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Received October 13, 1976

(1216)

## On weakly compact operators from some uniform algebras

by

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**Abstract.** We prove that weakly compact operators from a class of uniform algebras, containing many natural algebras of analytic functions on planar compact sets, behave exactly like weakly compact operators from  $C(K)$  spaces. In particular, a non-weakly compact operator from such an algebra is an isomorphism on some subspace isomorphic to  $c_0$ .

Recently S. V. Kislakov [15] and F. Delbaen [5], generalizing the results of A. Grothendieck [9] and A. Pełczyński [17] and [18], have proved that the disc algebra has the Pełczyński property and the Dunford-Pettis property. Let us recall that the Banach space  $X$  has the *Dunford-Pettis property* if every weakly compact operator defined on  $X$  transforms weakly convergent sequences into norm convergent sequences. This is equivalent to the following: for every  $(x_n) \subset X$ ,  $x_n \xrightarrow{w} 0$  and  $(x_n^*) \subset X_n^*$ ,  $x_n^* \xrightarrow{w} 0$  we have  $\lim x_n^*(x_n) = 0$ .

The Banach space  $X$  has the *Pełczyński property* if every non-weakly compact operator defined on  $X$  is an isomorphism when restricted to some subspace of  $X$ , isomorphic to  $c_0$ . We say that the series  $\sum f_n$ ,  $f_n \in X$  is *weakly unconditionally convergent* (and abbreviate it to “ $(f_n)$  is w.u.c.”) if for every  $x^* \in X^*$  we have  $\sum |x^*(f_n)| < \infty$ . With the use of this concept the Pełczyński property can be equivalently defined as follows: A Banach space  $X$  has the Pełczyński property if every set  $V \subset X^*$  such that  $\limsup_n \{ |x^*(f_n)| : x^* \in V \} = 0$  for every w. u. c.  $(f_n)$  in  $X$  is weakly relatively compact.

It is well known (cf. [5], [9], [17], [18], [20]) that if the Banach space  $X$  has both the Dunford-Pettis property and the Pełczyński property, then

- (a) The following are equivalent for an arbitrary Banach space  $E$ :
  - (a<sub>1</sub>) An operator  $T: X \rightarrow E$  is weakly compact.
  - (a<sub>2</sub>) An operator  $T: X \rightarrow E$  is strictly singular.
  - (a<sub>3</sub>) An operator  $T: X \rightarrow E$  is not an isomorphism when restricted to any subspace of  $X$  isomorphic to  $c_0$ .
  - (a<sub>4</sub>) An operator  $T: X \rightarrow E$  is unconditionally converging.
- (b) If  $T: X \rightarrow X$  is a weakly compact operator, then  $T^2$  is compact.

(c) Every complemented, infinite-dimensional subspace of  $X$  contains a subspace isomorphic to  $c_0$ .

Our main result is Theorem 2.4 and it is proved in Section 2. Section 1 contains two technical results. Lemma 1.3 allows us to use the result of Chaumat [4] and gives us some information on the isometric structure of the unit ball of certain abstract  $H_\infty$  spaces. Proposition 1.7 is the main analytical tool in our proof of Theorem 2.4.

The author would like to express his deep gratitude to Yu. A. Abramovič and S. V. Kislov for providing the prepublication copy of [15]. The author also thanks F. Delbaen for pointing out an inaccuracy in the earlier version of the proof of Proposition 1.7. Special thanks are due to A. Pełczyński for correcting many grammatical errors in the text of this paper and for writing the new introduction.

1. In this section we work in the context of  $H_\infty$  spaces of a  $w^*$ -Dirichlet algebra. The detailed study of  $w^*$ -Dirichlet algebras is presented in [12] and [21].

**DEFINITION 1.1.** Let  $(S, \mu)$  be a probability measure space. An algebra  $A \subset L_\infty(\mu)$ , containing constants, is called a  $w^*$ -Dirichlet algebra if  $\int fg d\mu = \int f d\mu \cdot \int g d\mu$  for all  $f, g \in A$ , and  $A + \bar{A}$  is dense in  $L_\infty(\mu)$  in the  $\sigma(L_\infty(\mu), L_1(\mu))$  topology.

By  $H_p(\mu)$ ,  $1 \leq p < \infty$ , we denote the  $L_p(\mu)$  closure of  $A$  and  $H_\infty(\mu)$  is the  $\sigma(L_\infty(\mu), L_1(\mu))$  closure of  $A$  in  $L_\infty(\mu)$ . It was shown in [12] and [21] that all essential measure-theoretical results valid for classical Hardy spaces are also valid for  $H_p(\mu)$ . In particular, there is a conjugation operator  $\sim$  defined on  $L_1^R(\mu)$  which has the following properties (cf. [12]):

- (1)  $\sim$  is of weak 1-1 type;
- (2)  $\sim$  is continuous on  $L_p^R(\mu)$ ,  $1 < p < \infty$ ;
- (3) if  $f \in L_p^R(\mu)$  and  $\tilde{f} \in \overline{L_p^R(\mu)}$ ,  $1 \leq p < \infty$ , then  $f + i\tilde{f} \in H_p(\mu)$ ;
- (4) if  $f \in A$  then  $\text{Ref} + i\overline{\text{Ref}} \in A$  and  $f - (\text{Ref} + i\overline{\text{Ref}})$  is a constant function.

Throughout this paper  $L_p^R(\mu)$  denotes the  $L_p$ -space of real-valued functions. Analogously,  $C_R(K)$  denotes the space of all real-valued continuous functions on a compact Hausdorff space  $K$ .

Our first goal is to generalize the result of Amar-Lederer [1] to the case of  $H_\infty(\mu)$  spaces.

**LEMMA 1.2.** Let  $(A, \mu)$  be a  $w^*$ -Dirichlet algebra and let  $f \in H_p(\mu)$  for all  $p < \infty$  and  $\text{Ref} \leq -1$ . Then  $1|f \in H_\infty(\mu)$ .

**Proof.** We first prove that  $f \cdot A$  is dense in  $H_{4/3}(\mu)$ . Otherwise, by the Riesz projection theorem, there would be an  $h \in H_4(\mu)$ ,  $\|h\|_4 = 1$  such that  $\int f \cdot a \cdot \bar{h} d\mu = 0$  for all  $a \in A$ . Let us consider the sequence  $(a_n) \subset A$

such that  $\|h - a_n\|_4 \rightarrow 0$ ,  $\|a_n\|_4 \leq \|h\|_4$ . Then

$$\|f a_n \bar{h} - f|h|^2\|_1 = \int |f(a_n - h)\bar{h}| d\mu \\ \leq \left( \int |f|^2 d\mu \right)^{1/2} \left( \int |a_n - h|^4 d\mu \right)^{1/4} \left( \int |\bar{h}|^4 d\mu \right)^{1/4} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

because  $f \in H_2(\mu)$ . So  $\int f \cdot a_n \cdot \bar{h} d\mu \rightarrow \int f|h|^2 d\mu \neq 0$ , because  $\text{Ref} \leq -1$  and  $|h| \neq 0$ . This contradiction proves the claim.

Now let us take the sequence  $(f \cdot a_n)$ ,  $a_n \in A$ , such that  $\|f \cdot a_n\|_{4/3} \leq 1$  and  $\|f a_n - 1\|_{4/3} \rightarrow 0$ . Since  $|f^{-1}| \leq 1$ , we have

$$\|f^{-1}(f \cdot a_n - 1)\|_{4/3} \leq \|f \cdot a_n - 1\|_{4/3} \rightarrow 0;$$

equivalently,  $\|a_n - f^{-1}\|_{4/3} \rightarrow 0$ . Hence  $f^{-1} \in H_{4/3}(\mu)$ . Since  $f^{-1} \in L_\infty(\mu)$ , we infer (cf. [21], 2.1 (viii)) that  $f^{-1} \in H_\infty(\mu)$ .

Now we will consider  $H_\infty(\mu)$  as a subalgebra of  $\mathcal{O}(\mathfrak{M}(L_\infty))$ , where  $\mathfrak{M}(L_\infty)$  is the maximal ideal space of the algebra  $L_\infty(\mu)$ . We will identify the measure  $\mu$  with the corresponding measure on  $\mathfrak{M}(L_\infty)$ .

**LEMMA 1.3** (Amar-Lederer). Let  $(A, \mu)$  be a  $w^*$ -Dirichlet algebra. Every closed set  $K \subset \mathfrak{M}(L_\infty)$  with  $\mu(K) = 0$  is contained in a peak set  $\tilde{K}$  for  $H_\infty(\mu)$  with  $\mu(\tilde{K}) = 0$ .

**Proof.** Since  $\mathfrak{M}(L_\infty)$  is totally disconnected and  $\mu(K) = 0$ , there exists a sequence of closed and open sets  $E_n \subset \mathfrak{M}(L_\infty)$  such that  $\mu(E_n) \leq 2^{-n}$ ,

$E_0 = \mathfrak{M}(L_\infty)$  and  $K \subset \bigcap_{n=0}^{\infty} E_n$ . The sets  $E_n$  can be identified with Borel

subsets of  $S$ . Consider the function  $s = \sum_{n=0}^{\infty} -(n+1)\chi_{E_n}$ . Since  $s \in L_p^R(\mu)$

for all  $p < \infty$ , we infer that  $f = (s + i\tilde{s}) \in H_p(\mu)$  for all  $p < \infty$ , and obviously  $\text{Ref} \leq -1$ . Hence, by Lemma 1.2,  $f^{-1} \in H_\infty(\mu)$ . Thus  $h = \exp f^{-1} \in H_\infty(\mu)$ . We claim that  $h$  peaks on certain set  $\tilde{K}$  with  $K \subset \tilde{K}$  and  $\mu(\tilde{K}) = 0$ . To this end note that

$$|h| = \exp \text{Ref}^{-1} = \exp \frac{s}{s^2 + \tilde{s}^2} \leq 1,$$

and

$$h = \exp \left( \frac{s}{s^2 + \tilde{s}^2} - \frac{i\tilde{s}}{s^2 + \tilde{s}^2} \right).$$

From this formula we see that  $h = 1$  on the set  $\tilde{K} = \bigcap_{n=1}^{\infty} V_n$  where  $V_n = \{t \in S: s^2(t) + \tilde{s}^2(t) \geq n\}$ . It is easily seen that for  $t \in \mathfrak{M}(L_\infty)$ ,  $t \notin \bigcap_{n=1}^{\infty} V_n$  we have  $|h(t)| < 1$ . Moreover,  $\mu(\tilde{K}) = 0$ . Since  $E_n \subset V_n$ , we have  $K \subset \tilde{K}$ . This completes the proof of the lemma.

Lemma 1.3 and a result of Chaumat [4] yield

**THEOREM 1.4.** *If  $(A, \mu)$  is a  $w^*$ -Dirichlet algebra, then  $L_1(\mu)/H_\infty^\perp$  has the Dunford-Pettis property and it is weakly sequentially complete. Here*

$$H_\infty^\perp = \{f \in L_1(\mu) : \int g \cdot f d\mu = 0 \text{ for all } g \in H_\infty(\mu)\}.$$

More general results were obtained by Delbaen [6]. Using Lemma 1.3, we can extend onto  $H_\infty(\mu)$  the Amar-Lederer [1] characterization of exposed points of the unit ball of  $H_\infty$ . Repeating their argument, we obtain the following (cf. [11], Theorem 13)

**THEOREM 1.5.** *Let  $x \in H_\infty(\mu)$ ,  $\|x\| = 1$ . Then the following conditions are equivalent:*

- (1)  $x$  is an exposed point of the unit ball of  $H_\infty(\mu)$ ;
- (2)  $x$  is regularly exposed, i.e. there exists an  $f \in L_1(\mu)$  such that  $\|f\| = 1$ ,  $\int f \cdot x d\mu = 1$  and  $|\int f \cdot g d\mu| < 1$  for all  $g \in H_\infty(\mu)$ ,  $\|g\| = 1$ ,  $g \neq \lambda x$ ,  $|\lambda| = 1$ ;
- (3)  $\mu\{t \in S : |x(t)| = 1\} > 0$ .

Now let us recall that if  $A \subset C(K)$  is a uniform algebra and  $\mu$  is a unique representing measure for  $\Phi \in \mathfrak{M}(A)$ , then  $A \subset L_\infty(\mu)$  is a  $w^*$ -Dirichlet algebra (cf. [3], Theorem 4.2.10). In the proof of the next proposition we shall use the following

**LEMMA 1.6.** *If  $A \subset C(K)$  is a uniform algebra, then a  $\Phi \in \mathfrak{M}(A)$  admits a unique representing measure  $\mu$  if and only if*

$$\sup \{ \operatorname{Re} \Phi(f) : f \in A \text{ and } \operatorname{Re} f \leq a \} = \int a d\mu$$

for every  $a \in C_R(K)$ .

The proof of this lemma can be found in [3], Corollary 2.2.4.

The next proposition is a generalization of a result of Havin [10], who proved it for  $H_\infty$ . Our proof is similar to that of [10]. Kislakov [15] used an analogous result for the disc algebra to prove that it satisfies the Pelczyński property.

**PROPOSITION 1.7.** *Let  $A \subset C(K)$  be a uniform algebra,  $\Phi \in \mathfrak{M}(A)$ , and  $\mu$  a unique representing measure for  $\Phi$ . Then for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for every closed subset  $e$  of  $K$  with  $\mu(e) < \delta$  there exist  $k_e$  and  $K_e \in A$  such that*

- (a)  $|k_e(t)| + |K_e(t)| \leq 1$  for all  $t \in K$ ;
- (b)  $\sup_{t \in e} |K_e(t) - 1| \leq \varepsilon$ ,  $\int |K_e| d\mu < \varepsilon$ ;
- (c)  $\sup_{t \in e} |k_e(t)| < \varepsilon$ ,  $\int |1 - k_e| d\mu < \varepsilon$ .

**Proof.** Let us pick an  $a \in C_R(K)$  so that

$$\begin{aligned} a(t) &= -\mu(e)^{-1/2} \quad \text{for } t \in e, \\ -\mu(e)^{-1/2} &\leq a \leq -\mu(e)^{1/2}, \\ \mu\{t : a(t) < -\mu(e)^{1/2}\} &< 2\mu(e). \end{aligned}$$

Clearly,  $\int |a| d\mu < 2\sqrt{\mu(e)}$ , because  $\|\mu\| = \mu(K) = 1$ . By Lemma 1.6, we can find an  $h \in A$  such that  $\operatorname{Re} h \leq a$  and  $\int |\operatorname{Re} h| d\mu < 2\sqrt{\mu(e)}$ . Let us write  $\operatorname{Re} h = \omega$ ; we may assume without loss of generality that  $h = \omega + i\tilde{\omega}$ . Since  $\operatorname{Re} h < 0$ , we infer that  $h^{-1} \in A$ . We put  $K_e = \exp h^{-1}$ . Clearly,  $\|K_e\| \leq 1$ . For  $t \in e$  we have

$$|h^{-1}(t)| = \frac{1}{\sqrt{\omega(t)^2 + \tilde{\omega}(t)^2}} \leq \frac{1}{|\omega(t)|} \leq \frac{1}{|a(t)|} = \sqrt{\mu(e)}.$$

Hence

$$\sup_{t \in e} |\exp h^{-1}(t) - 1| \rightarrow 0 \quad \text{as } \mu(e) \rightarrow 0.$$

Let us put  $U_e = \{t \in K : |\tilde{\omega}(t)| \geq \mu(e)^{3/8}\}$ . Since the operator  $\sim$  is of weak type 1-1, there is an absolute constant  $C > 0$  such that

$$\mu(U_e) \leq C\mu(e)^{-3/8} \int |\omega(t)| d\mu \leq 2C\mu(e)^{1/8}.$$

Next let us put  $V = \{t \in K : \omega(t) < -2\sqrt{\mu(e)}\} = \{t \in K : |\omega(t)| > 2\sqrt{\mu(e)}\}$ . The following easy calculation shows that  $\mu(V) \leq 2\mu(e)$ :

$$\begin{aligned} 2\sqrt{\mu(e)} &\geq \int |\omega| d\mu = \int_{V \setminus e} |\omega| d\mu + \int_e |\omega| d\mu + \int_{K \setminus V} |\omega| d\mu \\ &\geq \sqrt{\mu(e)} + (\mu(V) - \mu(e))2\sqrt{\mu(e)} + (1 - \mu(V))\sqrt{\mu(e)}. \end{aligned}$$

Thus  $\mu(V) \leq 2\mu(e)$ .

Let us observe that for  $t \in (K \setminus V) \setminus U_e$  we have

$$\begin{aligned} |\operatorname{Re} h^{-1}(t)| &= \frac{\omega(t)}{|\omega(t) + i\tilde{\omega}(t)|^2} \geq \frac{|\omega(t)|}{\omega^2(t) + \mu(e)^{6/8}} \\ &\geq \frac{\sqrt{\mu(e)}}{4\mu(e) + \mu(e)^{6/8}} \geq \frac{1}{4\sqrt{\mu(e)} + \mu(e)^{1/4}} \geq \frac{1}{5\mu(e)^{1/4}}. \end{aligned}$$

Hence

$$\begin{aligned} \int |K_e| d\mu &= \int \exp(\operatorname{Re} h^{-1}) d\mu \leq \left( \int_V + \int_{U_e} + \int_{(K \setminus V) \setminus U_e} \right) \exp(\operatorname{Re} h^{-1}) d\mu \\ &\leq \mu(V) + \mu(U_e) + \exp(-5\mu(e)^{-1/4}) \\ &\leq 2\mu(e) + 2C\mu(e)^{1/8} + \exp(-5\mu(e)^{-1/4}). \end{aligned}$$

Therefore,  $\int |K_e| d\mu \rightarrow 0$  as  $\mu(e) \rightarrow 0$ . This completes the proof of (b).

Now we consider the function  $\beta = |\log(1 - |K_e|)| \in C_R(K)$ . By Lemma 1.6 there is a  $v_e \in C_R(K)$   $v_e + i\tilde{v}_e \in A$  and  $v_e \geq \beta$  and  $\int v_e d\mu \leq 2 \int \beta d\mu$ . Now we put  $k_e = \exp(-v_e - i\tilde{v}_e)$ . Clearly,

$$|K_e(t)| + |k_e(t)| = |K_e(t)| + \exp(-v_e) \leq |K_e| + \exp(-\beta) = 1,$$

which verifies (a). Moreover, (a) and (b) imply  $\sup_{t \in \mathbb{C}} |k_e(t)| \leq \varepsilon$ .

To complete the proof we have to show that

$$\int |k_e - 1| d\mu \rightarrow 0 \quad \text{as} \quad \mu(e) \rightarrow 0.$$

To this end we shall need the following

**SUBLEMMA.** *There exists an absolute constant  $C > 0$  such that for all  $y \leq 0$*

$$[\log(1 - \exp y)]^2 \leq C(1 + |y|^{-1/4}).$$

The existence of such a  $C$  for  $y \leq -1$  is obvious and the existence for  $-1 \leq y \leq 0$  follows from the fact that

$$\lim_{y \rightarrow 0^-} |y|^{1/8} \log(1 - \exp y) = 0.$$

Now we shall show that  $\sup \{ \int |v_e|^2 d\mu : e \subset K, e \text{ closed} \} < \infty$ . Indeed, we have

$$\begin{aligned} \int |v_e|^2 d\mu &\leq \int |\log(1 - K_e)|^2 d\mu \leq \int \left| \log \left( 1 - \exp \frac{\omega}{\omega^2 + \tilde{\omega}^2} \right) \right|^2 d\mu \\ &\leq C + C \int \left| \frac{\omega}{\omega^2 + \tilde{\omega}^2} \right|^{-1/4} d\mu = C + C \int \sqrt[4]{|\omega| + \frac{\tilde{\omega}^2}{|\omega|}} d\mu \\ &\leq C + C \int |\omega|^{1/4} d\mu + C \int \frac{|\tilde{\omega}|^{1/2}}{|\omega|^{1/4}} d\mu \\ &\leq C + C \int |\omega|^{1/4} d\mu + C \mu(e)^{-1/8} \int \sqrt{|\tilde{\omega}|} d\mu \\ &\leq C + C \int |\omega|^{1/4} d\mu + C_1 C \mu(e)^{-1/8} \int |\omega| d\mu \\ &\leq C + 16C + 2C_1 C \mu(e)^{3/8} \leq 17C + 2C_1 C \end{aligned}$$

where  $C_1$  is an absolute constant whose existence is implied by the 1-1 weak type of the conjugation operator  $\sim$ . We also use the inequality  $\int |\omega|^{1/4} d\mu \leq (\int |\omega| d\mu)^{1/4} \leq (2\sqrt{\mu(e)})^{1/4} \leq 16$ .

Since  $\{v_e\}$  is bounded in  $L_2(\mu)$  and since  $v_e \rightarrow 0$  in measure as  $\mu(e) \rightarrow 0$ , we infer that  $\int |v_e| d\mu \rightarrow 0$  as  $\mu(e) \rightarrow 0$ . Because  $\sim$  is of the weak type 1-1, we infer that if  $\mu(e) \rightarrow 0$  then  $\tilde{v}_e \rightarrow 0$  in measure, and so  $v_e + i\tilde{v}_e \rightarrow 0$  in

measure. The last implies that  $k_e - 1 = \exp(-v_e - i\tilde{v}_e) - 1 \rightarrow 0$  in measure. Since  $\|k_e\|_\infty \leq 1$ , we get  $\int |k_e - 1| d\mu \rightarrow 0$  as  $\mu(e) \rightarrow 0$ . This completes the proof.

2. We start this section with several known lemmas.

**LEMMA 2.1** (cf. Grothendieck [9], Pełczyński [18]). *Let  $(\mu_n)$  be a non-weakly relatively compact sequence of Borel measures on a compact Hausdorff space  $K$ . Then there exist a  $\delta > 0$ , a subsequence  $(\mu_{n_k})$ , and a sequence  $(G_k)$  of mutually disjoint open sets such that  $|\mu_{n_k}(G_k)| \geq \delta$  for  $k = 1, 2, \dots$*

**LEMMA 2.2.** *Let  $(\mu_n)$  be a bounded sequence of Borel measures on a compact Hausdorff space  $K$ . Then there exist disjoint closed sets  $(e_k)$  and a subsequence of indices  $(n_k)$  such that  $\mu_{n_k}|_{(K \setminus e_k)}$  is weakly relatively compact.*

This lemma was formulated by Kislakov [15]. Its proof is a modification of the proof of Theorem 6 of [13]. The next lemma goes back to Orlicz; for a simple proof cf. [20], Lemma 7.1.

**LEMMA 2.3.** *If  $(x_n^*) \subset X^*$  is weakly relatively compact and  $(x_n) \subset X$  is w.u.c., then*

$$\limsup_{k, n} |x_n^*(x_k)| = 0.$$

Let us mention (cf. [18]) that  $(f_n) \subset C(K)$  is w.u.c. if and only if there exists a constant  $M$  such that  $\sum_{n=1}^{\infty} |f_n(t)| \leq M$  for all  $t \in K$ .

Now we are ready to prove the main result of the present paper.

**THEOREM 2.4.** *Let  $A$  be a uniform algebra on a compact Hausdorff space  $K$ . Assume that there exists a sequence  $(\Phi_n)_{n=1}^{\infty}$  in  $\mathfrak{M}(A)$  such that each  $\Phi_n$  admits a unique representing measure  $\mu_n$  on  $K$ , the measures  $\mu_n$  are pairwise singular and there is no measure  $\mu \neq 0$  belonging to the annihilator  $A^\perp = \{\mu \in C(K)^* : \mu|_A = 0\}$  and orthogonal to all the  $\mu_n$ 's.*

*Then  $A^*$  and  $A$  have the Dunford-Pettis property,  $A^*$  is weakly sequentially complete, and  $A$  has the Pełczyński property.*

**Proof.** It follows from [8], Chapter VI, § 2 that each  $v \in A^\perp$  admits a decomposition  $v = \sum_{n=1}^{\infty} v_n$ ,  $v_n \ll \mu_n$ ,  $v_n \in A^\perp$ . Since  $A^* = C(K)^*/A^\perp$ , we infer that

$$A^* = L_1 + \left( \sum_{n=1}^{\infty} L_1(\mu_n) / H_\infty^\perp(\mu_n) \right)_1$$

where  $L_1$  denotes the space of all measures singular with respect to all  $\mu_n$ 's. To see that  $A^*$  has the Dunford-Pettis property and is weakly sequentially complete we apply Theorem 1.4 and the easy observation that the above properties are stable under taking  $l_1$ -sums.  $A$  has the Dunford-Pettis property because  $A^*$  has.

Now let us concentrate on the Pelczyński property. Since the supporting functionals are norm dense in  $A^*$  (cf. [2]), it is enough to prove that for a given non-weakly relatively compact sequence  $(a_n^*) \subset A^*$ ,  $\|a_n^*\| = 1$ ,  $a_n^*$  supporting functionals, there exists a w.u.c.  $(f_n) \subset A$  such that

$$\limsup_{n, k} |a_k^*(f_n)| > 0.$$

Let  $a_n^*$  support  $g_n \in A$ ,  $\|g_n\| = 1$  and let  $\nu_n$  be the Hahn-Banach extension of  $a_n^*$  to  $O(K)$ .

Consider the probability measures  $\eta_n = g_n \cdot \nu_n$  and let  $\mu$  be their  $w^*$ -limit point.

Let  $\alpha = \sum_{n=1}^{\infty} 2^{-n} \mu_n$  and let us consider the decompositions:

$$\begin{aligned} \mu &= \mu^s + \mu^a, & \mu^a &\leq \alpha, & \mu^s &\perp \alpha, \\ \eta_n &= \eta_n^s + \eta_n^a, & \eta_n^a &\leq \alpha, & \eta_n^s &\perp \alpha. \end{aligned}$$

We shall consider the following three cases:

I. *The sequence  $(\eta_n^s)$  is not weakly relatively compact.* By Lemma 2.1, passing to a subsequence if necessary, we infer that there exists a sequence of disjoint open sets  $G_n$  such that  $\eta_n^s(G_n) > \delta$ . Since  $\eta_n^s \perp \alpha$ , we can find a closed subsets  $F_n \subset G_n$  such that  $\eta_n^s(F_n) > \delta$  and  $\alpha(F_n) = 0$ ,  $n = 1, 2, \dots$ . Hence  $F_n$  are peak interpolation sets for  $A$  and this implies that there are  $h_n \in A$  such that  $h_n|_{F_n} = 1$ ,  $|\int h_n d\eta_n^a| < 2^{-n}$ ,  $|\int h_n d\eta_n^s| > \delta$  and  $\sum_{n=1}^{\infty} |h_n(t)| \leq 2$  for all  $t \in K$ . We put  $f_n = g_n \cdot h_n$ . Then  $(f_n)$  is w.u.c. and

$$|a_n^*(f_n)| = \left| \int g_n h_n d\nu_n \right| = \left| \int h_n d\eta_n \right| \geq \left| \int h_n d\eta_n^s \right| - \left| \int h_n d\eta_n^a \right| \geq \delta - 2^{-n}.$$

II. *The sequence  $(\eta_n^s)$  is weakly relatively compact and  $\mu^s \neq 0$ .* Since  $\mu^s \perp \alpha$  and  $A^\perp \subset L_1(\alpha)$ , we infer that there exists a peak set  $F$  such that  $\|\mu^s\| - \mu^s(F) \leq 4^{-1} \|\mu^s\|$ . Let  $g$  be a function which peaks on  $F$ . Using easy induction, we define sequences of indices  $(n_k)$  and  $(r_k)$  so that

$$\sum_k |g^{n_{2k+1}}(t) - g^{n_{2k}}(t)| \leq 2 \quad \text{for all } t \in K$$

and

$$2 \left| \int g^{n_{2k}} d\eta_{r_k} \right| \geq \|\mu^s\| \quad \text{and} \quad 3 \left| \int g^{n_{2k+1}} d\eta_{r_k}^a \right| < \|\mu^s\|.$$

Finally, we put  $f_k = (g^{n_{2k+1}} - g^{n_{2k}})g_{r_k}$  for  $k = 1, 2, \dots$ . Clearly,  $(f_k)$  is w.u.c. Using Lemma 2.3 and the conditions of the inductive definition, one can show that  $\limsup_{k, n} |a_n^*(f_k)| > 0$ .

III. *The sequence  $(\eta_n^s)$  is weakly relatively compact and  $\mu^s = 0$ .* By Lemma 2.2 we can pass to a subsequence to ensure the existence of closed, disjoint sets  $E_n \subset K$  such that the restricted measures  $\eta_n^a|(K \setminus E_n)$  form

a weakly relatively compact sequence. Thus, by Lemma 2.3, it is enough to construct a w.u.c.  $(f_n)$  such that

$$\limsup_{k, n} |(\eta_n^a|_{E_n})(f_k)| > 0.$$

Let  $\hat{\mu}$  be the  $w^*$ -limit point of  $\eta_n^a|_{E_n}$ . If  $\hat{\mu}$  is not absolutely continuous with respect to  $\alpha$ , we can apply case II. The only case to be considered is  $\hat{\mu} \ll \alpha$ . Then  $\hat{\mu} = \sum_{n=1}^{\infty} h_n \mu_n$ . Since  $\hat{\mu} \neq 0$ , we may assume without loss of generality that  $\mu_n$ 's are enumerated so that  $\|h_1 \cdot \mu_1\| > \delta$  for some  $\delta > 0$ .

Let  $\sigma = \sum_{n=2}^{\infty} h_n \mu_n$ . We can find a closed sets  $F_n$  such that  $\mu_1(F_n) = 0$  and  $\|\sigma\| - |\sigma|(F_n) < 2^{-n}$  for  $n = 1, 2, \dots$ . Next, for  $e_n = E_n \cup F_n$ , we pick  $k_n = k_{e_n}$  and  $K_n = K_{e_n}$  satisfying the conditions of Proposition 1.7 applied for  $\mu = \mu_1$ .

Let us observe that

$$\lim_n \int f k_n d\mu_1 = \int f d\mu_1 \quad \text{for all } f \in L_1(\mu_1),$$

$$\lim_{n \in e_n} |K_n(t) - 1| = 0,$$

and  $\int g d\hat{\mu}$  is the limit point of the sequence  $\int g d\eta_n^a|_{E_n}$ , for every  $g \in C(K)$ .

Using the above properties, we construct a sequence of indices  $(n_j)$  such that

$$(a) \quad \left| \int \left( \prod_{j=1}^s k_{n_j} \right) h_1 d\mu_1 - \int h_1 d\mu_1 \right| \leq \left( \sum_{j=1}^s 8^{-j} \right) \delta \leq 4^{-1} \delta,$$

$$(b) \quad \left| \int \left( \prod_{j=1}^s k_{n_j} \right) d\hat{\mu} - \int \left( \prod_{j=1}^s k_{n_j} \right) d\eta_{n_{s+1}}^a \right| \leq 2^{-s},$$

$$(c) \quad |k_{n_j}|_{e_{n_j}}| < 2^{-j} \quad \text{and} \quad |K_{n_j}|_{e_{n_j}} - 1| < 2^{-j}.$$

Now we put  $w_s = \prod_{j=1}^s k_{n_j}$  and  $f_s = w_s K_{n_{s+1}}$ .

Claim 1.  $(f_s)$  is w.u.c. Indeed,

$$\begin{aligned} \sum_{s=1}^n |f_s| &= \sum_{s=1}^n |w_s| |K_{n_{s+1}}| \leq \sum_{s=1}^n |w_s| (1 - |k_{n_{s+1}}|) = \sum_{s=1}^n (|w_s| - |w_{s+1}|) \\ &= |w_1| - |w_{n+1}| \leq 2. \end{aligned}$$



Claim 2.  $\lim_{s \rightarrow \infty} |(\eta_{n_{s+1}}^\alpha | E_{n_{s+1}}) f_s| \geq \frac{3}{4} \delta$ . Indeed, we have

$$\begin{aligned} |(\eta_{n_{s+1}}^\alpha | E_{n_{s+1}})(f_s)| &= \left| \int_{E_{n_{s+1}}} w_s K_{n_{s+1}} d\eta_{n_{s+1}}^\alpha \right| \\ &= \left| \int_{E_{n_{s+1}}} w_s d\eta_{n_{s+1}}^\alpha + \int_{E_{n_{s+1}}} w_s (K_{n_{s+1}} - 1) d\eta_{n_{s+1}}^\alpha \right| \\ &\geq \left| \int_{E_{n_{s+1}}} w_s d\eta_{n_{s+1}}^\alpha \right| - 2^{-s-1} \geq \left| \int w_s d\mu \right| - 2^{-s} - 2^{-s-1} \\ &= \int w_s h_1 d\mu_1 + \int w_s d\sigma - 2^{-s} - 2^{-s-1} \\ &\geq \left| \int w_s h_1 d\mu_1 \right| - \left| \int w_s d\sigma \right| - 2^{-s} - 2^{-s-1} \\ &\geq \frac{3}{4} \delta - 2 \cdot 2^{-s} - 2^{-s-1} - 2^{-n_s}. \end{aligned}$$

In the last inequality we have used the fact that  $h_1 \geq 0$ , which yields  $\int h_1 d\mu_1 = \|h_1\|_1$ , and the estimate

$$\begin{aligned} \left| \int_K w_s d\sigma \right| &\leq \left| \int_{K \setminus F_{n_s}} w_s d\sigma \right| + \left| \int_{F_{n_s}} w_s d\sigma \right| \\ &\leq |\sigma|(K \setminus F_{n_s}) + \max_{t \in F_{n_s}} |w_s(t)| \cdot \|\sigma\| \leq 2^{-n_s} + 2^{-s}. \end{aligned}$$

The proof of Theorem 2.4 is complete.

Remark 2.5. (a) The proof of case II follows the construction of Kahane [14] and Chaumat [4].

(b) Our proof of the Dunford–Pettis property does not depend on the separability of the annihilator (cf. Delbaen [6]).

(c) The fact that  $A^*$  is weakly sequentially complete follows directly from the fact that  $A$  satisfies the Pełczyński property (cf. [17]). We do not have to use Chaumat's result.

Now we list some concrete algebras which satisfy the assumptions of Theorem 2.4. Let  $K$  be a compact subset of the complex plane. Let  $P(K)$  be the uniform algebra of complex-valued functions, which are uniformly approximated on  $K$  by polynomials in  $z$ ,  $A(K)$  — the algebra of functions continuous on  $K$  and analytic on  $\text{Int} K$ , and  $R(K)$  — the algebra of functions which are uniformly approximated on  $K$  by rational functions with poles off  $K$ .

Then we have the following

**THEOREM 2.6.** *Let  $K$  be a compact subset of a complex plane. The following algebras, when considered on their Shilov boundaries, satisfy the assumptions of Theorem 2.4.*

(a)  $P(K)$  for every compact  $K$ .

(b)  $R(K)$ , whenever it is a Dirichlet algebra.

(c)  $A(K)$ , whenever the complement of  $K$  is connected.

Therefore these algebras have the Dunford–Pettis property and the Pełczyński property and their duals are weakly sequentially complete.

**Proof.** All those facts follow from the theorems contained in Chapter II of [8]: (a) follows from 3.4, 1.4 and 8.5, (b) follows from 8.5 and (c) follows from Mergelyan theorem 9.1.

For conditions which ensure that  $R(K)$  is a Dirichlet algebra see [16].

**Remark 2.7.** Let us observe that the conclusion of Theorem 2.4 is invariant under Banach space isomorphisms while the assumptions are not. For example, the algebra  $R(K_0)$  where  $K_0 = \{z \in \mathbb{C} : 1/2 \leq |z| \leq 1\}$  does not satisfy the assumptions of Theorem 2.4. On the other hand, it is quite easy to see that  $R(K_0)$  is, as a Banach space, isomorphic to the disc algebra, and so the conclusions of Theorem 2.4 hold for  $R(K_0)$ .

It would be interesting to have any non-trivial information on the linearly-topological classification of the algebras  $A(K)$ ,  $P(K)$ , and  $R(K)$ .

**Remark 2.8.** If an algebra  $A$  satisfies the assumptions of Theorem 2.4, it behaves exactly like a  $C(K)$ -space as far as the characterization of weakly compact operators from the algebra is concerned. However, the results of [22] and [19] show that, unless  $A = C(K)$ ,  $A$  is not isomorphic to any  $\mathcal{L}_\infty$ -space. In fact the local properties of such algebras are rather different from the properties of  $C(K)$  spaces, cf. [20].

**Remark 2.9.** F. Delbaen [7] generalized our Proposition 1.7. Namely, he replaced the assumption that  $\mu$  is a unique representing measure by the assumption that the set of representing measures is weakly compact.

**Added in proof.** A much simpler proof of Proposition 1.7 can be obtained using the proof of the lemma from the paper J. B. Garnett, *Two remarks on interpolation by bounded analytic functions*, Banach spaces of analytic functions, Springer Verlag Lecture Notes in Math. 604 (1977).

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Received October 21, 1976

(1217)

## The Pełczyński property for some uniform algebras

by

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**Abstract.** Let  $A$  be a separable uniform algebra on its Shilov boundary  $X$ . If there are no singular orthogonal measures and if each element of the spectrum has a weakly compact set of representing measures, then  $A$  has the Pełczyński property. The theorem can be applied in the case of Banach algebras of analytic functions on suitable compact sets of the plane.

**1. Introduction.** In [11] Wojtaszczyk proved that some uniform algebras have the Pełczyński property. For the disc algebra this property was already known (see Kislakov [9] and Delbaen [2]). The present paper generalizes the results of [11] in two ways:

- (i) the separability of the annihilator is dropped, and
- (ii) the unicity of the representing measure is replaced by an assumption of weak compactness.

For any unexplained notion on uniform algebras we refer to Gamelin [5].

**2. Wilken algebras.** If  $A$  is a point separating subalgebra of  $\mathcal{C}(X)$  ( $X$  a compact space) such that  $1 \in A$ , then we say that  $A$  is a *uniform algebra*. For simplicity we assume that  $X$  is the Shilov boundary of  $A$ . A positive measure  $m$  on  $X$  is *multiplicative* if  $\int f \cdot g \, dm = \int f \, dm \cdot \int g \, dm$  for all  $f$  and  $g$  in  $A$ . From the Hahn–Banach theorem we learn that every nonzero multiplicative linear functional on  $A$  can be represented by such a measure. If  $\mu$  is any measure on  $X$ , then  $\mu$  is called *orthogonal* when  $\int f \, d\mu = 0$  for all  $f \in A$ .

**DEFINITION.** An algebra is called a *Wilken algebra* if the only orthogonal measure which is singular to all multiplicative measures is the zero measure. (Wilken (see [5]) proved that  $R(K)$  is such an algebra).

**Notation.** If  $X$  is a compact metric space which is the Shilov boundary for the uniform algebra  $A \subset \mathcal{C}(X)$ , then we denote by  $\partial X$  the set of peak points for  $A$ . This set is equal to the Choquet boundary of  $A$ .

**THEOREM 1.** Let  $A$  be a Wilken algebra on the compact metric space  $X$ .