y\|^p \leq \|x\|^p + \|y\|^p, \quad x, y \in X,$$

then  $X$  is called a *p-Banach space* (or if  $p = 1$  a Banach space).

A quasi-Banach space  $X$  is said to be *p-convex* for some  $0 < p \leq 1$  if there is a constant  $A$  such that

$$(2.0.5) \quad \|x_1 + x_2 + \dots + x_n\| \leq A(\|x_1\|^p + \dots + \|x_n\|^p)^{1/p}$$

for  $x_1, \dots, x_n \in X$ . If  $X$  is *p-convex*, it may be equivalently quasi-normed to be a *p-Banach space*. A theorem of Aoki and Rolewicz (see [17]) states that every quasi-Banach space is *p-convex* for some  $p > 0$ . We shall repeatedly exploit this by assuming the quasi-norm on a given space satisfies (2.0.4) for some  $p > 0$ .

We denote by  $(\varepsilon_n: n \in \mathbf{N})$  a sequence of independent random variables (or measurable functions) on  $[0, 1]$  such that  $\lambda(\varepsilon_n = +1) = \lambda(\varepsilon_n = -1) = \frac{1}{2}$  where  $\lambda$  is Lebesgue measure. We then say that a quasi-Banach space  $X$  is *type p* ( $0 < p \leq 2$ ) ([12], [13]) if for some constant  $K < \infty$  we have

$$(2.0.6) \quad \left( \int_0^1 \left\| \sum_{i=1}^n \varepsilon_i(t) x_i \right\|^p dt \right)^{1/p} \leq K \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p}.$$

If  $X$  is *p-convex*, then  $X$  is certainly *type p*.

We remark here that Kahane [4] shows that for a Banach space  $X$  and  $0 < p < q < \infty$  there is a constant  $K = K(p, q)$  such that

$$(2.0.7) \quad \left( \int_0^1 \left\| \sum \varepsilon_i x_i \right\|^p dt \right)^{1/p} \leq \left\{ \int_0^1 \left\| \sum \varepsilon_i x_i \right\|^q dt \right\}^{1/q} \leq K \left\{ \int_0^1 \left\| \sum \varepsilon_i x_i \right\|^p dt \right\}^{1/p}.$$

This means that we can change the exponent on the left of (2.0.6) without altering the definition.

In fact, (2.0.7) holds for quasi-Banach spaces; the modifications in Kahane's argument are minor but we include a proof for completeness.

**THEOREM 2.1.** *Let  $X$  be a quasi-Banach space. Then (2.0.7) holds.*

**Proof.** Let  $\tilde{L}_0(X)$  be the space of  $X$ -valued simple functions on  $[0, 1]$  equipped with the topology of convergence in measure. Let  $\text{Rad}(X)$  be the subspace of functions of the form  $\varepsilon_1 x_1 + \dots + \varepsilon_n x_n$  for  $x_1, \dots, x_n \in X$  and  $n \in \mathbf{N}$ . We show that on  $\text{Rad}(X)$ , the  $L_0$ -topology coincides with the stronger topology induced by any quasi-norm

$$f \rightarrow \left\{ \int_0^1 \|f(t)\|^p dt \right\}^{1/p}.$$

We see this, we need only show that the set of  $f \in \text{Rad}(X)$  with  $\lambda(\|f\| \geq 1) < \frac{1}{8}$  is bounded in each  $L_p$ -norm.

Suppose  $f = \varepsilon_1 x_1 + \dots + \varepsilon_n x_n$  and

$$\lambda(\|f\| > r) = \alpha.$$

Let

$$M(t) = \max_{1 \leq k \leq n} \left\| \sum_{i=1}^k \varepsilon_i(t) x_i \right\|, \quad 0 \leq t \leq 1,$$

$$N(t) = \max_{1 \leq k \leq n} \left\| \sum_{i=k}^n \varepsilon_i(t) x_i \right\|, \quad 0 \leq t \leq 1.$$

Let  $A_k$  ( $1 \leq k \leq n$ ) be the set of  $t$  such that

$$\left\| \sum_{i=1}^l \varepsilon_i(t) x_i \right\| < Kr, \quad 1 \leq l \leq k-1,$$

$$\left\| \sum_{i=1}^k \varepsilon_i(t) x_i \right\| \geq Kr$$

(where  $K$  is the modulus of concavity of the quasi-norm given by (2.0.3)).

Since  $f$  has the same distribution as

$$f^* = \varepsilon_1 x_1 + \dots + \varepsilon_k x_k - \varepsilon_{k+1} x_{k+1} - \dots - \varepsilon_n x_n$$

and

$$\begin{aligned} \lambda(A_k \cap (\|f + f^*\| \geq 2Kr)) &\leq \lambda(A_k \cap (\|f\| \geq r)) + \lambda(A_k \cap (\|f^*\| \geq r)) \\ &= 2\lambda(A_k \cap (\|f\| \geq r)) \end{aligned}$$

and hence

$$\lambda(A_k) \leq 2\lambda(A_k \cap (\|f\| \geq r)),$$

so that, summing over  $k$ ,

$$\lambda(M > Kr) \leq 2\alpha.$$

Similarly,  $\lambda(N > Kr) \leq 2\alpha$ .

Now if  $t \in A_k$  and  $\|f\| \geq 2K^2r$ ,

$$\left\| \sum_{i=1}^k \varepsilon_i x_i \right\| \geq Kr$$

and

$$\left\| \sum_{i=k}^n \varepsilon_i x_i \right\| \geq Kr \quad (\text{since } \left\| \sum_{i=1}^{k-1} \varepsilon_i x_i \right\| < Kr).$$

Hence

$$\lambda(A_k \cap (\|f\| \geq 2K^2r)) \leq \lambda\left(A_k \cap \left(\left\| \sum_{i=k}^n \varepsilon_i x_i \right\| \geq Kr\right)\right) \leq 2\alpha\lambda(A_k)$$

since these sets are independent. Summing over  $k$ ,

$$\lambda(\|f\| \geq 2K^2r) \leq 4\alpha^2.$$

Thus if

$$\lambda(\|f\| \geq 1) < \frac{1}{8},$$

$$\lambda(\|f\| \geq (2K^2)^n) < 4^{2^n-1} \left(\frac{1}{8}\right)^{2^n} < \left(\frac{1}{2}\right)^{2^n}$$

and

$$\int_0^1 \|f\|^p dt < 1 + \sum_{n=0}^{\infty} (2K^2)^{np} \left(\frac{1}{2}\right)^{2^n} = S_p < \infty.$$

The *galb*  $G(X)$  of a quasi-Banach space is the space of all sequences  $\{a_n\}$  such that if  $\|x_n\| \leq 1$ , then  $\{\sum_{k=1}^n a_k x_k\}$  is bounded.  $G(X)$  is a quasi-Banach space when quasi-normed by

$$\|(a_1, a_2, \dots)\| = \sup_{\|x_k\| \leq 1} \sup_n \left\| \sum_{k=1}^n a_k x_k \right\|.$$

$X$  is said to be *galbed* by a space of sequences  $E$  if, given  $\|x_n\| \leq 1$ , and  $\{a_n\} \in E$ , then  $\sum_{k=1}^n a_k x_k$  is bounded (see Turpin [17] for a more detailed study of these notions).

An  $F$ -space is a complete metric topological vector space. A twisted sum of two  $F$ -spaces  $X$  and  $Y$  is a space  $Z$  which has a closed subspace  $X_0 \cong X$  such that  $Z/X_0 \cong Y$ . Thus there is a short exact sequence  $0 \rightarrow X \rightarrow Z \rightarrow Y \rightarrow 0$ . If every twisted sum of  $X$  and  $Y$  is a direct sum (i.e.  $Z \cong X \oplus Y$  in the natural way) then we say that  $(X, Y)$  *splits* (the order is important here). If  $(R, X)$  splits, then  $X$  is a  $\mathcal{K}$ -space ([8]).

If  $X$  is a locally convex  $\mathcal{K}$ -space, then every twisted sum of  $Y$  and  $X$  with  $Y$  locally convex is also locally convex ([6], Theorem 4.10) and this property characterizes locally convex  $\mathcal{K}$ -spaces.

**3. Logconvex spaces and related classes.** Let  $\varphi$  denote the Orlicz function

$$\varphi(t) = \begin{cases} t \left(1 + \log \frac{1}{t}\right), & 0 \leq t \leq 1, \\ t, & t \geq 1 \end{cases}$$

(where  $0 \log \infty = 0 \log 0 = 0$  by convention). Then  $l_\varphi$  is a locally bounded but non-locally convex Orlicz sequence space. The quasi-norm inducing the topology on  $l_\varphi$  may be given by

$$\|x\|_\varphi = \sup \left\{ \xi : \sum_{i=1}^{\infty} \varphi(\xi^{-1}|x_i|) \leq 1 \right\}.$$

Our first result gives an equivalent quasi-norm.

**THEOREM 3.1.** *An equivalent quasi-norm on  $l_\varphi$  is given by*

$$\|x\|_\varphi^* = \|x\|_1 + \sum_{i=1}^{\infty} |x_i| \log \frac{\|x\|_1}{|x_i|}$$

where  $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ .

**Proof.** Since  $\|\cdot\|_\varphi^*$  is easily seen to be homogeneous, it suffices to show that

$$0 < \inf(\|x\|_\varphi^* : \|x\|_\varphi = 1) \leq \sup(\|x\|_\varphi^* : \|x\|_\varphi = 1) < \infty.$$

If  $\|x\|_\varphi = 1$ , then  $\sum \varphi(|x_i|) = 1$  and hence

$$\sum |x_i| \left(1 + \log \frac{1}{|x_i|}\right) = 1.$$

Hence  $\|x\|_1 \leq 1$  and

$$\|x\|_\varphi^* \leq \|x\|_1 + \sum_{i=1}^{\infty} |x_i| \log \frac{1}{|x_i|} = \|x\|_\varphi = 1.$$

Conversely,

$$\|x\|_\varphi^* = \|x\|_\varphi - \|x\|_1 \log \|x\|_1 = 1 - \|x\|_1 \log \|x\|_1 \geq 1 - \frac{1}{e}$$

since  $0 \leq \|x\|_1 \leq 1$ .

**DEFINITION 3.2.** A quasi-Banach space  $X$  is *logconvex* if it is galbed by  $l_\varphi$ , i.e., whenever  $x_n \in X$  and

$$\sum \varphi(\|x_n\|) < \infty$$

then  $\sum x_n$  converges.

EXAMPLE. The space  $l_\varphi$  itself is logconvex; this follows easily from the fact that  $\varphi$  is submultiplicative at 0 (cf. Turpin [19], p. 79).

THEOREM 3.3. A quasi-Banach space  $X$  is logconvex if and only if for some constant  $C$  and any  $x_1, \dots, x_n \in X$

$$(3.3.1) \quad \|x_1 + x_2 + \dots + x_n\| \leq C \left[ \sum_{i=1}^n \|x_i\| \left( 1 + \log \frac{S}{\|x_i\|} \right) \right]$$

where  $S = \sum_{i=1}^n \|x_i\|$ .

Remark. (3.3.1) is equivalent to

$$(3.3.2) \quad \|x_1 + \dots + x_n\| \leq C \left( 1 + \sum_{i=1}^n \|x_i\| \log \frac{1}{\|x_i\|} \right)$$

whenever  $\|x_1\| + \dots + \|x_n\| \leq 1$ .

Proof. Let  $I$  be an infinite set with  $|I| = |X|$  and let  $(x_i: i \in I)$  be the unit ball of  $X$ . If  $X$  is logconvex the map  $T: l_\varphi(I) \rightarrow X$  (where  $l_\varphi(I)$  is the generalized sequence space of all  $(\xi_i: i \in I)$  such that  $\sum \varphi(|\xi_i|) < \infty$  defined by

$$T(\xi) = \sum_{i \in I} \xi_i x_i$$

is well-defined and continuous. Hence for some  $C < \infty$

$$\|T(\xi)\| \leq C \left( \|\xi\|_1 + \sum_{i \in I} |\xi_i| \log \frac{\|\xi\|_1}{|\xi_i|} \right)$$

and (3.3.1) follows easily.

Conversely, if (3.3.1) holds and

$$\sum_{n=1}^{\infty} \varphi(\|x_n\|) \leq 1,$$

then

$$\sum_{n=1}^{\infty} \|x_n\| \left( 1 + \log \frac{1}{\|x_n\|} \right) \leq 1,$$

and hence

$$\left\| \sum_{n=k+1}^l x_n \right\| \leq C \sum_{n=k+1}^l \|x_n\| \left( 1 + \frac{1}{\log \|x_n\|} \right) \rightarrow 0 \quad \text{as } k, l \rightarrow \infty$$

so that  $\sum x_n$  converges.

$L(1, \infty)$  denotes the space of measurable functions on  $[0, 1]$  such that

$$\|f\| = \sup \lambda \{ |f| > \lambda \} < \infty.$$

THEOREM 3.4. The space  $L(1, \infty)$  is logconvex. [Added in proof: see [20].]

Proof. Suppose  $x_1, \dots, x_n \in L(1, \infty)$  and let

$$f = x_1 + \dots + x_n.$$

Suppose also  $\|x_1\| + \dots + \|x_n\| = 1$ .

Fix  $\tau$ ,  $0 < \tau < 1$  and let  $A \subset (0, 1)$  be a set of measure  $\tau$ . For each  $i = 1, 2, \dots, n$  let

$$E_i = \{ |x_i| > 2\tau^{-1} \}.$$

The  $\lambda(E_i) \leq \frac{1}{2}\tau\|x_i\|$ ; hence if  $E = E_1 \cup \dots \cup E_n$ , then  $\lambda(E) \leq \frac{1}{2}\tau$ . Now

$$\begin{aligned} \inf_{t \in A} |f(t)| &\leq \inf_{t \in A \setminus E} |f(t)| \leq \frac{2}{\tau} \int_{A \setminus E} |f(t)| dt \\ &\leq \frac{2}{\tau} \sum_{i=1}^n \int_{A \setminus E} |x_i(t)| dt \leq \frac{2}{\tau} \sum_{i=1}^n \int_{A \setminus E_i} |x_i(t)| dt \\ &\leq \frac{2}{\tau} \sum_{i=1}^n \int_A \min(|x_i(t)|, 2\tau^{-1}) dt \leq \frac{2}{\tau} \sum_{i=1}^n \int_0^\tau \min\left(\frac{\|x_i\|}{u}, 2\tau^{-1}\right) du \\ &\leq \frac{2}{\tau} \sum_{i=1}^n \left( \|x_i\| + \int_{1/2\tau\|x_i\|}^\tau \frac{\|x_i\|}{u} du \right) = \frac{2}{\tau} \sum_{i=1}^n \left( \|x_i\| + \|x_i\| \log \frac{2}{\|x_i\|} \right) \\ &= \frac{1}{\tau} \left( 2 \log 2 + 2 + 2 \sum_{i=1}^n \|x_i\| \log \frac{1}{\|x_i\|} \right). \end{aligned}$$

Hence

$$\|f\| \leq (2 \log 2 + 2) + 2 \sum_{i=1}^n \|x_i\| \log \frac{1}{\|x_i\|}$$

and so  $L(1, \infty)$  is logconvex.

EXAMPLE. Let  $(g_n: n = 1, 2, \dots)$  be a sequence of independent random variables each with the Cauchy distribution (i.e., with probability density function

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \quad -\infty < x < \infty.$$

Then  $(|g_n|: n \in \mathbb{N})$  is bounded in  $L(1, \infty)$  and so if  $a_n \geq 0$ ,  $\sum a_n |g_n|$  converges in  $L(1, \infty)$  if

$$(3.4.1) \quad \sum_{n=1}^{\infty} a_n \left(1 + \log \frac{1}{a_n}\right) < \infty$$

and then

$$(3.4.2) \quad \sum_{n=1}^{\infty} a_n |g_n(t)| < \infty \text{ a.e.}$$

L. Schwartz [18] shows that (3.4.1) is equivalent to (3.4.2). See Kahane [4] p. 97 for a similar example.

We now give another characterization of logconvex spaces; for this we require the following lemma.

LEMMA 3.5. Suppose  $\varepsilon > 0$  and

$$C_\varepsilon = \log \left[ \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{1+\varepsilon} \right] \quad (= \log \zeta(1+\varepsilon)).$$

If  $\xi_1 \geq \xi_2 \geq \dots \geq \xi_n \geq 0$  and

$$\xi_1 + \xi_2 + \dots + \xi_n = 1$$

then

$$(3.5.1) \quad \sum_{k=1}^n \xi_k \log k \leq \sum_{k=1}^n \xi_k \log \frac{1}{\xi_k} \leq (1+\varepsilon) \sum_{k=1}^n \xi_k \log k + C_\varepsilon.$$

Proof. Since  $\xi_k \leq 1/k$  the first inequality is clear. To prove the second, fix  $n$  and let  $C_n$  be the maximum of

$$F(\xi_1, \xi_2, \dots, \xi_n) = \sum_{k=1}^n \xi_k \log \frac{1}{\xi_k} - (1+\varepsilon) \sum_{k=1}^n \xi_k \log k$$

subject to  $\xi_1 \geq \dots \geq \xi_n \geq 0$  and  $\xi_1 + \dots + \xi_n = 1$ . Then for some  $(u_1, \dots, u_n)$ ,  $F(u_1, \dots, u_n) = C_n$ .

We claim first that  $u_1 > u_2 > u_3 > \dots > u_n > 0$ . For if  $l \leq n$  in the first index such that  $u_l = 0$  then a small increase in  $u_l$  and decrease in  $u_{l-1}$  increases  $F$ ; a similar argument shows that  $u_i \neq u_j$  if  $i \neq j$ . It follows that  $(u_1, \dots, u_n)$  is a local maximum of  $F$  subject to the single condition  $\xi_1 + \dots + \xi_n = 1$ . Hence there is a Lagrange multiplier  $\lambda$  such that

$$\frac{\partial F}{\partial \xi_k}(u_1, \dots, u_n) = \lambda, \quad k = 1, 2, \dots, n,$$

i.e.,

$$\log \frac{1}{u_k} - (1+\varepsilon) \log k = \lambda + 1.$$

Here  $u_k = e^{-1-\lambda} \left(\frac{1}{k}\right)^{1+\varepsilon}$  and so

$$\left(\sum_{k=1}^n \left(\frac{1}{k}\right)^{1+\varepsilon}\right) = e^{\lambda+1}, \quad F(u_1, \dots, u_n) = (\lambda-1) = C_n,$$

and hence  $C_n \leq C_\varepsilon$  and the result is proved.

THEOREM 3.6. A quasi-Banach space  $X$  is logconvex if and only if for some  $C < \infty$ , whenever  $x_1, \dots, x_n \in X$

$$(3.6.1) \quad \|x_1 + \dots + x_n\| \leq C \sum_{k=1}^n \|x_k\| (1 + \log k).$$

Proof. This follows immediately from the preceding lemma and Theorem 3.3.

Remark. This theorem essentially means that the Orlicz space  $l_\varphi$  is identical to the Lorentz space of all sequences  $(a_n)$  such that  $\sum a_n^*(1 + \log n) < \infty$  where  $(a_n^*)$  is decreasing re-arrangement of  $(|a_n|)$ . See [11] for similar results for convex Orlicz spaces and Lorentz spaces.

#### 4. Type in quasi-Banach spaces.

THEOREM 4.1. Suppose  $1 < p \leq 2$ ; then a quasi-Banach space of type  $p$  is convex.

Proof. Clearly if

$$b_n = \sup_{\|x_i\| \leq 1} \inf_{\sigma_i = \pm 1} \|\sigma_1 x_1 + \sigma_2 x_2 + \dots + \sigma_n x_n\|,$$

then  $b_n = o(n)$ , and the result follows from Theorem 2.5 of [6].

THEOREM 4.2. Suppose  $0 < p < 1$ ; then a quasi-Banach space  $X$  of type  $p$  is  $p$ -convex.

Proof. We can and do suppose  $X$  is an  $r$ -Banach space where  $0 < r < p$ . For each  $n \in \mathbb{N}$ , let  $d_n$  be the least constant such that

$$\|x_1 + \dots + x_n\| \leq d_n (\|x_1\|^p + \dots + \|x_n\|^p)^{1/p}$$

for  $x_1, \dots, x_n \in X$ . Suppose for any  $n$

$$\left\{ \int_0^1 \left\| \sum_{i=1}^n \varepsilon_i(t) x_i \right\|^p dt \right\}^{1/p} \leq C \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p}.$$

Then for any  $x_1, \dots, x_n \in X$  there exists  $\sigma_i = \pm 1$  ( $1 \leq i \leq n$ ) such that

$$\|\sigma_1 x_1 + \dots + \sigma_n x_n\| \leq C \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p}.$$



We may suppose that if  $F = \{i: \sigma_i = -1\}$  then  $\sum_{i \in F} \|x_i\|^p \leq \frac{1}{2} \sum_{i=1}^n \|x_i\|^p$ . Then

$$\left\| \sum_{i \in F} x_i \right\| \leq 2^{-1/p} d_n \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p},$$

and hence

$$\left\| \sum_{i=1}^n x_i \right\|^r \leq \left\| \sum_{i=1}^n \sigma_i x_i \right\|^r + 2^r \left\| \sum_{i \in F} x_i \right\|^r \leq (C^r + 2^{r(1-1/p)} d_n^r) \left( \sum_{i=1}^n \|x_i\|^p \right)^{r/p}.$$

Thus

$$d_n^r \leq C^r + 2^{r(1-1/p)} d_n^r$$

so that

$$d_n \leq \frac{C}{[1 - (\frac{1}{2})^{r(1/p-1)}]^{1/r}}.$$

As  $\{d_n\}$  is bounded,  $X$  is  $p$ -convex.

Remark. As is easily seen the hypothesis actually used in the proof is that  $X$  satisfies

$$\min_{\sigma_i = \pm 1} \|\sigma_1 x_1 + \dots + \sigma_n x_n\| \leq C (\|x_1\|^p + \dots + \|x_n\|^p)^{1/p}.$$

The same argument shows that if  $b_n(X) = O(n^{1/p})$  ( $p < 1$ ) then  $a_n(X) = O(n^{1/p})$  where

$$b_n(X) = \sup_{\|x_i\| \leq 1} \min_{\sigma_i = \pm 1} \|\sigma_1 x_1 + \dots + \sigma_n x_n\|,$$

$$a_n(X) = \sup_{\|x_i\| \leq 1} \|x_1 + \dots + x_n\|.$$

We do not know, however, if  $a_n = O(n^{1/p})$  implies  $X$  is  $p$ -convex when  $p < 1$ .

When  $p = 1$  the above proof breaks down and we shall see later that a type 1 space need not be convex. It is tempting to conjecture that a type 1 space must at least be logconvex in view of the following theorem (the converse is clearly false—consider  $l_p$ ).

**THEOREM 4.3.** *Let  $X$  be a type 1 quasi-Banach space isomorphic to a subspace of  $L_0$ . Then  $X$  is log convex.*

**Proof.** By Nikisin's theorem ([7], [14]),  $X$  embeds in  $L(1, \infty)$ . Now apply Theorem 3.5.

We have not been able to substantiate this conjecture and have only the following, whose proof we omit. It depends on rather more delicate handling of the argument in Theorem 4.2.

**THEOREM 4.4.** *Suppose  $X$  is a type 1 quasi-Banach space; then for some  $C < \infty$  and any  $x_1, \dots, x_n \in X$*

$$\|x_1 + \dots + x_n\| \leq C(1 + \log n)(\|x_1\| + \dots + \|x_n\|).$$

In view of the above results we shall only consider type when  $1 \leq p \leq 2$ .

**5. Twisted sums.** Suppose  $X$  and  $Y$  are quasi-Banach spaces and  $Z$  is a twisted sum of  $Y$  and  $X$ , so that  $Z$  has a subspace isomorphic to  $Y$  such that  $Z/Y \cong X$ . Then (cf. [6], [8]) there is a map  $F: X \rightarrow Y$  satisfying

$$(5.0.1) \quad F(tx) = tF(x), \quad t \in \mathbf{R}, x \in X,$$

$$(5.0.2) \quad \|F(x_1 + x_2) - F(x_1) - F(x_2)\| \leq K(\|x_1\| + \|x_2\|), \quad x_1, x_2 \in X,$$

where  $K$  is independent of  $x_1, x_2$ , such that  $Z$  is isomorphic to the space  $Y \oplus_F X$ , i.e., the Cartesian sum  $Y \oplus X$  quasinormed by

$$\|(y, x)\| = \|y - F(x)\| + \|x\|.$$

Conversely, given any such *quasilinear map*  $F$  satisfying (5.0.1) and (5.0.2), then  $Y \oplus_F X$  is a twisted sum of  $Y$  and  $X$ .

Suppose then  $F: X \rightarrow Y$  is a fixed quasilinear map. We define for a finite subset  $\{x_1, \dots, x_n\}$  of  $X$

$$\Delta(x_1, \dots, x_n) = F(x_1 + \dots + x_n) - \sum_{i=1}^n F(x_i).$$

We now state the properties of  $\Delta$ .

**LEMMA 5.1.** (1) *If  $A_1, A_2, \dots, A_m$  are disjoint subsets of  $\{1, 2, \dots, n\}$  such that  $A_1 \cup \dots \cup A_m = \{1, 2, \dots, n\}$  and*

$$u_i = \sum_{j \in A_i} x_j,$$

*then*

$$\Delta(x_1, \dots, x_n) = \Delta(u_1, u_2, \dots, u_m) + \sum_{i=1}^m \Delta(x_j: j \in A_i).$$

$$(2) \Delta(x) = 0.$$

$$(3) \|\Delta(x_1, x_2)\| \leq K(\|x_1\| + \|x_2\|).$$

$$(4) \text{ For some } s, 0 < s \leq 1 \text{ and } M < \infty$$

$$\|\Delta(x_1, \dots, x_n)\| \leq M(\|x_1\|^s + \dots + \|x_n\|^s)^{1/s}$$

for  $x_1, \dots, x_n \in X$ .

**Proof.** (1)–(3) are obvious and (4) is shown in [6].

Now suppose that  $W$  is any quasi-Banach space. We define  $d_n = d_n(W)$  to be a least constant such that

$$\|w_1 + \dots + w_n\| \leq d_n(\|w_1\| + \dots + \|w_n\|)$$

whenever  $w_1, \dots, w_n \in W$ . We also define  $\delta_n = \delta_n(W)$  to be the least constant such that

$$\left\{ \int_0^1 \left\| \sum_{i=1}^n \varepsilon_i w_i \right\|^2 dt \right\}^{1/2} \leq \delta_n \left( \sum_{i=1}^n \|w_i\|^2 \right)^{1/2}.$$

The sequence  $\{\delta_n\}$  has been studied for Banach space in [1], [2] and [13]. It is easy enough to see that both sequences  $\{d_n\}$  and  $\{\delta_n\}$  are submultiplicative ( $d_{mn} \leq d_m d_n$ ,  $\delta_{mn} \leq \delta_m \delta_n$ ).

For a quasilinear map  $F: X \rightarrow Y$  we define  $c_n = c_n(F)$  to be the least constant such that

$$\|A(x_1, \dots, x_n)\| \leq c_n (\|x_1\| + \dots + \|x_n\|), \quad x_1, \dots, x_n \in X,$$

and  $\gamma_n = \gamma_n(F)$  to be the least constant such that

$$\left\{ \int_0^1 \|A(\varepsilon_1 x_1, \varepsilon_2 x_2, \dots, \varepsilon_n x_n)\|^2 dt \right\}^{1/2} \leq \gamma_n (\|x_1\|^2 + \dots + \|x_n\|^2)^{1/2}, \quad x_1, \dots, x_n \in X.$$

Our first result is simply a generalization of a result of Enflo, Lindenstrauss and Pisier.

**THEOREM 5.2.** Suppose  $0 < r \leq 1$  and  $Y$  is an  $r$ -Banach space. For  $m, n \in \mathbb{N}$

$$(5.2.1) \quad c_{mn}^r \leq c_m^r d_n^r(X) + c_n^r d_m^r(Y),$$

$$(5.2.2) \quad \gamma_{mn}^r \leq \gamma_m^r \delta_n^r(X) + \gamma_n^r \delta_m^r(Y).$$

**Proof.** (5.2.1). Suppose  $x_1, x_2, \dots, x_{mn} \in X$ . Let

$$u_i = \sum_{(i-1)n+1}^{in} x_j, \quad i = 1, 2, \dots, m.$$

Then  $\|A(u_1, \dots, u_m)\| \leq c_m \sum_{i=1}^m \|u_i\| \leq c_m d_n(X) \sum_{i=1}^{mn} \|x_i\|$  and

$$\begin{aligned} \left\| \sum_{i=1}^m A(x_j: (i-1)n < j \leq in) \right\| &\leq d_m(Y) \sum_{i=1}^m \|A(x_j: (i-1)n < j \leq in)\| \\ &\leq d_m(Y) c_n \sum_{i=1}^{mn} \|x_i\| \end{aligned}$$

and (5.2.1) follows from Lemma 5.1.

(5.2.2). Let

$$u_i(t) = \sum_{(i-1)n+1}^{in} \varepsilon_j(t) x_j.$$

Then

$$\int_0^1 \|A(u_1(t), u_2(t), \dots, u_m(t))\|^2 dt = \int_0^1 \int_0^1 \|A(\varepsilon_1(s) u_1(t), \dots, \varepsilon_m(s) u_m(t))\|^2 ds dt$$

(by symmetry)

$$\leq \int_0^1 \gamma_m^2 \sum_{i=1}^m \|u_i(t)\|^2 dt \leq \gamma_m^2 \delta_n^2(X) \sum_{i=1}^{mn} \|x_i\|^2.$$

Also, by a similar argument,

$$\int_0^1 \left\| \sum_{i=1}^m A(x_j: (i-1)n < j \leq in) \right\|^2 dt \leq \delta_m^2(Y) \gamma_n^2 \sum_{i=1}^{mn} \|x_i\|^2$$

and (5.2.2) now follows from Lemma 5.1 and the convexity of the  $L_{2/r}$ -norm.

**Lemma 5.3.** If  $p > 0$ , there exists  $a = a(p) \geq 0$  and  $C = C(p)$  such that for  $x_1, \dots, x_n \in X$

$$(5.3.1) \quad \|A(x_1, \dots, x_n)\| \leq C \left( \sum_{k=1}^n k^a \|x_k\|^p \right)^{1/p}$$

**Proof.** By Lemma 5.1 we can find  $s > 0$  and  $M < \infty$  such that

$$\|A(x_1, \dots, x_n)\| \leq M \left( \sum_{k=1}^n \|x_k\|^s \right)^{1/s}.$$

Thus for  $0 < p \leq s$ ,  $a = 0$  will suffice. Now suppose  $s < p < \infty$ , and choose  $\theta > 1$ . Let  $c = \left( \sum_{k=1}^{\infty} k^{-\theta} \right)^{-1}$ . Then  $\sum_{k=1}^{\infty} ck^{-\theta} = 1$  and hence

$$\left( \sum_{k=1}^n \|x_k\|^s \right)^{1/s} = \left( \sum_{k=1}^n ck^{-\theta} \|x_k\|^s c^{-1} k^{\theta} \right)^{1/s} \leq \left( \sum_{k=1}^n ck^{-\theta} \|x_k\|^p c^{-p/s} k^{\theta p/s} \right)^{1/p}$$

and hence  $a = \theta(p/s - 1)$  will suffice.

## 6. Twisted sums with unequal convexity.

**LEMMA 6.1.** Suppose  $\mu > \nu \geq 0$ , and  $Y$  is an  $r$ -Banach space.

(6.1.1) If  $d_n(Y) = O(n^\mu)$  and  $d_n(X) = O(n^\nu)$ , then  $c_n = O(n^\mu)$ .

(6.1.2) If  $\delta_n(Y) = O(n^\mu)$  and  $\delta_n(X) = O(n^\nu)$ , then  $\gamma_n = O(n^\mu)$ .

**Remark.** The roles of  $X$  and  $Y$  may be interchanged in this lemma.

**Proof.** Suppose  $d_n(Y) \leq an^\mu$  and  $d_n(X) \leq bn^\nu$ . Select  $N$  so that  $bN^{\nu-\mu} < 1$ . Let  $\theta_k = (c_{N^k} N^{-k\mu})^r$ . Then

$$c_{N^k}^r \leq c_{N^{k-1}}^r d_N^r(X) + c_N^r d_{N^{k-1}}^r(Y)$$



so that

$$\theta_k \leq (bN^{p-\mu})^r \theta_{k-1} + a\theta_1$$

and hence  $\{\theta_k\}$  is bounded. Hence  $c_n = O(n^\mu)$ .

(6.1.2) has a similar proof.

**THEOREM 6.2.** Suppose that  $0 < p, q \leq 1$  and  $p \neq q$ , and that  $X$  is a  $p$ -convex quasi-Banach space and  $Y$  is a  $q$ -convex quasi-Banach space. Then any twisted sum  $Y \oplus_F X$  of  $Y$  and  $X$  is  $\min(p, q)$ -convex.

**Proof.** The case  $q > p$  is proved in [6]. We therefore assume that  $q < p$  and that  $Y$  is a  $q$ -Banach space and  $X$  is a  $p$ -Banach space. Then

$$d_n(Y) \leq n^{1/q-1}, \quad d_n(X) \leq n^{1/p-1},$$

and hence for any quasilinear map  $F: X \rightarrow Y$

$$c_n(F) \leq Cn^{1/q-1}$$

for some  $C$ .

Now suppose  $x_1, \dots, x_n \in X$  is non-zero and

$$\|x_1\|^q + \dots + \|x_n\|^q = 1.$$

Let  $A_m = \{i: 2^{-m} < \|x_i\| \leq 2 \cdot 2^{-m}\}$ ,  $m = 1, 2, 3, \dots$ . Then for some  $N$ ,  $A_1, \dots, A_N$  partitions  $\{1, 2, \dots, n\}$ . Let

$$u_m = \sum_{i \in A_m} x_i.$$

Then, if we make the convention  $\Delta(\emptyset) = 0$  and  $\sum_{\emptyset} x_i = 0$ ,

$$\|\Delta(x_1, \dots, x_n)\|^q \leq \|\Delta(u_1, \dots, u_N)\|^q + \sum_{i=1}^N \|\Delta(x_j: j \in A_i)\|^q.$$

Now

$$\|\Delta(x_j: j \in A_i)\| \leq C|A_i|^{1/q-1} \sum_{j \in A_i} \|x_j\| \leq 2C|A_i|^{1/q} 2^{-i},$$

so that

$$\|\Delta(x_j: j \in A_i)\|^q \leq (2C)^q |A_i| 2^{-iq} \leq (2C)^q \sum_{j \in A_i} \|x_j\|^q.$$

Hence

$$\sum_{i=1}^N \|\Delta(x_j: j \in A_i)\|^q \leq (2C)^q \sum_{i=1}^N |A_i| 2^{-iq} = (2C)^q.$$

Now by Lemma 5.3 there is a constant  $M$  and  $\alpha \geq 0$  so that

$$\|\Delta(w_1, \dots, w_l)\|^p \leq M \sum_{k=1}^l \|w_k\|^p, \quad w_1, \dots, w_l \in X.$$

Hence

$$\begin{aligned} \|\Delta(u_1, \dots, u_N)\|^p &\leq M \sum_{k=1}^N \|u_k\|^p \leq M \sum_{k=1}^N \|u_k\|^q \sum_{i \in A_k} \|x_i\|^q \\ &\leq M^* \sum_{i=1}^n \|x_i\|^q \left( \log \frac{2}{\|x_i\|} \right)^q \end{aligned}$$

where  $M^* = M/\log 2$ .

Now  $\sup_{0 < \xi \leq 1} \xi^{p-q} (\log(2/\xi))^q = 0 < \infty$  and hence if  $M^{**} = \theta M^*$

$$\|\Delta(u_1, \dots, u_N)\|^p \leq M^{**} \sum_{i=1}^n \|x_i\|^q = M^{**}.$$

Hence

$$\|\Delta(x_1, \dots, x_n)\|^q \leq (M^{**})^q + (2C)^q$$

where both  $M^{**}$  and  $C$  are independent of  $x_1, \dots, x_n$ . We conclude that for any  $y_1, \dots, y_n$

$$\|\Delta(x_1, \dots, x_n)\|^q \leq D \sum_{i=1}^n \|x_i\|^q.$$

Now suppose  $(y_i, x_i) \in Y \oplus_F X$ . Then

$$\begin{aligned} \left\| \left( \sum y_i, \sum x_i \right) \right\|^q &= \left( \left\| \sum y_i - F\left(\sum x_i\right) \right\|^q + \left\| \sum x_i \right\|^q \right) \\ &\leq \left( \left\| \sum y_i - F\left(\sum x_i\right) \right\|^q + \left\| \sum x_i \right\|^q \right) \\ &\leq \left( \left\| \sum (y_i - F(x_i)) \right\|^q + \left\| \Delta(x_1, \dots, x_n) \right\|^q + \left\| \sum x_i \right\|^q \right) \\ &\leq 2D \left( \sum \|x_i\|^q + \sum \|y_i - F(x_i)\|^q \right) \\ &\leq 2^{1/q-1} D \sum \|(x_i, y_i)\|^q \end{aligned}$$

and so  $Y \oplus_F X$  is  $q$ -convex.

**THEOREM 6.4.** Suppose that  $X$  is a quasi-Banach space of type  $p$  ( $1 \leq p \leq 2$ ) and  $Y$  is a quasi-Banach space of type  $q$  ( $1 \leq q \leq 2$ ). Then if  $q < p$ , any twisted sum  $Y \oplus_F X$  is of type  $q$ .

**Proof.** This proof mimics Theorem 6.2. We assume that  $Y$  is an  $r$ -Banach space where  $r \leq 1$ ; of course, if  $q > 1$ , we may take  $r = 1$ . We suppose that if  $x_1, \dots, x_n \in X$  and  $y_1, \dots, y_n \in Y$ , then

$$\begin{aligned} \left\{ \int_0^1 \left\| \sum_{i=1}^n e_i(t) x_i \right\|^p dt \right\}^{1/p} &\leq c \left( \sum \|x_i\|^p \right)^{1/p}, \\ \left\{ \int_0^1 \left\| \sum_{i=1}^n e_i(t) y_i \right\|^q dt \right\}^{1/q} &\leq c \left( \sum \|y_i\|^q \right)^{1/q}, \end{aligned}$$

and that

$$\|\Delta(x_1, \dots, x_n)\| \leq M \left( \sum_{k=1}^n k^a \|x_k\|^p \right)^{1/p}$$

as in Lemma 5.3.

We have  $\delta_n(Y) = O(n^{1/q-1/2})$  and  $\delta_n(X) = O(n^{1/p-1/2})$  and hence

$$\gamma_n(I^p) \leq C n^{1/q-1/2}, \quad n \in \mathbb{N}.$$

Suppose  $x_1, \dots, x_n \in X$  are non-zero and

$$\|x_1\|^q + \dots + \|x_n\|^q = 1.$$

Let  $A_m = \{i: 2^{-m} < \|x_i\| \leq 2 \cdot 2^{-m}\}$  and suppose  $A_1, \dots, A_N$  positions  $\{1, 2, \dots, N\}$ . As before we make the convention  $\Delta(\emptyset) = 0$  and  $\sum_{i \in \emptyset} x_i = 0$ . Then if

$$u_i(t) = \sum_{j \in A_i} \varepsilon_j(t) x_j, \quad 0 \leq t \leq 1,$$

$$\Delta(\varepsilon_1(t)x_1, \dots, \varepsilon_n(t)x_n) = \Delta(u_1(t), \dots, u_N(t)) + \sum_{i=1}^N \Delta(\varepsilon_i(t)x_j: j \in A_i).$$

Now by symmetry

$$\begin{aligned} \int_0^1 \left\| \sum_{i=1}^N \Delta(\varepsilon_j(t)x_j: j \in A_i) \right\|^q dt &= \int_0^1 \int_0^1 \left\| \sum_{i=1}^N \varepsilon_j(s) \Delta(\varepsilon_j(t)x_j: j \in A_i) \right\|^q ds dt \\ &\leq c^q \int_0^1 \sum_{i=1}^N \left\| \Delta(\varepsilon_j(t)x_j: j \in A_i) \right\|^q dt \\ &\leq c^q \sum_{i=1}^N \left\{ \int_0^1 \left\| \Delta(\varepsilon_j(t)x_j: j \in A_i) \right\|^2 dt \right\}^{q/2} \\ &\leq c^q \sum_{i=1}^N C^q |A_i|^{1-q/2} \left( \sum_{j \in A_i} \|x_j\|^q \right)^{q/2} \\ &\leq 2^{q/2} c^q C^q \sum_{i=1}^N |A_i| 2^{-iq} \leq 2^{q/2} c^q C^q \sum_{i=1}^n \|x_i\|^q \leq 2^{q/2} c^q C^q. \end{aligned}$$

Also

$$\begin{aligned} \int_0^1 \|\Delta(u_1(t), \dots, u_N(t))\|^p dt &\leq M^p \cdot \int_0^1 \sum_{k=1}^N k^a \|u_k(t)\|^p dt \leq M^p c^p \sum_{k=1}^N k^a \sum_{j \in A_k} \|x_j\|^p \\ &\leq M^* \sum_{i=1}^n \|x_i\|^p \left( \log \frac{2}{\|x_i\|} \right)^a \leq M^{**} \end{aligned}$$

as in Theorem 6.2, where  $M^{**}$  is independent of  $x_1, \dots, x_n$ . Hence

$$\begin{aligned} \int_0^1 \| \Delta(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \|^q dt &\leq \left\{ \int_0^1 \| \Delta(u_1(t), \dots, u_N(t)) \|^q dt \right\}^{r/q} + \\ &\quad + \left\{ \int_0^1 \left\| \sum_{i=1}^N \Delta(\varepsilon_j x_j: j \in A_i) \right\|^q dt \right\}^{r/q} \\ &\leq (M^{**})^r + 2^{r/2} c^r C^r. \end{aligned}$$

It follows easily that for any  $x_1, \dots, x_n$

$$\left\{ \int_0^1 \| \Delta(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \|^q dt \right\}^{1/q} \leq D \left\{ \sum \|x_i\|^q \right\}^{1/q}.$$

Now suppose  $(y_i, x_i) \in Y \oplus_p X$  ( $1 \leq i \leq n$ ). Then

$$\begin{aligned} \int_0^1 \left\| \sum_{i=1}^n \varepsilon_i(t) (y_i, x_i) \right\|^q dt &= \int_0^1 \left( \left\| \sum \varepsilon_i(t) y_i - F \left( \sum \varepsilon_i(t) x_i \right) \right\|^q + \left\| \sum \varepsilon_i(t) x_i \right\|^q \right) dt \\ &\leq \int_0^1 \left\| \sum \varepsilon_i(t) y_i - F \left( \sum \varepsilon_i(t) x_i \right) \right\|^q dt + c \left( \sum \|x_i\|^q \right)^{1/q}, \end{aligned}$$

$$\left\| \sum \varepsilon_i(t) y_i - F \left( \sum \varepsilon_i(t) x_i \right) \right\| \leq 2^{1/r-1} \left( \left\| \sum \varepsilon_i(t) (y_i - F(x_i)) \right\| + \|\Delta(x_1, \dots, x_n)\| \right)$$

so

$$\begin{aligned} \int_0^1 \left\| \sum \varepsilon_i(t) (y_i, x_i) \right\|^q dt &\leq 2^{1/r-1} c \left( \sum \|y_i - F(x_i)\|^q \right)^{1/q} + (2^{1/r-1} D + c) \left( \sum \|x_i\|^q \right)^{1/q} \\ &\leq (2^{1/r-1} (c + D) + c) \left( \sum (\|x_i\| + \|y_i - F(x_i)\|)^q \right)^{1/q} \\ &\leq K \left( \sum \|(y_i, x_i)\|^q \right)^{1/q}, \end{aligned}$$

i.e.,  $Y \oplus_p X$  is type  $q$ .

We now show that a type 1 space need not be convex.

In [6], [15] and [16] it is shown that one can construct a non-convex twisted sum of  $\mathcal{R}$  and a Banach space. This is type 1 by the next theorem.

**THEOREM 6.5.** *Suppose  $X$  is a type  $p$  quasi-Banach space and  $Y$  is a type  $q$  Banach space where  $q > p$ . Then any twisted sum  $Y \oplus_p X$  is type  $p$ .*

**Proof.** Here our techniques are rather different. We use a result of Kahane [4] that there is a constant  $K = K(p, q)$  such that for any

elements  $u_1, \dots, u_n$  of  $X$

$$\left\{ \int_0^1 \left\| \sum \varepsilon_i u_i \right\|^q dt \right\}^{1/q} \leq K \left\{ \int_0^1 \left\| \sum \varepsilon_i u_i \right\|^p dt \right\}^{1/p}.$$

[For the case  $p = 1$ , we apply Theorem 2.1.]

Suppose  $c$  has the same meaning as in Theorem 6.4. Let  $\theta_n$  be the least constant such that

$$\left\{ \int \Delta(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \right\}^{1/q} \leq \theta_n \left\{ \sum_{i=1}^n \|x_i\|^p \right\}^{1/p}.$$

Suppose  $\|x_1\|^p + \dots + \|x_n\|^p = 1$  and  $\|x_n\|^p \geq 1/N$ . Then

$$\Delta(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) = \Delta(u(t), \varepsilon_n(t) x_n) + \Delta(\varepsilon_1 x_1, \dots, \varepsilon_{n-1} x_{n-1})$$

where  $u(t) = \sum_{i=1}^{n-1} \varepsilon_i(t) x_i$ . Now

$$\left\{ \int_0^1 \Delta(u(t), \varepsilon_n(t) x_n) \right\}^{p/q} \leq \theta_2^p \left\{ \int_0^1 (\|u(t)\|^p + \|x_n\|^p)^{q/p} dt \right\}^{p/q}$$

by the usual symmetrization argument.

Hence

$$\begin{aligned} \left\{ \int_0^1 \Delta(u(t), \varepsilon_n(t) x_n) \right\}^{p/q} &\leq \theta_2^p \left[ \left\{ \int_0^1 \|u(t)\|^q dt \right\}^{p/q} + \|x_n\|^p \right] \\ &\leq \theta_2^p \left[ K^p c^p \sum_{i=1}^{n-1} \|x_i\|^p + \|x_n\|^p \right] \leq K^p \theta_2^p c^p. \end{aligned}$$

Hence if  $\|x_1\|^p + \dots + \|x_n\|^p = 1$  and  $\max \|x_i\|^p \geq N^{-1}$ ,

$$\left\{ \int_0^1 \Delta(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \right\}^{1/q} \leq \theta_{n-1} (1 - 1/N)^{1/p} + K \theta_2 c.$$

Now suppose  $\|x_1\|^p + \dots + \|x_n\|^p = 1$  and  $\max \|x_i\|^p < N^{-1}$ . Then it is possible to subdivide  $\{1, 2, \dots, n\}$  into  $N$  sets  $A_1, \dots, A_N$  such that

$$\sum_{i \in A_j} \|x_i\|^p \leq 2/N, \quad j = 1, 2, \dots, N.$$

Then let

$$u_j(t) = \sum_{i \in A_j} \varepsilon_i(t) x_i,$$

$$\Delta(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) = \Delta(u_1(t), \dots, u_N(t)) + \sum_{i=1}^N \Delta(\varepsilon_i x_i: i \in A_j),$$

and by symmetrization

$$\begin{aligned} \left\{ \int_0^1 \Delta(u_1(t), \dots, u_N(t)) \right\}^{p/q} &\leq \theta_N^p \left\{ \int_0^1 \left( \sum_{i=1}^N \|u_i(t)\|^p \right)^{q/p} dt \right\}^{p/q} \\ &\leq \theta_N^p \sum_{i=1}^N \left\{ \int_0^1 \|u_i(t)\|^q dt \right\}^{p/q} \\ &\leq \theta_N^p \sum_{i=1}^N K^p c^p \sum_{i \in A_j} \|x_i\|^p \leq \theta_N^p K^p c^p, \end{aligned}$$

$$\left\{ \int_0^1 \left\| \sum_{i=1}^N \Delta(\varepsilon_i x_i: i \in A_j) \right\|^q dt \right\}^{1/q} \leq c \left\{ \sum_{i=1}^N \int_0^1 \Delta(\varepsilon_i x_i: i \in A_j) \right\}^{1/q}$$

(by symmetrization)

$$\begin{aligned} &\leq c \theta_n \left\{ \sum_{i=1}^N \left( \sum_{i \in A_j} \|x_i\|^q \right)^{p/q} \right\}^{1/q} \\ &\leq c \theta_n \left\{ \sum_{i=1}^N \left( \frac{2}{N} \right)^{q/p-1} \sum_{i \in A_j} \|x_i\|^p \right\}^{1/q} \\ &\leq \left( \frac{2}{N} \right)^{1/p-1/q} c \theta_n. \end{aligned}$$

Hence

$$\left\{ \int_0^1 \Delta(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \right\}^{1/q} \leq c K \theta_N + \left( \frac{2}{N} \right)^{1/p-1/q} c \theta_n.$$

Thus

$$\theta_n \leq \max \left\{ \theta_n \left( 1 - \frac{1}{N} \right)^{1/p} + K \theta_2 c; c K \theta_N + \left( \frac{2}{N} \right)^{1/p-1/q} c \theta_n \right\}.$$

If we choose  $N$  so that

$$c \left( \frac{2}{N} \right)^{1/p-1/q} < 1,$$

this implies a bound on  $\theta_n$ .

The fact that  $\theta_n$  is bounded implies that  $Y \oplus_p X$  is type  $p$  in the usual way, as in Theorem 6.4.

**7. Twisted sums with equal convexity.** Since the twisted sum of two Banach spaces may not be convex we may ask what class it does belong to. It turns out that we can give a complete answer to this. We require first the following lemma. We use the notation of Section 5.

LEMMA 7.1. Suppose  $\mu \geq 0$  and  $Y$  is a Banach space.

(7.1.1) If  $\bar{d}_n(X) = 1$  and  $\bar{d}_n(Y) = 1$ , then  $c_n = O(\log n)$ .

(7.1.2) If  $\delta_n(X) \leq n^\mu$  and  $\delta_n(Y) = O(n^\mu)$  (or  $\delta_n(X) = O(n^\mu)$  and  $\delta_n(Y) \leq n^\mu$ ), then  $\gamma_n = O(n^\mu \log n)$ .

Proof. We prove only (7.1.2) (as the same argument then proves (7.1.1)). If  $\delta_n(X) \leq n^\mu$  and  $\delta_n(Y) \leq Cn^\mu$ , then by Theorem 5.2

$$\frac{\gamma_{mn}}{(mn)^\mu} \leq \frac{\gamma_m}{m^\mu} + C \frac{\gamma_n}{n^\mu}$$

and hence  $\gamma_n n^{-k\mu} \leq Ck\gamma_n n^{-\mu}$  and the result follows easily.

THEOREM 7.1. Suppose  $X$  and  $Y$  are Banach spaces. Then any twisted sum  $Y \oplus_F X$  is logconvex.

Proof. Here we have  $\bar{d}_n(X) = \bar{d}_n(Y) = 1$  for all  $n$  and hence  $c_n \leq C(\log n + 1)$ . Induction on  $n$  as in Lemma 3.2 of [6] shows that

$$\|\Delta(x_1, \dots, x_n)\| \leq M \sum_{k=1}^n k \|x_k\|, \quad x_1, \dots, x_n \in X,$$

in this case, for some  $M$  independent of  $x_1, \dots, x_n$ .

Suppose  $\|x_1\| + \|x_2\| + \dots + \|x_n\| = 1$  and suppose  $\|x_1\| \geq \|x_2\| \geq \dots \geq \|x_n\| > 0$ . Let  $N_k$  be the greatest suffix such that  $\|x_{N_k}\| > 2^{-k}$  ( $k = 1, 2, \dots$ ) and let

$$u_k = \sum_{i=N_{k-1}+1}^{N_k} x_i, \quad k = 1, 2, \dots$$

(where  $N_0 = 0$ ). Suppose  $N_l = n$ . Then

$$\Delta(x_1, \dots, x_n) = \Delta(u_1, \dots, u_l) + \sum_{k=1}^l \Delta(x_{N_{k-1}+1}, \dots, x_{N_k}),$$

$$\begin{aligned} \left\| \sum_{k=1}^l \Delta(x_{N_{k-1}+1}, \dots, x_{N_k}) \right\| &\leq C \sum_{k=1}^l (1 + \log(N_k - N_{k-1})) \sum_{N_{k-1}+1}^{N_k} \|x_i\| \\ &\leq C + C \sum_{k=1}^l \log N_k \sum_{N_{k-1}+1}^{N_k} \|x_i\|. \end{aligned}$$

Clearly,  $N_k 2^{-k} \leq 1$  so that  $N_k \leq 2/\|x_i\|$  for  $N_{k-1}+1 < i \leq N_k$ . Hence

$$\left\| \sum_{k=1}^l \Delta(x_{N_{k-1}+1}, \dots, x_{N_k}) \right\| \leq C + C \log 2 + C \sum_{i=1}^n \|x_i\| \log \frac{1}{\|x_i\|}.$$

Also

$$\begin{aligned} \|\Delta(u_1, \dots, u_l)\| &\leq M \sum_{k=1}^l k \|u_k\| \leq \frac{M}{\log 2} \sum_{i=1}^n \|x_i\| \log \frac{2}{\|x_i\|} \\ &\leq M + \frac{M}{\log 2} \sum_{i=1}^n \|x_i\| \log \frac{1}{\|x_i\|}. \end{aligned}$$

Thus

$$\|\Delta(x_1, \dots, x_n)\| \leq M + C + C \log 2 + \left( C + \frac{M}{\log 2} \right) \sum_{i=1}^n \|x_i\| \log \frac{1}{\|x_i\|}$$

whenever  $\sum \|x_i\| = 1$ . Hence for general  $x_1, \dots, x_n$

$$\|\Delta(x_1, \dots, x_n)\| \leq B_1 \sum_{i=1}^n \|x_i\| + B_2 \sum_{i=1}^n \|x_i\| \log \frac{S}{\|x_i\|}$$

where  $S = \sum_{i=1}^n \|x_i\|$ .

Now it easily follows that  $Y \oplus_F X$  is logconvex for

$$\begin{aligned} \left\| \sum_{i=1}^n (y_i, x_i) \right\| &= \left\| \sum y_i - F \left( \sum x_i \right) \right\| + \left\| \sum x_i \right\| \\ &= \left\| \sum (y_i - F(x_i)) \right\| + \|\Delta(x_1, \dots, x_n)\| + \left\| \sum x_i \right\| \\ &\leq \sum_{i=1}^n \|(y_i, x_i)\| + B_1 \sum \|x_i\| + B_2 \sum \|x_i\| \log \frac{S}{\|x_i\|}, \end{aligned}$$

and the result follows from the fact that the function

$$\Phi(\xi_1, \dots, \xi_n) = \sum_{i=1}^n \xi_i + \sum_{i=1}^n \xi_i \log \frac{\sum \xi_j}{\xi_i}, \quad \xi_1, \dots, \xi_n \geq 0$$

is monotone in each  $\xi_i$ , and  $\|x_i\| \leq \|(y_i, x_i)\|$ .

Remark. If we take  $X = Y = l_1$  and  $F: l_1 \rightarrow l_1$  is given on the finitely non-zero sequences by

$$F(x) = \left( x_n \log \frac{\|x\|}{\|x_n\|} \right),$$

then  $l_1 \oplus_F l_1$  contains  $l_\varphi$  (where as usual  $\varphi(x) = x(1 + \log(1/x))$  near zero). (See [8].) This shows that the result of Theorem 7.1 is best possible.

THEOREM 7.2. A quasi-Banach space is logconvex if and only if it is the quotient of a subspace of a twisted sum of two Banach spaces.

**Proof.** By Theorem 7.1 such a space must be logconvex. The above example generalized to  $L_1(I)$  for arbitrary index sets  $I$  enables one to obtain  $L_p(I)$  as a subspace of a twisted sum of Banach spaces and hence every logconvex space as a quotient (cf. [19] or the method of Theorem 3.3).

**DEFINITION 7.3.** We say a quasi-Banach space  $X$  is of *logtype*  $p$  ( $1 \leq p \leq 2$ ) if for some constant  $C < \infty$  we have

$$(7.3.1) \quad \left\{ \int_0^1 \left\| \sum \varepsilon_i(t) x_i \right\|^p dt \right\}^{1/p} \leq C \left( 1 + \sum \|x_i\|^p \left( \log \frac{1}{\|x_i\|} \right)^p \right)^{1/p}$$

whenever  $\|x_1\|^p + \dots + \|x_n\|^p = 1$ .

**Remark.** In order that  $X$  is of logtype  $p$  it is sufficient that

$$(7.3.2) \quad \left\{ \int_0^1 \left\| \sum \varepsilon_i(t) x_i \right\|^p dt \right\}^{1/p} \leq C' \left\{ \sum \|x_k\|^p (1 + \log k)^p \right\}^{1/p}.$$

To see this arrange  $x_k$  so that  $\|x_k\|$  decreases. Then if  $\sum \|x_k\|^p = 1$ ,  $\|x_k\|^p \leq k^{-1}$  and hence

$$\log \frac{1}{\|x_k\|} \geq \frac{1}{p} \log k.$$

We will see later that (7.3.1) and (7.3.2) are equivalent; of course, for  $p = 1$  this is immediate from Section 3, and for  $1 < p \leq 2$  could be established directly in a similar manner. However our indirect methods also establish this result without difficulty.

**DEFINITION 7.4.** A Banach space  $X$  is of *exact type*  $p$  if

$$\|x_1 - x_2\|^p + \|x_1 + x_2\|^p \leq 2(\|x_1\|^p + \|x_2\|^p), \quad x_1, x_2 \in X.$$

**Remarks.** If  $p = 1$ , this is automatic. If  $p = 2$ , it implies that  $X$  is a Hilbert space, for in this case

$$\|2x_1\|^2 + \|2x_2\|^2 \leq 2(\|x_1 - x_2\|^2 + \|x_1 + x_2\|^2)$$

and hence the parallelogram law holds; then apply a result of Jordan and von Neumann [3]. For  $1 < p < 2$  it is sufficient that  $X$  is a quotient of a subspace of an  $L_p$ -space.

**THEOREM 7.5.** Suppose  $X$  and  $Y$  are Banach spaces of type  $p$  where  $1 \leq p \leq 2$ . Suppose that either  $X$  or  $Y$  is of exact type  $p$ . Then any twisted sum  $Z = Y \oplus_p X$  satisfies

$$\left\{ \int_0^1 \left\| \sum_{i=1}^n \varepsilon_i(t) z_i \right\|^p dt \right\}^{1/p} \leq C \left\{ \sum_{k=1}^n \|z_k\|^p + \sum \|z_k\|^p (\log k)^p \right\}^{1/p}$$

and hence is of logtype  $p$ .

**Proof.** If  $X$  is of exact type  $p$ , then

$$\left\{ \int \left\| \sum \varepsilon_i(t) x_i \right\|^p dt \right\}^{1/p} \leq \left( \sum \|x_i\|^p \right)^{1/p}$$

for  $x_1, \dots, x_n$  so that  $\delta_n(w) \leq n^{1/p-1/2}$ . Hence Lemma 7.1 implies that

$$\gamma_n \leq B n^{1/p-1/2} (\log n + 1).$$

By Lemma 5.3 there exists  $\alpha > 0$  and  $M < \infty$  such that

$$\|A(x_1, \dots, x_n)\| \leq M \left( \sum_{k=1}^n k^\alpha \|x_k\|^p \right)^{1/p}$$

for  $x_1, \dots, x_n \in X$ .

Now suppose  $\|x_1\|^p + \dots + \|x_n\|^p = 1$  and  $\|x_1\| \geq \dots \geq \|x_n\| > 0$ . Let  $N_k$  be the greatest suffix such that  $\|x_{N_k}\| > 2^{-k}$  and let

$$u_k(t) = \sum_{N_{k-1}+1}^{N_k} \varepsilon_i(t) x_i, \quad k = 1, 2, \dots$$

Suppose  $N_1 = n$ . Then

$$\begin{aligned} A(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) &= A(u_1(t), \dots, u_1(t)) + \\ &+ \sum_{k=1}^{N_k} A(\varepsilon_{N_{k-1}+1} x_{N_{k-1}+1}, \dots, \varepsilon_{N_k} x_{N_k}). \end{aligned}$$

Now let

$$\begin{aligned} \alpha &= \left\{ \int_0^1 \left\| \sum_{k=1}^l A(\varepsilon_{N_{k-1}+1} x_{N_{k-1}+1}, \dots, \varepsilon_{N_k} x_{N_k}) \right\|^p dt \right\}^{1/p} \\ &\leq B_2 \left\{ \int_0^1 \sum_{k=1}^l \|A(\varepsilon_{N_{k-1}+1} x_{N_{k-1}+1}, \dots, \varepsilon_{N_k} x_{N_k})\|^p dt \right\}^{1/p} \end{aligned}$$

by the symmetrization argument, where  $B_2$  is the type  $p$  constant of  $Y$ . Hence

$$\begin{aligned} \alpha &\leq B B_2 \left\{ \sum_{k=1}^l (N_k - N_{k-1})^{1-p/2} [\log(N_k - N_{k-1}) + 1]^p \left( \sum_{i=N_{k-1}+1}^{N_k} \|x_i\|^2 \right)^{p/2} \right\}^{1/p} \\ &\leq B B_2 \left\{ \sum_{k=1}^l [\log(N_k - N_{k-1}) + 1]^p \sum_{i=N_{k-1}+1}^{N_k} \|x_i\|^p \right\}^{1/p}. \end{aligned}$$

Now observe

$$\begin{aligned} \sum_{m+1}^n (1 + \log k)^p &\geq \int_m^n (1 + \log x)^p dx \\ &= (n-m)(1 + \log n)^p - p \int_m^n \frac{x-m}{x} (1 + \log x)^{p-1} dx \\ &\geq (n-m)(1 + \log n)^p - p(n-m)(1 + \log n)^{p-1} \\ &\geq \frac{1}{2}(n-m)(1 + \log n)^p \end{aligned}$$

provided  $n \geq n_0$  where  $n_0$  depends on  $p$ .

Hence there is a constant  $c$  such that  $c > 0$  and

$$\sum_{k=m+1}^n (1 + \log k)^p \geq c(n-m)(1 + \log n)^p$$

for all  $n, m$ . Thus we have

$$\begin{aligned} \sum_{N_{k-1}+1}^{N_k} (1 + \log j)^p \|x_j\|^p &\geq 2^{-kp} \sum_{N_{k-1}+1}^{N_k} (1 + \log j)^p \\ &\geq c2^{-kp}(N_k - N_{k-1})(1 + \log N_k)^p \\ &\geq c2^{-p}(1 + \log N_k)^p \sum_{N_{k-1}+1}^{N_k} \|x_j\|^p. \end{aligned}$$

Thus

$$a \leq 2c^{-1/p} B B_2 \left\{ \sum_{k=1}^n \|x_k\|^p (1 + \log k)^p \right\}^{1/p}.$$

Now we shall show that if

$$b = \left\{ \int_0^1 \left\| \Delta(u_1(t), \dots, u_l(t)) \right\|^p dt \right\}^{1/p},$$

then

$$b \leq D \left\{ \sum_{k=1}^n \|x_k\|^p (1 + \log k)^p \right\}^{1/p}$$

for some  $D$  independent of  $x_1, \dots, x_n$ . We have

$$(7.5.1) \quad b \leq M \left\{ \int_0^1 \sum_{k=1}^l k^\alpha \|u_k(t)\|^p dt \right\}^{1/p} \leq M B_3 \left\{ \sum_{k=1}^l k^\alpha \sum_{N_{k-1}+1}^{N_k} \|x_i\|^p \right\}^{1/p}$$

where  $B_3$  is the type  $p$  constant of  $X$ . Hence

$$(7.5.2) \quad b \leq M_1 \left\{ \sum_{i=1}^n \|x_i\|^p \left( \log \frac{2}{\|x_i\|} \right)^{\alpha/p} \right\}^{1/p}$$

(for some constant  $M_1$ )

$$\leq M_2 \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p}$$

where  $\frac{1}{2}p < q < p$ . Let  $\theta = q/(p-q) > 1$ . Then

$$\begin{aligned} b &\leq M_2 \left( \sum_{k=1}^n k^{q/p} \|x_k\|^{q/p} \right)^{1/p} \\ &\leq M_2 \left( \sum_{k=1}^n k \|x_k\|^p \right)^{q/p^2} \left( \sum_{k=1}^n k^{-\theta} \right)^{1/p - q/p^2} \leq M_3 \left( \sum_{k=1}^n k \|x_k\|^p \right)^{1/p}. \end{aligned}$$

Combining

$$a + b \leq M_4 \left( \sum_{k=1}^n k \|x_k\|^p \right)^{1/p}$$

so that

$$\left\{ \int \left\| \Delta(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \right\|^p dt \right\}^{1/p} \leq M_4 \left( \sum_{k=1}^n k \|x_k\|^p \right)^{1/p}$$

and this must hold for *any*  $x_1, \dots, x_n \in X$ . Returning to (7.5.1) it is clear that (after a symmetrization argument) we may now take  $\alpha = 1$ , and in (7.5.2)

$$b \leq M_5 \left\{ \sum_{i=1}^n \|x_i\|^p \log \frac{2}{\|x_i\|} \right\}^{1/p}.$$

By Lemma 3.5

$$b \leq M_6 \left\{ \sum_{k=1}^n \|x_k\|^p (1 + \log k) \right\}^{1/p}$$

and combining we now have the estimate

$$\left\{ \int_0^1 \left\| \Delta(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \right\|^p dt \right\}^{1/p} \leq C \left\{ \sum_{k=1}^n \|x_k\|^p (1 + \log k)^p \right\}^{1/p}.$$

We omit the verification this implies the desired property of  $Z = Y \oplus_p X$ .

Remark. We observe that Theorem 7.5 is best possible, in the sense that each  $p$ ,  $1 \leq p \leq 2$ , there is a twisted sum  $Z_p$  of  $l_p$  and itself and a constant  $c > 0$  such that if  $\xi_1^p + \dots + \xi_n^p = 1$  there are  $z_i \in Z_p$ ,  $i = 1, 2, \dots, n$  with  $\|z_i\| = \xi_i$  and

$$\left\{ \int_0^1 \left\| \sum_{i=1}^n \varepsilon_i z_i \right\|^p dt \right\}^{1/p} \geq c \left( 1 + \sum_{i=1}^n \|z_i\|^p \left( \log \frac{1}{\|z_i\|} \right)^p \right)^{1/p}.$$



Indeed, consider the spaces  $Z_p$  of [9]. Then  $Z_p = l_p \oplus_F l_p$  where  $F: l_p^0 \rightarrow l_p^0$  is defined by

$$F(x) = \left( x_n \log \frac{\|x\|}{|x_n|} \right).$$

If  $e_n$  is the  $n$ th basis vector of  $l_p$ , then

$$\begin{aligned} \left\| \left( 0, \sum \pm \xi_n e_n \right) \right\| &= \left\| F \left( \sum \pm \xi_n e_n \right) \right\| + \left\| \sum \pm \xi_n e_n \right\| \\ &= \left( \sum \xi_n^p \left( \log \frac{1}{\xi_n} \right)^{p/1/p} \right)^{1/p} + \left( \sum \xi_n^p \right)^{1/p} \\ &\geq \left( 1 + \sum \xi_n^p \left( \log \left( \frac{1}{\xi_n} \right) \right)^{p/1/p} \right)^{1/p}. \end{aligned}$$

Similarly this implies

$$\left( 1 + \sum \xi_n^p \left( \log \left( \frac{1}{\xi_n} \right) \right)^{p/1/p} \right)^{1/p} \leq C' \left( \sum \xi_n^p (1 + \log n)^p \right)^{1/p}$$

so that (7.3.1) and (7.3.2) are equivalent.

**8. Twisted sums of  $l_1$  and  $\mathbf{R}$ .** We now recall two ways of forming a twisted sum  $\mathbf{R} \oplus_F l_1$ . One method due to the author is by defining  $F_1: l_1^0 \rightarrow \mathbf{R}$  by

$$F_1(x) = \sum_{n=1}^{\infty} \tilde{x}_n \log n, \quad x \geq 0,$$

where  $(\tilde{x}_n)$  is the decreasing rearrangement of  $x$ , and

$$F_1(x) = F_1(x^+) - F_1(x^-)$$

where  $x = x^+ - x^-$  where  $x^+ \geq 0$ ,  $x^- \geq 0$  and  $|x^+| \wedge |x^-| = 0$ ; see [6]. The other functional due to Ribe [15] is  $F_2: l_1^0 \rightarrow \mathbf{R}$  given by

$$F_2(x) = \sum_{n=1}^{\infty} x_n \log \frac{\|x\|}{|x_n|}.$$

(Actually Ribe uses the equivalent functional

$$\tilde{F}_2(x) = \sum_{n=1}^{\infty} x_n \log \frac{1}{x_n} + \left( \sum_{n=1}^{\infty} x_n \right) \log \left( \sum_{n=1}^{\infty} x_n \right).$$

$\| \cdot \|_1$  and  $\| \cdot \|_2$  denote the induced norms on  $\mathbf{R} \oplus l_1$ . Then if  $e_n$  is the  $n$ th basis vector of  $l_1$  and  $t_n \geq 0$  ( $n = 1, 2, \dots, N$ )

$$\left\| \left( 0, \sum_{i=1}^N t_i e_i \right) \right\|_1 = \sum_{n=1}^N |t_n| + \sum_{n=1}^N \tilde{t}_n \log n$$

and

$$\left\| \left( 0, \sum_{n=1}^N t_n e_n \right) \right\|_2 = \sum_{n=1}^N (t_n) + \sum_{n=1}^{\infty} t_n \log \frac{(\sum t_k)}{t_n}.$$

Thus we see for both  $F_1$  and  $F_2$  we have examples to show

**THEOREM 8.1.** *There is a twisted sum of  $\mathbf{R}$  and  $l_1$  where  $\text{galb}$  is  $l_p$ .*

In view of this we remark that these two twisted sums are not projectively equivalent ([9]) in the sense that there is no isomorphism  $S: \mathbf{R} \oplus_{F_1} l_1 \rightarrow \mathbf{R} \oplus_{F_2} l_1$  such that the diagram

$$\begin{array}{ccc} \mathbf{R} \oplus_{F_1} l_1 & \xrightarrow{S} & \mathbf{R} \oplus_{F_2} l_1 \\ \downarrow & & \downarrow \\ l_1 & \xrightarrow{\alpha} & l_1 \end{array}$$

commutes where  $\alpha \neq 0$ . For (see [9]), projective equivalence implies the existence of  $\alpha \neq 0$ ,  $N < \infty$  and a linear map  $t: l_1^0 \rightarrow \mathbf{R}$  so that

$$|F_1(\alpha x) - F_2(x) - t(x)| \leq N \|x\|, \quad x \in l_1^0.$$

Since  $F_1(e_n) = F_2(e_n) = 0$  for all  $n$ , this would imply  $t(e_n)$  bounded so that  $t$  is continuous and

$$|F_1(\alpha x) - F_2(x)| \leq (N + \|t\|) \|x\|, \quad x \in l_1^0.$$

Now by Lemma 3.5 we see the only possible value of  $\alpha$  is  $\alpha = 1$ . Thus

$$|F_1(x) - F_2(x)| \leq (N + \|t\|) \|x\|.$$

Now let  $x_N = \sum_{n=1}^N \frac{1}{n} e_n$ .

$$F_1(x_N) = \sum_{n=1}^N \frac{\log n}{n},$$

$$F_2(x_N) = \sum_{n=1}^N \frac{1}{n} (\log n + \log S_N)$$

where  $S_N = \sum_{n=1}^N \frac{1}{n}$ . Hence

$$F_2(x_N) - F_1(x_N) = S_N \log S_N$$

while  $\|x_N\| = S_N \rightarrow \infty$  and so we have a contradiction.

To conclude this short section we consider the following:

DEFINITION 8.2. An operator  $\tilde{T}: l_1 \rightarrow l_1$  is *liftable* if for every twisted sum  $\mathbf{R} \oplus_{\mathcal{F}} l_1$  there is a map  $\tilde{T}: l_1 \rightarrow \mathbf{R} \oplus_{\mathcal{F}} l_1$  such that the diagram

$$\begin{array}{ccc} & \mathbf{R} \oplus_{\mathcal{F}} l_1 & \\ \tilde{T} \nearrow & \downarrow & \\ l_1 & \xrightarrow{T} & l_1 \end{array}$$

commutes.

THEOREM 8.3. Let  $T: l_1 \rightarrow l_1$  be given by  $Te_n = d_n e_n$  where  $\|T\| = \sup |d_n| < \infty$ . Then the following are equivalent:

- (a)  $T$  is liftable.
- (b) For some  $\tau < \infty$ ,  $\sum_{n=1}^{\infty} \exp(-\tau/|d_n|) < \infty$ .
- (c)  $[d_n] \in c_0$  and if  $\tilde{d}_n^*$  is the decreasing re-arrangement of  $(d_n)$ , then  $\{\tilde{d}_n^* \log n\}$  is bounded.
- (d)  $T(l_1) \subset l_p$ .
- (e) For any logconvex space  $X$  and quotient map  $q: X \rightarrow l_1$  there is an operator  $S: l_1 \rightarrow X$  such that  $qS = T$ .

Proof. It clearly suffices to consider the case  $d_n \geq 0$ . Note first that if  $d_n \rightarrow 0$ , then there is a projection  $P$  onto a subspace isomorphic to  $l_1$  such that  $P = ST$  for some bounded  $S$ . Then  $T$  is liftable so is  $P$  and this clearly contradicts the fact that  $(\mathbf{R}, l_1)$  does not split. Here we may assume  $d_n \rightarrow 0$  and then we may suppose  $\{d_n\}$  decreasing.

(a)  $\Rightarrow$  (c). Consider

$$\begin{array}{ccc} & \mathbf{R} \oplus_{\mathcal{F}} l_1 & \\ \tilde{T} \nearrow & \downarrow a & \\ l_1 & \xrightarrow{T} & l_1 \end{array}$$

Suppose  $\tilde{T}e_n = (e_n, d_n e_n)$ . Then

$$\|\tilde{T}e_n\|_1 = |e_n| + |d_n| \leq \|\tilde{T}\|$$

and

$$\begin{aligned} \|\tilde{T}(e_1 + \dots + e_n)\|_1 &= \sum_{k=1}^n d_k + \left| \sum_{k=1}^n e_k - \sum_{k=1}^n d_k \log k \right| \\ &\geq \sum_{k=1}^n d_k + \sum_{k=1}^n d_k \log k - \sum_{k=1}^n e_k \\ &\geq n d_n \log n - n \|\tilde{T}\|. \end{aligned}$$

Hence

$$d_n \log n \leq 2 \|\tilde{T}\|.$$

(c)  $\Rightarrow$  (b). If  $d_n \leq b(\log n)^{-1}$ ,  $n \geq 2$ , then if  $\tau > b$

$$\exp\left(-\frac{\tau}{d_n}\right) \leq n^{-(\tau/b)}.$$

(b)  $\Rightarrow$  (d). Suppose  $|t_1| + \dots + |t_n| \leq 1$ . Then

$$\begin{aligned} \|T(t_1 e_1 + \dots + t_n e_n)\|_p^* &= \sum_{i=1}^n d_i |t_i| + \sum_{i=1}^n d_i |t_i| \log \frac{\sum_{j=1}^n d_j |t_j|}{d_i |t_i|} \\ &= S + S \log S - \sum_{i=1}^n d_i |t_i| \log d_i - \\ &\quad - \sum_{i=1}^n d_i |t_i| \log |t_i| \end{aligned}$$

where  $S = \sum_{i=1}^n d_i |t_i| \leq \|T\|$ . Also  $-d_i \log d_i \leq e^{-1}$ . Hence

$$\|T(t_1 e_1 + \dots + t_n e_n)\|_p^* \leq \|T\| + \|T\| \log \|T\| + e^{-1} + \sum_{i=1}^n d_i |t_i| \log \frac{1}{|t_i|}.$$

Now suppose  $\xi_1, \dots, \xi_n \geq 0$  are chosen to maximize  $\psi(\xi_1, \dots, \xi_n) = \sum_{i=1}^n d_i \xi_i \log \frac{1}{\xi_i}$  subject to  $\xi_1 + \dots + \xi_n = 1$ .

Then there is a Lagrange multiplier  $\lambda$  such that if  $\xi_i \neq 0$ ,

$$d_i \log \frac{1}{\xi_i} - d_i = \lambda,$$

i.e.,

$$\log \frac{1}{\xi_i} = 1 + \frac{\lambda}{d_i}$$

so that

$$\xi_i = e^{-(1+\lambda/d_i)}.$$

Let  $A = \{i: \xi_i > 0\}$ . Then

$$\sum_{i \in A} e^{-(1+\lambda/d_i)} = 1,$$

$$\psi(\xi_1, \dots, \xi_n) = \sum_{i \in A} d_i e^{-(1+\lambda/d_i)} (1 + \lambda/d_i) \leq \|T\| + \lambda.$$

Now

$$\sum_{i \in A} e^{-\lambda/d_i} = c.$$

Hence

$$\sum_{i=1}^{\infty} e^{-\lambda/d_i} \geq e.$$

Since for some  $\tau < \infty$

$$\sum e^{-\lambda/d_i} < \infty,$$

there exists  $\lambda_0$  such that  $\lambda > \lambda_0$  implies

$$\sum_{i=1}^{\infty} e^{-\lambda/d_i} < e.$$

Thus  $\gamma \leq \lambda_0$  and so

$$\|T(t_1 e_1 + \dots + t_n e_n)\|_{\varphi}^* \leq \|T\| (2 + \log \|T\|) + e^{-1} + \lambda_0.$$

Hence  $T$  maps  $l_1$  into  $l_{\varphi}$ .

(d)  $\Rightarrow$  (e).

$$\begin{array}{ccc} & & X \\ & \nearrow S & \downarrow \sigma \\ l_1 & \xrightarrow{T_0} l_{\varphi} & \xrightarrow{J} l_1 \end{array}$$

$T$  factors  $T = JT_0$  where  $J: l_{\varphi} \rightarrow l_1$  is the inclusion map. The existence of a lift  $S$  follows from the fact that  $X$  is logconvex.

(c)  $\Rightarrow$  (a). Theorem 7.1.

**9. Orlicz sequence spaces.** We recall that an  $F$ -space  $X$  is a  $\mathcal{X}$ -space if  $(R, X)$  splits ([6], [8]). In this section we classify completely those locally banded Orlicz sequence spaces  $l_p \subset l_1$  which are  $\mathcal{X}$ -spaces. It is known ([6]) that  $l_p$  is a  $\mathcal{X}$ -space if  $0 < p < 1$  and fails to be a  $\mathcal{X}$ -space if  $p = 1$ .

We shall suppose throughout that  $f$  is a twice-differentiable strictly increasing Orlicz function with  $f(1) = 1$  such that  $af(x)$  is convex (cf. [5]); these assumptions may be made without loss of generality. We also suppose that  $f$  satisfies the  $\Delta_2$ -condition, i.e. for some  $K$

$$f(2x) \leq Kf(x), \quad 0 \leq x < \infty.$$

We define

$$\alpha_f = \sup\{p: \exists M, f(ax) \leq Ma^p f(x), 0 < a, x < 1\},$$

$$\beta_f = \inf\{p: \exists M, f(ax) \geq Ma^p f(x), 0 < a, x < 1\}.$$

Since  $l_f$  is locally bounded,  $\alpha_f > 0$ , and the  $\Delta_2$ -condition implies  $\beta_f < \infty$ .

Since  $l_f \subset l_1$ , we shall suppose

$$f(x) \leq Mx, \quad 0 \leq x \leq 1,$$

for some  $M$ .

Now let  $h: R \rightarrow R$  be defined by

$$h(x) = x \int_x^1 \frac{f(t)}{t^2} dt, \quad x > 0,$$

$$h(0) = 0,$$

$$h(x) = -h(-x), \quad x < 0.$$

LEMMA 9.1.  $h$  has the following properties:

(i)  $h$  is continuous, and twice differentiable for  $x \neq 0$ .

(ii)  $h''(u) \leq 0$ ,  $u \neq 0$ .

(iii) If  $u + v + w = 0$ ,

$$(9.1.1) \quad h(u) + h(v) + h(w) \leq 2(f(|u|) + f(|v|) + f(|w|)).$$

(iv) If  $0 \leq a \leq 1$  and  $x \in R$ ,

$$(9.1.2) \quad |h(ax) - ah(x)| \leq f(x).$$

Proof. (i) For continuity at 0, observe if  $0 < x < 1$

$$h(x) \leq x \int_x^1 \frac{M}{t} dt \leq Mx \log \frac{1}{x}.$$

The other assertion is clear.

$$(ii) \quad h'(u) = \int_u^1 \frac{f(t)}{t^2} dt - \frac{f(u)}{u}, \quad u > 0,$$

$$h''(u) = -\frac{f'(u)}{u} \leq 0.$$

(iii) Suppose without loss of generality  $u > 0$ ,  $v > 0$  and  $w = -(u+v)$ .

Since  $h'' \leq 0$ ,

$$h(u+v) \leq h(u) + h(v) \leq 2h\left(\frac{1}{2}(u+v)\right),$$

so that

$$0 \leq h(u) + h(v) + h(w) \leq 2h\left(\frac{1}{2}(u+v)\right) - h(u+v),$$

$$2h\left(\frac{1}{2}(u+v)\right) - h(u+v) = (u+v) \int_{\frac{1}{2}(u+v)}^{u+v} \frac{f(x)}{x^2} dx \leq 2f(u+v).$$

Hence

$$h(u) + h(v) + h(w) \leq 2f(|w|) \leq 2(f(|u|) + f(|v|) + f(|w|)).$$

(iv) For  $x > 0$ .

$$h(ax) - ah(x) = ax \int_{ax}^x \frac{f(t)}{t^2} dt \leq axf(x) \int_{ax}^x \frac{1}{t^2} dt = f(x)(1-a) \leq f(x).$$

LEMMA 9.2. Suppose for some  $B < \infty$  we have for  $0 \leq x \leq 1$

$$h(x) \leq Bf(x).$$

Then  $\beta_f < 1$ .

Proof. Let  $C_f \subset C[0, 1]$  be defined by

$$C_f = \overline{\text{co}}\{f_t : 0 < t \leq 1\}$$

where

$$f_t(x) = \frac{f(tx)}{f(t)}$$

(cf. [5], [10]). Since  $l_f \subset l_1$ ,  $\alpha_f \leq 1$ . If  $\beta_f \geq 1$ , then  $x \in C_f$  ([10]).

Now

$$\int_x^1 \frac{f(t)}{t^2} dt \leq \frac{f(x)}{x}$$

and if  $0 < s \leq 1$

$$\begin{aligned} \int_x^1 \frac{f_s(t)}{t^2} dt &= \int_x^1 \frac{f(st)}{t^2 f(s)} dt = \int_{sx}^f \frac{sf(u)}{u^2 f(s)} du \leq \frac{s}{f(s)} \int_{sx}^1 \frac{f(u)}{u^2} du \\ &\leq B \frac{s}{f(sx)} \frac{f(sx)}{sx} = \frac{Bf_s(x)}{x}, \quad 0 < x \leq 1. \end{aligned}$$

Hence, if  $g \in C_f$

$$\int_x^1 \frac{g(t)}{t^2} dt \leq B \frac{g(x)}{x}, \quad 0 < x \leq 1.$$

In particular if  $\beta_f \geq 1$ , we may let  $g(t) = t$

$$\int_x^1 \frac{1}{t} dt \leq B, \quad 0 < x \leq 1$$

and this contradiction shows  $\beta_f < 1$ .

THEOREM 9.3. Suppose  $f$  is a Orlicz function satisfying the  $\Delta_2$ -condition and that  $l_f$  is locally bounded and contained in  $l_1$ . Then  $l_f$  is a  $\mathcal{K}$ -space if and only if  $\beta_f < 1$ .

Proof. If  $\beta_f < 1$ ,  $l_f$  is a  $\mathcal{K}$ -space [6]. Conversely, suppose  $l_f$  is a  $\mathcal{K}$ -space. We define

$$H: l_f^0 \rightarrow \mathbf{R}$$

(where  $l_f^0$  is the finitely non-zero sequences in  $l_f$ ) by

$$H(x) = \sum_{i=1}^{\infty} h(x_i) \quad \text{if} \quad \sum_{i=1}^{\infty} f(|x_i|) = 1$$

and extend so that

$$H(ax) = aH(x), \quad a \in \mathbf{R}.$$

We first assert that  $H: l_f^0 \rightarrow \mathbf{R}$  is quasilinear. To see this we show that if  $u, v, w \in l_f^0$

$$u + v + w = 0$$

and

$$\|u\| + \|v\| + \|w\| \leq 1,$$

then

$$|H(u) + H(v) + H(w)| \leq 9.$$

Indeed,

$$\left| H(u) - \sum_{i=1}^{\infty} h(u_i) \right| \leq 1$$

and similarly for  $v, w$  while

$$\left| \sum_{i=1}^{\infty} h(u_i) + h(v_i) + h(w_i) \right| \leq 6.$$

Now  $l_f$  is a  $\mathcal{K}$ -space so that there a linear map  $\psi: l_f^0 \rightarrow \mathbf{R}$  such that

$$\sup_{\|x\| \leq 1} |H(x) - \psi(x)| < \infty.$$

Thus,  $\{\psi(e_n): n = 1, 2, \dots\}$  is bounded since  $H(e_n) = 0$  and so  $\psi$  is continuous. Hence,

$$|H(x)| \leq L\|x\|, \quad x \in l_f^0,$$

for some  $L < \infty$ .

Suppose  $0 < \xi \leq 1$ ; choose  $n$  so that  $n \in \mathbf{N}$  and  $\frac{1}{2} < n\xi \leq 1$ . Choose  $\eta$  so that  $f(\eta) = 1 - n\xi$  and let

$$x = \xi(e_1 + \dots + e_n) + \eta e_{n+1}.$$

Then  $\|x\| = 1$  and

$$H(x) \geq n\xi \int_{\xi}^1 \frac{f(t)}{t^2} dt \geq \frac{1}{2} \frac{\xi}{f(\xi)} \int_{\xi}^1 \frac{f(t)}{t^2} dt.$$

Hence,

$$\int_{\xi}^1 \frac{f(t)}{t} dt \leq 2L \frac{f(\xi)}{\xi}, \quad 0 < \xi \leq 1,$$

and so by Lemma 9.2,  $\beta_f < 1$ .

Suppose now  $f$  is submultiplicative at 0. Then we say  $X$  is  $f$ -convex if  $\sum f(\|x_i\|) < \infty$  implies  $\sum x_i$  converges, i.e.,  $X$  is galbed by  $l_f$ .

**COROLLARY 9.4.** *Suppose  $f$  is submultiplicative at 0; then every twisted sum of  $\mathbf{R}$  and a  $f$ -convex space is also  $f$ -convex if and only if*

$$\beta_f = \lim_{x \rightarrow 0} \frac{\log f(x)}{\log x} < 1.$$

**Proof.** Observe that  $l_f(I)$  (for an index set  $I$ ) is projective among  $f$ -convex, i.e.

$$\begin{array}{ccc} & & X \\ & \nearrow \tilde{T} & \\ l_f(I) & \xrightarrow{\pi} & X/N \end{array}$$

Hence if every twisted sum of  $\mathbf{R}$  and  $l_f$  is  $f$ -convex, then  $l_f$  is a  $\mathcal{K}$ -space. Conversely, suppose  $l_f$  is a  $\mathcal{K}$ -space and  $X$  is any  $f$ -convex space and  $Y = \mathbf{R} \oplus_f X$  is a twisted sum of  $\mathbf{R}$  and  $X$ . Then there is a quotient map  $T: l_f(I) \rightarrow X$  for some index set  $I$ . Now  $l_f(I)$  is also a  $\mathcal{K}$ -space (this is easy to show) and so there is a lift  $\tilde{T}: l_f(I) \rightarrow Y$ . If  $\tilde{T}$  fails to be surjective  $Y$  splits, while if  $\tilde{T}$  is surjective,  $Y$  is  $f$ -convex.

**Remark.** In this case  $\beta_f < 1$  implies  $f(x) \geq cx^p$  for some  $p < 1$ , and  $c > 0$  for all  $0 \leq x \leq 1$ .

**10. Locally convex  $\mathcal{K}$ -spaces.** Let  $X$  be a metrizable locally convex space. Let  $\|\cdot\|_n$  be a sequence of semi-norms on  $X$  which define the topology of  $X$  and such that

$$\|x\|_n \leq \|x\|_{n+1}, \quad n \in \mathbf{N}.$$

Define a map  $F: X \rightarrow \mathbf{R}$  be quasilinear if for some  $n \in \mathbf{N}$ ,  $K < \infty$

$$F(tx) = tF(x), \quad t \in \mathbf{R}, x \in X,$$

$$|F(x+y) - F(x) - F(y)| \leq K(\|x\|_n + \|y\|_n)$$

if  $F$  is quasilinear we define  $\mathbf{R} \oplus_f X$  to be the space  $\mathbf{R} \oplus X$  equipped with the quasi-semi-norms

$$\|(t, x)\|_m^* = |t - F(x)| + \|x\|_m, \quad m \geq n.$$

Then if  $q: \mathbf{R} \oplus_f X \rightarrow X$  is given by

$$q(t, x) = x,$$

$q$  is a quotient map and  $q^{-1}(0) = \{(t, 0), t \in \mathbf{R}\}$ . Thus we have a short exact sequence  $0 \rightarrow \mathbf{R} \rightarrow \mathbf{R} \oplus_f X \rightarrow X \rightarrow 0$  and  $\mathbf{R} \oplus_f X$  is a twisted sum of  $\mathbf{R}$  and  $X$ . It is easy to show that if  $X$  is complete, then so is  $\mathbf{R} \oplus_f X$  (since it is such a twisted sum).

The twisted sum  $\mathbf{R} \oplus_f X$  will split if and only if there is a linear map  $\psi: X \rightarrow \mathbf{R}$  such that

$$|F(x) - \psi(x)| \leq M \|x\|_m, \quad x \in X,$$

for some  $m \in \mathbf{N}$  and  $M < \infty$ .

**THEOREM 10.1.** *Let  $X$  be a Fréchet space (complete metrizable locally convex space) and suppose an  $F$ -space  $Y$  is a twisted sum of  $\mathbf{R}$  and  $X$ . Then there exists a quasilinear map  $F: X \rightarrow \mathbf{R}$  such that  $Y$  is isomorphic to  $\mathbf{R} \oplus_f X$  (as a twisted sum).*

**Proof.** It is convenient to write  $Y = \mathbf{R} \oplus X$  algebraically so that the quotient map  $q: Y \rightarrow X$  is given by  $q(t, x) = x$ .

Let  $\{V_n: n \in \mathbf{N}\}$  be a base of balanced neighborhoods of 0 such that  $V_{n+1} + V_{n+1} \subset V_n$  for all  $n$  and  $V_1 \cap \mathbf{R}e$  is bounded where  $e = (1, 0)$ . Let  $|\cdot|_n$  be the Minkowski functional of  $V_n$ . Then we have

$$(10.1.1) \quad |(t, x)|_n \leq |(t, x)|_{n+1}, \quad t \in \mathbf{R}, x \in X,$$

$$(10.1.2) \quad |(s+t, x+y)|_n \leq |(s, x)|_{n+1} + |(t, y)|_{n+1},$$

$$(10.1.3) \quad |(1, 0)|_n = \alpha_n \quad \text{where} \quad \alpha_1 > 0 \text{ and } \alpha_n \uparrow \infty.$$

Also since  $q$  is open, there exist increasing sequences  $\{m(n)\}$  and  $\{\beta_n\}$  such that

$$(10.1.4) \quad \text{For } x \in X \text{ there exists } t_n \in \mathbf{R}$$

$$|(t_n, x)|_n \leq \beta_n \|x\|_{m(n)}.$$

In view of (10.1.4) there is a map  $F_n: X \rightarrow \mathbf{R}$  such that  $F(tx) = tF(x)$ ,  $t \in \mathbf{R}$  and

$$|(F_n(x), x)|_n \leq \beta_n \|x\|_{m(n)}.$$

Now if  $n > p > 1$ ,

$$\begin{aligned} |F_n(x) - F_p(x)| &= \alpha_{p-1}^{-1} |(F_n(x) - F_p(x), 0)|_{p-1} \\ &\leq \alpha_{p-1}^{-1} (|(F_n(x), x)|_p + |(F_p(x), x)|_p) \\ (10.1.5) \quad &\leq \alpha_{p-1}^{-1} (\beta_n + \beta_p) \|x\|_{m(n)}. \end{aligned}$$

Also if  $n \geq 3$  and  $x, y \in X$

$$(10.1.6) \quad |(F_n(x) + F_n(y), x + y)|_{n-1} \leq \beta_n(\|x\|_{m(n)} + \|y\|_{m(n)}).$$

$$(10.1.7) \quad |(F_{n-1}(x + y), x + y)|_{n-1} \leq \beta_{n-1}(\|x\|_{m(n)} + \|y\|_{m(n)}).$$

Hence combining (10.1.6) and (10.1.7) with (10.1.2)

$$\alpha_{n-2}|F_n(x) + F_n(y) - F_n(x + y)| \leq (\beta_n + \beta_{n-1})(\|x\|_{m(n)} + \|y\|_{m(n)}).$$

Thus

$$|F_n(x) + F_n(y) - F_n(x + y)| \leq C_n(\|x\|_{m(n)} + \|y\|_{m(n)}).$$

In particular  $F \equiv F_3$  in quasilinear, and for  $n \geq 3$

$$|F_n(x) - F(x)| \leq D_n\|x\|_{m(n)}.$$

Thus

$$\begin{aligned} |(u, x)|_n &\leq \alpha_{n+1}|u - F_n(x)| + \beta_{n+1}\|x\|_{m(n+1)} \\ &\leq \alpha_{n+1}|u - F(x)| + (\alpha_{n+1}D_n + \beta_{n+1})\|x\|_{m(n+1)} \\ &\leq A_n(|u - F(x)| + \|x\|_{m(n+1)}). \end{aligned}$$

Hence the identity  $i: \mathbf{R} \oplus_F X \rightarrow Y$  is continuous. By the closed graph theorem  $i$  is an isomorphism.

**THEOREM 10.2.** Any nuclear Fréchet space is a  $\mathcal{K}$ -space.

**Proof.** If  $X$  is a nuclear Fréchet space and  $F: X \rightarrow \mathbf{R}$  is quasilinear, then there is a Hilbertian semi-norm  $\|\cdot\|_n$  on  $X$  such that

$$|F(x + y) - F(x) - F(y)| \leq \|x\|_n + \|y\|_n.$$

Since a Hilbert space is a  $\mathcal{K}$ -space ([6]), there is a linear map  $\psi: X \rightarrow \mathbf{R}$  such that

$$|F(x) - \psi(x)| \leq M\|x\|_n.$$

Let  $(a_{mn})$  be a matrix with non-negative entries such that  $a_{m+1,n} \geq a_{m,n}$  for all  $n$  and for each  $n$  there exists  $m$  with  $a_{mn} > 0$ . Then the Köthe sequence space  $l_1[a_{mn}]$  is the space of sequences  $(x_n)$  such that

$$\|x\|_m = \sum_{n=1}^{\infty} |a_{mn}| |x_n| < \infty, \quad m = 1, 2, \dots$$

**THEOREM 10.3.**  $l_1[a_{mn}]$  is a  $\mathcal{K}$ -space if and only if given  $m \in \mathbf{N}$  there exists  $\tau < \infty$  and  $r > m$  with

$$(10.3.1) \quad \sum_{n=1}^{\infty} \exp\left(-\tau \frac{a_{rn}}{a_{mn}}\right) < \infty \quad (0/0 = \infty).$$

**Proof.** If (10.3.1) fails, there exists  $m$  such that for all  $r > m$  and  $\tau < \infty$

$$(10.3.2) \quad \sum_{n=1}^{\infty} \exp\left(-\tau \frac{a_{rn}}{a_{mn}}\right) = \infty.$$

Define  $F: l_1^0[a_{mn}] \rightarrow \mathbf{R}$  by

$$F(x) = \sum_{n=1}^{\infty} a_{mn} x_n \log \frac{\|x\|_m}{a_{mn}|x_n|}$$

(for finitely non-zero sequences). If  $\mathbf{R} \oplus_{F, l_1^0[a_{mn}]}$  splits then there is a linear map  $\psi: l_1^0[a_{mn}] \rightarrow \mathbf{R}$  such that

$$|F(x) - \psi(x)| \leq K\|x\|_r$$

for some  $r > m$ . This implies

$$\psi(e_n) \leq K a_{rn}$$

and so

$$|\psi(x)| \leq K\|x\|_r,$$

so that

$$|F(x)| \leq 2K\|x\|_r.$$

Hence

$$\sum_{n=1}^{\infty} a_{mn}|x_n| \log \frac{\|x\|_m}{a_{mn}|x_n|} \leq 2K \sum_{n=1}^{\infty} a_{rn}|x_n|.$$

This means the diagonal map  $\{x_n\} \rightarrow \{d_n x_n\}$  maps  $l_1$  into  $l_r$  where

$$d_n = \frac{a_{mn}}{a_{rn}} \quad (= 0 \text{ if } a_{rn} = 0).$$

Hence by Theorem 8.3 we have contradicted (10.3.2).

Conversely if (10.3.1) holds and  $F$  is quasilinear, then if

$$|F(x + y) - F(x) - F(y)| \leq K(\|x\|_m + \|y\|_m),$$

choose  $r > m$  to satisfy (10.3.1). Let

$$d_n = \frac{a_{mn}}{a_{rn}}.$$

Then  $D: \{x_n\} \rightarrow \{d_n x_n\}$  is liftable. If we define  $G: l_1 \rightarrow \mathbf{R}$  by

$$G(x) = F(\{a_{mn}^{-1} x_n\}) \quad (1/0 = 0!).$$



$G$  is quasilinear on  $l_1$  and hence there is a linear map  $\tilde{D}$

$$\begin{array}{ccc} & \mathbf{R} \oplus_a l_1 & \\ \tilde{D} \swarrow & \downarrow & \\ l_1 & \xrightarrow{D} & l_1 \end{array}$$

If  $\tilde{D}x = (\psi(x), Dx)$ , then

$$|\psi(x) - G(Dx)| \leq C\|x\|, \quad x \in l_1.$$

Hence

$$|\psi\{a_{rn}x_n\} - F(x)| \leq C\|x\|_r, \quad x \in l_1[a_{mn}]$$

so that  $\mathbf{R} \oplus_F l_1[a_{mn}]$  splits.

Remark. Condition (10.3.1) is thus a topological invariant of  $l_1[a_{mn}]$ .

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