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Received December 21, 1976 (1237)

STUDIA MATHEMATICA, T. LXV. (1979)

On the sum of two Brownian paths

by

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Abstract. We study a mapping property of the random function of two variables given by $X(s)+X(t)$. This process has a complicated dependence structure, and a combinatorial estimation of product measures is used, in place of martingales. The property is suggested by the Peano curve and the known modulus of continuity of Brownian motion.

Let X denote Brownian motion on the half-line $t \geqslant 0$, let $Z(s,t) = X(s)+X(t)$ on the quadrant $s \geqslant 0$, $t \geqslant 0$, and let F be a compact set in this quadrant.

THEOREM. *If the Hausdorff dimension of F exceeds $1/2$, then for almost all paths X, $Z(F)$ has an interior point.*

Before entering upon the proof, we point out how the present theorem differs from previous results, including [3]. From the viewpoint of probability theory, we observe that the process Z has a complicated dependence structure, for example an identity $Z(a,b)+Z(c,d) = Z(a,d)+Z(b,c)$. We did not succeed in finding a proof based on martingale inequalities, but rely on a direct estimation of moments; the obstacle to a proof following [3] is precisely the presence of relations like the one cited. In the calculation of moments we require a new estimate for the product measures of certain sets in $F\times \ldots \times F$, which may be of interest for Gaussian processes in general. Finally, the process Z seems to be intractable by the method of calculating individual Fourier coefficients, e.g. [1]. If, for example $F = F_1\times F_2$, wherein $\dim F_1 = \dim F_2 = 0$, then the sets $X(F_1)$ and $X(F_2)$ are subject to no workable restriction; indeed we can find F_1 and F_2 so that the additive groups generated by $X(F_1)$ and $X(F_2)$ have Hausdorff dimension 0 for almost all paths X.

1. In this paragraph we make a few preliminary reductions and write down the integrals whose estimation is the main burden of the proof. The quadrant $s\geqslant 0$, $t\geqslant 0$ is covered by its subsets $s=t$, $0\leqslant s<t$, $0\leqslant t<s$. Then F meets one of these sets in a subset of the same dimension as F itself, say F_0, F_1, F_2. In case $\dim F_0 = \dim F > 1/2$, we observe that $Z(s,t) = 2X(s)$ on F_0, and this possibility is easily included in the remaining ones. If $\dim F_1 = \dim F > 1/2$, then F_1 (being a countable union of closed sets) contains a compact set of the same dimension, included in a product set $[\varepsilon_1, \varepsilon_2]\times[\varepsilon_3, \varepsilon_4]$, with $0<\varepsilon_1<\ldots<\varepsilon_4$. After a harmless change of variable, we assume $F \subseteq [1,2]\times[3,4]$.

F carries a probability measure μ with a Lipschitz condition $\mu(S) \leqslant C(\operatorname{diam} S)^a$ for all open sets S, and a certain $a > 1/2$ [2]. We shall prove that the transform of μ by Z, i.e. $\lambda(S) \equiv \mu\big(Z^{-1}(S)\big)$, admits a continuous density. In doing so we follow a method from Fourièr analysis used in [3].

We choose and fix a function Φ of class C^∞, vanishing off $(-2,2)$ and equal to 1 on $(-1,1)$ and write $\psi(x) = \Phi(2x)-\Phi(x)$. Then we set, for real u, and $R>1$,

$$I(R, u) = \int R\hat{\psi}\big(RZ(s,t)-u\big)\mu(ds\,dt).$$

As explained in [3] it will be sufficient to obtain the following inequalities

$$\|I(R,u)\|_{2r}^{2r} \leqslant B_r R^{-\beta r},$$

in which $\beta > 0$, and B_r depends on r but not on u or R. The $2r$-th power of $|I(r,u)|$ is the integral over $F^{(2r)}$ of

$$R^{2r}\hat{\psi}\big(RZ(s_1,t_1)-u\big)\hat{\psi}\big(RZ(s_2,t_2)-u\big)^- \times \ldots \tag{1}$$

Here there are $2r$ factors $\hat{\psi}$ or $\hat{\psi}^-$, whose arguments are $RZ(s_i,t_i)-u$.

Let $\eta > 0$, to be chosen later; then the product is exceedingly small unless we have $|RZ(s_i,t_i)-u| < R^\eta$; $1\leqslant i\leqslant 2r$. Our plan is to estimate the probability of this event as a function of $(s_1,t_1),\ldots,(s_{2r},t_{2r})$. Integration of this function, with the factor R^{2r}, yields an estimate $R^{2r\eta}$. To improve this, we calculate the expected value of the product (1) by different means, and find a negligible value except on a certain set $H_R \subseteq F^{(2r)}$. Then we integrate the probability found before over H_R and improve the estimate to $R^{-r\beta}$. Of these steps, the calculation of the expected value, and the estimation of the product measure of H_R are the keys to the theorem. It is the unusual form of H_R that distinguishes this result from its simpler version in one dimension.

2. In this paragraph we consider the events A_i: $|RZ(s_i,t_i)-u| < R^\eta$. We introduce the functions $d_1(s,t) = 1$ and $d_i(s,t) = \min(|s_i-s_j|,\ 1\leqslant j < i)+\min(|t_i-t_k|,\ 1\leqslant k<i)$ for $2\leqslant i\leqslant 2r$. Then $Z(s_i,t_i) = A_i+B_i$, where A_i is a Gaussian variable in the span of $Z(s_j,t_j)$, $1\leqslant j<i$, and B_i is orthogonal to that space (and hence independent of it). We claim that $\sigma^2(B_i)\geqslant cd_i$, an evident assertion for $i=1$. In the verification of this we use the formalism of stochastic integrals, an isometry of $L^2(0,+\infty)$ onto a Hilbert space of Gaussian variables. Writing $Q(u) = 1$ for $u>0$, $Q(u) = 0$ for $u<0$, we observe that $Z(s,t)$ corresponds to the function, of $u>0$, $Q(t-u)+Q(s-u)$. If now f in $L^2(0,+\infty)$ is constant on an interval of length p, $0<p<1$, around t_i, then $\|Q(t_i-u)+Q(s_i-u)-f(u)\|_2^2 > p$. The same is true for s_i in place of t_i, and this proves the claim. Using conditional probabilities in succession we find that

$$P\big(|RZ(s_i,t_i)-u| < R^\eta,\ 1\leqslant i\leqslant 2r\big) \leqslant CR^{(\eta-1)2r}(d_1d_2\ldots d_{2r})^{-1/2}.$$

We write $d(S,T)^{-1/2}$ for the factor on the right, and observe that $d(S,T)^{-1/2}$ belongs to $L^1(\mu^{2r})$, by Fubini's theorem and the Lipschitz condition on μ. In fact $d(S,T)^{-1/2}$ is in $L^p(\mu^{2r})$ provided $1<p<2a$, again by Fubini's theorem; this observation will be important later.

3. In this paragraph we give a direct estimation of the expected value of the random function (1). We define the functions

$$d_i^*(s,t) = \min(|s_j-s_i|,\ j\neq i)+\min(|t_k-t_i|,\ k\neq i).$$

Then $Z(s_i,t_i) = A_i^*+B_i^*$, where A_i^* is a Gaussian variable in the span of $Z(s_j,t_j)$, $j\neq i$, and B_i^* is orthogonal to that space. As before, $\sigma^2(B_i^*)\geqslant cd_i^*$. The conditional expectation of $\hat{\psi}\big(RZ(s_i,t_i)-u\big)$, given the field of $A(s_j,t_j)$, $j\neq i$ is then

$$(2\pi)^{-1/2}\int\hat{\psi}(R\sigma y+v)\exp(-y^2/2)dy = \int\psi(s)e^{-iv\xi}\exp(-1/2R^2\sigma^2)s\,ds.$$

Inasmuch as $\psi(s) = 0$ for $|s|<1/2$, the integral is exceedingly small, uniformly with respect to v, if $R\sigma > R^\eta$. In different terms, the expected valued of the entire product (1) is exceedingly small unless $(s_1,t_1),\ldots$ $\ldots,(s_{2r},t_{2r})$ belongs to the set H_R defined by the inequality $d_i^* < R^{2\eta-2}$ $(1\leqslant i\leqslant 2r)$.

4. Recalling the definition of the functions $d_1^*,\ldots,d_{2r}^*$ we shall prove that the product measure of a set of the type $d_i^*\leqslant h$ $(1\leqslant i\leqslant 2r)$ is $O(h^{ar/2})$. To apply this to the main theorem we take $h = R^{2\eta-2}$.

Since $d_1^*\leqslant h$, for each element in our set, we can find $i_1\geqslant 2$ and $j_1\geqslant 2$ so that $|s_1-s_{i_1}|\leqslant h$ and $|t_1-t_{j_1}|\leqslant h$. From the sequence (s_1,t_1), $(s_2,t_2),\ldots,(s_{2r},t_{2r})$ we discard (s_1,t_1), (s_{i_1},t_{i_1}) and (s_{j_1},t_{j_1}). Among the remaining elements of the sequence, we repeat this process, using $2h$ in place of h. That is, we find (s_p,t_p), (s_q,t_q), (s_m,t_m), with $q\neq p$, $m\neq p$, and $|s_p-s_q|<2h$, $|t_p-t_m|<2h$. If this is possible, we discard the terms involved, and continue with $2h$, etc. Suppose that this process can be continued for $N\geqslant 1$ steps; the elements so chosen form a set of product measure $\ll h^{-aN}$. Thus we can suppose that $N<r/2$. The elements already chosen we label (s_i,t_i), $i\in I_1$, and the remainder we label with I_2. We consider elements (s_i,t_i), $i\in I_2$, for which we can find $j\in I_1$ and $k\in I_1$, so that $|s_i-s_j|<h$ and $|t_i-t_k|<h$. Since the sets I_1 and I_2 are disjoint, we obtain a set of measure $\ll h^{ar/2}$, if we assume that the number of indices i,

with this property, is at least $r/2$. Thus we assume the opposite situation prevails.

To recapitulate, I_2 has more than $3r/2$ elements. To each element i in I_2 there are indices j, k such that $|s_i - s_j| < h$ and $|t_i - t_k| < h$. At least one of j, k belongs to I_1, by the construction of I_1. For at least r elements i of I_2 (we call these I_2') at least one of j, k belongs to I_2. We write I_2'' if $j \in I_1$ and $k \in I_2$, and I_2''' otherwise. Then one of the sets I_2'', I_2''' has at least $r/2$ members.

Assuming that I_2'' has at least $r/2$ members, we finally attain a contradiction. For I_1 has fewer than $r/2$ members, so that I_2'' contains two elements i_0 and i_{00} such that for some j, we have $|s_{i_0} - s_j| < h$ and $|s_{i_{00}} - s_j| < h$, whence $|s_{i_0} - s_{i_{00}}| < 2h$. Also, $|t_{i_0} - t_k| < h$, with some k in I_2. Thus (s_{i_0}, t_{i_0}) is included in the first method of selection, a contradiction.

5. To complete our estimation of $\|I(R, u)\|_{2r}^{2r}$, we recall that this was expressed as an integral over $F^{(2r)}$, and that the integral over $F^{(2r)} \setminus H_R$ was found to be negligible. The measure of H_R was just found to have order $(R^{2\eta-2})^{ar/2} = R^{(\eta-1)ar}$. The integrand, moreover, is in $L^p(\mu^{2r})$, for $1 < p < 2a$, and its norm in L^p has order $R^{\eta 2r}$. The integral over H_R, therefore has order $R^{\eta 2r} \cdot R^{(\eta-1)arq}$, wherein $q = (p-1)p^{-1} > 0$. As η decreases to 0, the exponent approaches $-arq$, so that $\|I(r, u)\|_{2r}^{2r}$ has magnitude $B_r R^{-cr}$, for any $c < aq$. The number was subject to the inequality $q < 1 - (2a)^{-1}$, so that c is subject to the inequality $c < a - 1/2$. This allows us to conclude that the density of the measure λ belongs to a certain Hölder class, depending on a.

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Received January 17, 1977 (1249)

STUDIA MATHEMATICA, T. LXV. (1979)

Singular integrals on generalized Lipschitz and Hardy spaces

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Abstract. Let $d(x, y)$ be a quasi-distance and μ a measure, both defined on X, such that (X, d, μ) is a normalized space of homogeneous type. Singular integral kernels are defined on (X, d, μ). Norm inequalities are given for the singular integral operators, associated with these kernels, acting on atomic Hardy spaces and their duals.

Introduction. Let X be a topological space and $d(x, y)$ a non-negative function defined on $X \times X$ satisfying:

(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) There exists a constant k such that

$$d(x, y) \leqslant k\big(d(x, z) + d(z, y)\big).$$

(iv) The balls with center at x and radius $r > 0$,

$$B(x, r) = \{y:\ d(x, y) < r\},$$

are a basis of neighbourhoods of x.

Moreover, we shall assume that there is a regular Borel measure μ such that for every ball $B(x, r)$, $x \in X$, $r > 0$, there exist two positive and finite constants c_1, c_2 such that

$$c_1 r \leqslant \mu\big(B(x, r)\big) \leqslant c_2 r. \tag{1}$$

This property of the measure μ implies that if $b > 0$ and $\varepsilon > 0$, then,

$$\int\limits_{d(x,z)>b>0} d(x, z)^{-1-\varepsilon} d\mu(x) \leqslant cb^{-\varepsilon}. \tag{2}$$

The triple (X, d, μ), satisfying the requirements above, shall be called a *normalized homogeneous space* (see [3]).

Let $\varphi(x)$ be a real or complex valued function on X, square integrable on bounded subsets of X. The mean value of $\varphi(x)$, on a ball B, $\mu(B)^{-1}\int_B \varphi(x)\, d\mu(x)$, shall be denoted by $m_B(\varphi)$. We shall say that this fun-