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Fixed-point theorems for mappings defined on unbounded sets in Banach spaces *

by

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Abstract. Let K be a closed convex subset of a uniformly convex Banach space X and $T: K \to K$ a nonexpansive (or more generally, a lipschitzian pseudo-contractive) mapping. It is shown that if there exists a point a in K for which the set $\{z \in K: \|z - Ta\| \leqslant \|z - a\|\}$ is bounded, then T has a fixed point in K. Related results include the fact that surjective nonexpansive mappings, and even surjective asymptotically nonexpansive mappings, always have fixed points when defined on sufficiently sharp cones in X.

1. Introduction. In this paper we study the problem of the existence of fixed points for mappings $T: K \to K$ where K is an unbounded closed convex subset of a Banach space (usually uniformly convex) and T is either a *nonexpansive mapping* (i.e., $\|Tx - Ty\| \leqslant \|x - y\|$, $x, y \in K$), or a mapping of more general type, specifically pseudo-contractive or asymptotically nonexpansive. (These mappings are defined later.) Our investigation is prompted by a recent paper of Goebel and Kuczumow [7] in which a similar problem is treated in l_2.

The theorem of Browder–Göhde–Kirk [1], [8], [12] always assures existence of a fixed point for nonexpansive mappings $T: K \to K$ where K is a *bounded* closed convex subset of a uniformly convex space X, while at the same time it is clear that for a wide class of unbounded closed convex sets K (e.g., subsets of Hilbert space which contain an infinite ray) nonexpansive mappings $T: K \to K$ may exist which fail to have a fixed point. However, it is shown in [7] that certain unbounded closed convex sets K in l_2 have the property: $\inf\{\|x - Tx\|: x \in K\} = 0$ for nonexpansive $T: K \to K$. To describe this class, suppose $B \subset l_2$ is a set of the type:

$$B = \{x = (x_1, x_2, \ldots) \in l_2: |x_i| \leqslant M_i\}$$

where the M_i are fixed positive real numbers. Obviously, such a set B is bounded if and only if $\sum_{i=0}^{\infty} M_i^2 < \infty$, and the principal result of [7] states that if K is a (possibly unbounded) convex set contained in such a set B and if $T: K \to K$ is nonexpansive, then $\inf\{\|x - Tx\|: x \in K\} = 0$.

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We do not succeed here in obtaining another class of unbounded closed convex sets whose members possess the above property (or the fixed point property) with respect to nonexpansive self-mappings. Indeed, the general question as to whether a closed convex set K in a Banach space must be bounded if it has the fixed point property with respect to nonexpansive self-mappings apparently remains open. Although we do show in Section 5 that certain unbounded convex sets may possess the fixed point property for *surjective* nonexpansive mappings, thus partially responding to the more general question, our basic results run peripheral to this taking as point of departure another result in [7] which asserts that if K is a closed convex subset of l_2 and if $T\colon K\to K$ is a nonexpansive mapping for which there exists a point $x\in K$ such that the set

$$G_x = \{z\in K\colon \langle z-x, Tx-x\rangle \geqslant 0\}$$

is bounded, then T has a fixed point in K. We begin our discussion by giving simple extensions of this result to much wider classes of spaces. Our development then turns to more intricate observations about the geometry of Banach spaces which we apply to obtain further extensions of our more basic results to wider classes of mappings. While these geometric observations (of Section 3) constitute a significant feature of this paper, we obtain as a result the fact that if K is an unbounded closed convex subset of a uniformly convex space X and if $T\colon K\to K$ is a lipschitzian pseudo-contractive mapping for which the set $\{z\in K\colon \|z-Ta\| \leqslant \|z-a\|\}$ is bounded for some $a\in K$, then T has a fixed point in K. As noted above, our results also yield a fixed point theorem for surjective nonexpansive mappings defined on sufficiently 'sharp' cones in X, a theorem we are able to extend to the class of asymptotically nonexpansive mappings under slightly strengthened assumptions on the cone. We conclude by giving a characterization of the types of cones to which our results apply.

2. Preliminary results. We begin with a definition which is used throughout the paper. For a specified set K in a normed linear space X and given points $x, y\in X$ we use $G(x, y)$ to denote the set of points of K which are nearer y than x, i.e.,

$$G(x, y) = \{z\in K\colon \|z-x\| \geqslant \|z-y\|\}. \tag{2.1}$$

The first theorem of this section extends Theorem 2 of [7] from l_2 to the class of spaces in which bounded closed convex sets have the fixed point property with respect to nonexpansive self-mappings. This class of spaces includes all uniformly convex spaces ([1], [8]) and more generally, all reflexive spaces whose bounded convex subsets possess 'normal structure' (see [12]).

THEOREM 2.1. *Let X be a Banach space whose bounded closed convex subsets have the fixed point property relative to nonexpansive self-mappings, let K be a closed convex subset of X, and suppose $T\colon K\to K$ is a nonexpansive mapping. If there exists $u\in K$ such that the set $G(u, Tu)$ is bounded, then T has a fixed point in K.*

Proof. We reduce the problem to the bounded case. Let $R = 4\sup\{\|z-Tu\|\colon z\in G(u, Tu)\}$, and let $S = B(Tu; R)\cap K$. Since $Tu\in S$, $S\neq\emptyset$. We distinguish two cases:

(1) Suppose $z\in S\cap G(u, Tu)$. Then $\|z-u\| \leqslant \|z-Tu\|+\|u-Tu\|$. Since $\frac{1}{2}(u+Tu)\in G(u, Tu)$, it follows that $\|u-Tu\| \leqslant \frac{1}{2}R$; thus $\|z-u\| \leqslant \frac{3}{4}R < R$ whence by nonexpansiveness of T, $\|Tz-Tu\| \leqslant R$.

(2) If $z\in S$ and $z\notin G(u, Tu)$, then $\|Tz-Tu\| \leqslant \|z-u\| < \|z-Tu\| \leqslant R$.

Thus in either case $Tz\in B(Tu; R)$ from which $T\colon S\to S$, completing the proof.

DEFINITION ([10]). Let X be a linear space with $K\subset X$. For $x\in K$, define the *inward set*, $I_K(x)$, of x with respect to K as follows:

$$I_K(x) = \{x+\lambda(z-x)\colon z\in K, \lambda\geqslant 1\}.$$

A mapping $T\colon K\to X$ is said to be *weakly inward* if $Tx\in\overline{I_K(x)}$ for each $x\in K$.

We now prove a substantial generalization of Theorem 2.1 for uniformly convex spaces.

THEOREM 2.2. *Let X be a uniformly convex Banach space, K a closed and convex subset of X, and $T\colon K\to X$ a weakly inward nonexpansive mapping. Suppose for some bounded set $A\subset K$ the set*

$$G(A) = \bigcap_{a\in A} G(a, Ta)$$

is either empty or bounded. Then T has a fixed point in K.

This theorem follows immediately from Theorem 2.3 below and a well-known fact about nonexpansive mappings in uniformly convex spaces: If T is defined on a closed convex subset D of a uniformly convex space X then the mapping $f = I-T$ is *demi-closed* on D, i.e., if $\{x_n\}\subset D$ satisfies $x_n\to x$ weakly while $f(x_n)\to y$ strongly, then $f(x) = y$ (cf. [3], [8]).

THEOREM 2.3. *Let X be a Banach space, K a closed convex subset of X, and suppose $T\colon K\to X$ is nonexpansive and weakly inward on K. Suppose for some bounded set $A\subset K$ that the set*

$$G(A) = \bigcap_{a\in A} G(a, Ta)$$

is either empty or bounded. Then there exists a bounded sequence $\{x_n\} \subset K$ *such that* $\|x_n - Tx_n\| \to 0$ *as* $n \to \infty$.

Proof. We may suppose without loss of generality that $0 \in K$. For $a \in (0,1)$ define $T_a\colon K \to X$ by $T_a x = aTx$. Then clearly T_a is a weakly inward contraction mapping and thus has a fixed point $x_a \in K$ by Theorem 2.1 of [4]. Suppose the set $\{x_a\colon a \in (0,1)\}$ is unbounded. Then it is possible to choose $a \in (0,1)$ so that

$$\sup_{a \in A} \|Ta\| \leqslant \inf_{a \in A} \|x_a - a\|,$$

and in addition if $G(A) \neq \emptyset$, then a may also be chosen so that

$$\|x_a\| > \sup\{\|x\|\colon x \in G(A)\}.$$

It follows that, for each $a \in A$,

$$\begin{aligned}\|x_a - Ta\| &= \|aTx_a - Ta\| \leqslant a\|Tx_a - Ta\| + (1-a)\|Ta\| \\ &\leqslant a\|x_a - a\| + (1-a)\|x_a - a\| = \|x_a - a\|.\end{aligned}$$

This implies $x_a \in G(A)$, a contradiction. Thus $M = \sup\{\|x_a\|\colon a \in (0,1)\} < \infty$ and we have

$$\|x_a - Tx_a\| = (a^{-1}-1)\|x_a\| \leqslant (a^{-1}-1)M,$$

yielding $\|Tx_a - x_a\| \to 0$ as $a \to 1$.

3. Geometric lemmas. Many of our subsequent results depend on facts concerning the geometry of uniformly convex spaces.

If K is a given subset of a normed linear space X, for $x, y \in K$ and $\varepsilon > 0$ we define $G(x,y)$ as in (2.1) and let:

$$\begin{aligned}R(x,y) &= \sup\{\|z-y\|\colon z \in G(x,y)\}, \\ E(x,y) &= \{z \in G(x,y)\colon \|z-x\| = \|z-y\|\}, \\ G(x,y;\varepsilon) &= \{z \in K\colon \|z-x\| > \|z-y\| - \varepsilon\}.\end{aligned} \tag{3.1}$$

The mapping $\delta\colon [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - \tfrac{1}{2}\|x+y\|\colon \|x\|, \|y\| \leqslant 1,\ \|x-y\| \geqslant \varepsilon\}$$

is called the *modulus of convexity of* X. It follows immediately that if $x, y \in X$ with $\|x\|, \|y\| \leqslant r$ and $\|x-y\| \geqslant a$, then

$$\tfrac{1}{2}\|x+y\| \leqslant (1 - \delta(a/r))r. \tag{3.2}$$

It is known [9] that for any Banach space X the function δ is nondecreasing, zero at 0, and continuous on $[0,2)$. Also, if $\varepsilon_0 = \sup\{\varepsilon\colon \delta(\varepsilon) = 0\}$, then clearly X is uniformly convex if and only if $\varepsilon_0 = 0$.

Finally, if $G(x,y)$ is bounded for $x, y \in X$, $x \neq y$, (relative to given $K \subset X$), then we define:

$$\varepsilon(x,y) = \delta(\|x-y\|/R(x,y))\|x-y\|. \tag{3.3}$$

Lemma 3.1. *Let K be a closed and convex subset of a uniformly convex space X, and suppose the set $G(x,y)$ is bounded for some pair x, y with $x \in X$, $y \in K$. Suppose $u, v \in X$ satisfy: $\|x-u\| \leqslant \varepsilon/2$, $\|y-v\| \leqslant \varepsilon/2$, where $\varepsilon = \varepsilon(x,y)$. Then:*

(a) $\frac{1}{2}[G(x,y;\varepsilon) + y] \subset G(x,y)$, *and*

(b) $G(u,v)$ *is bounded.*

In our proof of the above we shall need the following trivial fact (which holds in arbitrary spaces X).

Lemma 3.2. *For fixed $z, x \in X$, the mapping $t \mapsto \|tz - x\| - \|tz\|$ is non-increasing in t for $t \geqslant 0$.*

Proof. For $t, h \geqslant 0$,

$$\begin{aligned}\|(t+h)z - x\| - \|(t+h)z\| - [\|tz-x\| - \|tz\|] &= \|(t+h)z - x\| - \|tz-x\| - h\|z\| \\ &\leqslant \|hz\| - h\|z\| = 0.\end{aligned}$$

Proof of Lemma 3.1. (a) Set $\tilde{K} = K - y$, $\tilde{x} = x - y$, and define $\tilde{G}(\tilde{x},0)$, $\tilde{R}(\tilde{x},0)$, $\tilde{E}(\tilde{x},0)$, $\tilde{G}(\tilde{x},0;\varepsilon)$, and $\tilde{\varepsilon}(\tilde{x},0)$ as in (2.1), (3.1), and (3.2) but relative to the set $\tilde{K}$ rather than K. It follows that

$$\begin{aligned}\tilde{G}(\tilde{x},0) &= G(x,y) - y, \\ \tilde{R}(\tilde{x},0) &= R(x,y), \\ \tilde{G}(\tilde{x},0;\varepsilon) &= G(x,y;\varepsilon) - y, \\ \tilde{\varepsilon}(\tilde{x},0) &= \varepsilon(x,y).\end{aligned} \tag{3.4}$$

Thus to prove (a) it will suffice to show that

$$\tfrac{1}{2}\tilde{G}(\tilde{x},0;\varepsilon) \subset \tilde{G}(\tilde{x},0)$$

for $\varepsilon = \tilde{\varepsilon}(\tilde{x},0)$. We begin by showing

$$\|2z - \tilde{x}\| - \|2z\| \leqslant -\varepsilon \quad \text{for} \quad z \in \tilde{E}(\tilde{x},0). \tag{3.5}$$

Let $z \in \tilde{E}(\tilde{x},0)$ and set $r = \|z - \tilde{x}\| = \|z\|$. Then by (3.2),

$$\tfrac{1}{2}\|(\tilde{x}-z) + (-z)\| \leqslant (1 - \delta(\|\tilde{x}\|/r))r \leqslant (1 - \delta(\|x-y\|/R(x,y)))\|z\|$$

from which (since $\|w-y\| \geqslant \frac{1}{2}\|x-y\|$ for all $w \in E(x,y)$),

$$\begin{aligned}\|2z - \tilde{x}\| - \|2z\| &\leqslant (1-\delta)(\|x-y\|/R(x,y))\|2z\| - \|2z\| \\ &\leqslant -\delta(\|x-y\|/R(x,y))\|x-y\| \\ &\leqslant -\varepsilon.\end{aligned}$$

This establishes (3.5).

Now let $z \in \tilde{G}(\tilde{x},0;\varepsilon)$. We consider two cases:

(1) $z \in \tilde{G}(\tilde{x}, 0)$. Then, since $0 \in \tilde{K}$ and $\tilde{K}$ is convex, $\frac{1}{2}z \in \tilde{K}$. By Lemma 3.2, $\|\frac{1}{2}z - \tilde{x}\| - \|\frac{1}{2}z\| \geq \|z - \tilde{x}\| - \|z\| \geq 0$ and thus $\frac{1}{2}z \in \tilde{G}(\tilde{x}, 0)$.

(2) $z \notin \tilde{G}(\tilde{x}, 0)$. In this case $\|z - \tilde{x}\| - \|z\| < 0$ and therefore there exists $\lambda < 1$ such that $\|\lambda z - \tilde{x}\| - \|\lambda z\| = 0$. This implies $\lambda z \in \tilde{E}(\tilde{x}, 0)$ and, in view of (3.5),

$$\|2\lambda z - \tilde{x}\| - \|2\lambda z\| \leq -\varepsilon.$$

Thus

$$\|z - \tilde{x}\| - \|z\| \geq -\varepsilon = -\varepsilon(x, y) \geq \|2\lambda z - \tilde{x}\| - \|2\lambda z\|.$$

By Lemma 3.2, $\frac{1}{2} \leq \lambda$, and another application of this lemma yields

$$\|\tfrac{1}{2}z - \tilde{x}\| - \|\tfrac{1}{2}z\| \geq \|\lambda z - \tilde{x}\| - \|\lambda z\| = 0.$$

In either case it follows that $\frac{1}{2}z \in \tilde{G}(\tilde{x}, 0)$, proving (a).

Part (b) of Lemma 3.1 follows immediately from (a) upon observing that for $\varepsilon = \varepsilon(x, y)$, $G(u, v) \subset G(x, y; \varepsilon)$.

LEMMA 3.3. *Let K be a closed and convex subset of a uniformly convex space X and suppose for some $x \in X$ and $y \in K$ the set $G(x, y)$ is bounded. If $\alpha > \beta \geq 0$, then the set $G(\alpha x + (1-\alpha)y, \beta x + (1-\beta)y)$ is bounded.*

Proof. As in the proof of Lemma 3.1 we may assume without loss of generality that $y = 0 \in K$ and use boundedness of $G(x, 0)$ to imply boundedness of $G(\alpha x, \beta x)$.

First note that for $\lambda > 1$, $w \in G(\lambda x, 0)$ implies $(1/\lambda)w \in G(x, 0)$, from which it follows that $R(\lambda x, 0) \leq \lambda R(x, 0)$. Thus

$$\varepsilon(\lambda x, y) = \delta\left(\frac{\|\lambda x\|}{R(\lambda x, 0)}\right)\|\lambda x\| \geq \delta\left(\frac{\|x\|}{R(x, 0)}\right)\|\lambda x\|$$

and thus we may choose λ_0 so large that $\lambda_0 \geq \alpha$ and $\frac{1}{2}\varepsilon(\lambda_0 x, 0) \geq \alpha\|x\|$. By Lemma 3.1(b), $G(\lambda_0 x, \alpha x)$ is bounded. We claim $G(\alpha x, \beta x) \subseteq G(\lambda_0 x, \alpha x)$, for if this is not the case then $u \in G(\alpha x, \beta x)$ exists such that $u \notin G(\lambda_0 x, \alpha x)$ from which $\|u - \beta x\| \leq \|u - \alpha x\|$ and $\|u - \alpha x\| > \|u - \lambda_0 x\|$. Thus $\mathrm{seg}(\beta x, \lambda_0 x) \subset B(u; \|u - \alpha x\|)$ and since $\alpha x \in \mathrm{seg}(\beta x, \lambda_0 x)$, this contradicts strict convexity of the norm of X.

4. Pseudo-contractive mappings. In this section we apply Lemma 3.1 to extend Theorem 2.1 to an important class of mappings which includes the nonexpansive mappings. Let X be a Banach space and $D \subset X$. A mapping $T\colon D \to X$ is said to be *pseudo-contractive* if for each $r > 0$ and $u, v \in D$,

$$\|u - v\| \leq \|(1+r)(u-v) - r(Tu - Tv)\|. \tag{4.1}$$

Fixed point theorems for pseudo-contractive mappings play an important role in nonlinear mapping theory because of their connection with the accretive transformations. (For a discussion, see [13]. We remark that a well-known result, due independently to Browder [3] and Kato [11], characterizes pseudo-contractive mappings as those mappings T for which the mapping $f = I - T$ is accretive, i.e., $\mathrm{Re}\langle fx - fy, j\rangle \geq 0$ for some $j \in J(x-y)$ where $J\colon X \to 2^{X^*}$ is the normalized duality mapping defined by:

$$J(x) = \{j \in X^*\colon \langle j, x\rangle = \|x\|^2, \|j\| = \|x\|\}.)$$

THEOREM 4.1. *Let K be a closed and convex subset of the uniformly convex space X and let $T\colon K \to K$ be a lipschitzian pseudo-contractive mapping. Suppose for some $a \in K$ the set $G(a, Ta)$ is bounded. Then T has a fixed point in K.*

Proof. Suppose T has Lipschitz constant k and select $\alpha \in (0, 1)$ so that $\alpha k < 1$. Then for each $y \in K$ the mapping $T_y\colon K \to K$ given by $T_y(x) = (1-\alpha)y + \alpha Tx$ is a contraction mapping and hence has a fixed point $F_\alpha(y)$ for each $y \in K$. Thus,

$$F_\alpha(y) = (1-\alpha)y + \alpha T F_\alpha(y), \quad y \in K. \tag{4.2}$$

A standard argument shows that the mapping $F_\alpha\colon K \to K$ is nonexpansive on K, for if $r > 0$, then

$$\|u - v\| \leq \|(1+r)(u-v) - r(Tu - Tv)\|$$

and if, in addition, r is chosen so small that $\alpha(1+r) > r$, then

$$\begin{aligned}\|F_\alpha(u) - F_\alpha(v)\| &\leq \big\|(1+r)\big(F_\alpha(u) - F_\alpha(v)\big) - r\big(TF_\alpha(u) - TF_\alpha(v)\big)\big\| \\ &= \Big\|(1+r)\big(F_\alpha(u) - F_\alpha(v)\big) - \frac{r}{\alpha}\big(F_\alpha(u) - F_\alpha(v)\big) - \\ &\qquad -(1-\alpha)(u-v)\Big\| \\ &\leq \left[\frac{\alpha + r(\alpha-1)}{\alpha}\right]\|F_\alpha(u) - F_\alpha(v)\| + \left[\frac{r(1-\alpha)}{\alpha}\right]\|u - v\|\end{aligned}$$

from which

$$\|F_\alpha(u) - F_\alpha(v)\| \leq \|u - v\|.$$

To find a fixed point for T it suffices by (4.2) to find a fixed point for F_α, and, in view of Theorem 2.1, this can be accomplished by showing $G(a, F_\alpha(a))$ is bounded (for fixed $\alpha \in (0,1)$ sufficiently small).

Since $F_\alpha(a) = (1-\alpha)a + \alpha TF_\alpha(a)$, it will follow from Lemma 3.3 that $G(a, F_\alpha(a))$ is bounded if $G(a, TF_\alpha(a))$ is bounded. But

$$\|F_\alpha(a) - a\| = \alpha\|a - TF_\alpha(a)\| \leq \alpha k\|a - F_\alpha(a)\| + \alpha\|a - Ta\|$$

and hence $\|F_\alpha(a) - a\| \leq (1-\alpha k)^{-1}\alpha\|a - Ta\|$. Thus, given $\varepsilon > 0$, it is possible to choose a so small that $\|Ta-TF_a(a)\|<\varepsilon$ and boundedness of $G(a, TF_a(a))$ now follows from Lemma 3.1(b).

5. Mappings defined on cones. If the mapping $T\colon K\to K$ is surjective, then certain convex sets K must always possess points u such that the set $G(u, Tu)$ is bounded. This is true in particular when K is sufficiently 'sharp' cone. In this section we prove fixed point theorems for surjective nonexpansive and asymptotically nonexpansive mappings defined on such cones. Then, in Section 6, we characterize precisely the classes of cones for which our results are valid (in spaces X for which both X and X^* are uniformly convex).

DEFINITION 5.1. Let X be a normed linear space. A convex cone C in X with vertex 0 is said to be *acute* if for each nonzero x in C the set $G(x, 0)$ is bounded.

We begin with a simple consequence of our previous results.

THEOREM 5.1. *Let C be an acute cone in a uniformly convex space X and suppose $T\colon C\to C$ is nonexpansive. If T is surjective, or more generally, if there exists $\beta\in[0,1)$ and $x\in C$ such that $Tx=\beta x$, then T has a fixed point in C.*

Proof. Suppose $Tx=\beta x$ for $x\in C$ and $\beta\in[0,1)$. Since C is acute, the set $G(x, 0)$ is bounded and thus by Lemma 3.3 the set $G(x, Tx)$ $=G(x,\beta x)$ is bounded. The theorem now follows from Theorem 2.1.

Remarks. (1) In view of Theorem 4.1, Theorem 5.1 remains true if T is lipschitzian and pseudo-contractive rather than nonexpansive.

(2) The assumption $T\colon C\to C$ is also stronger than necessary (for nonexpansive T) in Theorem 5.1 because by appealing to Theorem 2.2 it is clear that the assumption $T\colon C\to X$ is weakly inward on C suffices.

Using Lemma 3.1 we are able to extend Theorem 5.1 to a wider class of mappings, but for a (presumably) more restricted class of cones.

DEFINITION 5.2. A convex cone C with vertex 0 in a normed linear space X is said to be *uniformly acute* if

$$\sup\{R(x,0)\colon x\in C,\ \|x\|=1\}<\infty. \tag{5.1}$$

Recall that a mapping $T\colon D\to D$ is *asymptotically nonexpansive* ([7]) if there exists a sequence $\{a_n\}$ of real numbers with $a_n\to 1$ such that, for each $x, y\in D$ and each n,

$$\|T^n x-T^n y\|\leqslant a_n\|x-y\|.$$

It is shown in [6] that such a mapping always has a fixed point for D a bounded closed convex subset of a uniformly convex space.

THEOREM 5.2. *Let X be a uniformly convex Banach space with C a uniformly acute cone in X, and suppose $T\colon C\to C$ is asymptotically nonexpansive and surjective. Then T has a fixed point in C.*

We derive Theorem 5.2 from the following

THEOREM 5.3. *Let X be a uniformly convex Banach space with K a closed and convex subset of X, and suppose that T is an asymptotically nonexpansive mapping of K into itself. Suppose further that there exists a sequence $\{a_n\}\subset K$ such that*

(i) $G(a_n, T^n a_n)$ is bounded for each n;

(ii) $\limsup_{n\to\infty}[\|a_n-T^n a_n\|/R(a_n, T^n a_n)]>0$.

Then T has a fixed point in K.

Proof of Theorem 5.2 from Theorem 5.3. Under the assumptions of Theorem 5.2 there exists for each n a point $a_n\in C$ such that $T^n a_n=0$. Suppose $M=\sup\{R(x,0)\colon x\in C,\ \|x\|=1\}$. Obviously, $\|a_n\|^{-1}G(a_n, T^n a_n)$ $=G(\|a_n\|^{-1}a_n, 0)$ so we obtain $R(a_n, T^n a_n)\leqslant\|a_n-T^n a_n\|M$. Thus, $\|a_n-T^n a_n\|/R(a_n, T^n a_n)\geqslant M^{-1}>0$, $n=1,2,\ldots$, and the assumptions of Theorem 5.3 hold.

Proof of Theorem 5.3. Since X is uniformly convex, (i) implies:

$$\limsup_{n\to\infty}[\varepsilon(a_n, T^n a_n)/R(a_n, T^n a_n)]>0. \tag{5.2}$$

Thus, for N sufficiently large,

$$4(a_N-1)R(a_N, T^N a_N)<\varepsilon(a_N, T^N a_N). \tag{5.3}$$

Set $\varepsilon=\varepsilon(a_N, T^N a_N)$ and $R=a_N R_0$ where

$$R_0=2\sup[\{\|z-T^N a_N\|\colon z\in G(a_N, T^N a_N;\varepsilon)\}, \|a_N-T^N a_N\|]. \tag{5.4}$$

By Lemma 3.1(a) and (i), $R_0<\infty$. Also Lemma 3.1(a) yields for all $z\in G(a_N, T^N a_N;\varepsilon)$:

$$\tfrac12(z+T^N a_N)\in G(a_N, T^N a_N).$$

Thus

$$\begin{aligned}\tfrac12 R_0&=\sup[\{2\|\tfrac12(z+T^N a_N)-T^N a_N\|\colon z\in G(a_N, T^N a_N;\varepsilon)\}, \|a_N-T^N a_N\|]\\&\leqslant\max[2R(a_N, T^N a_N), \|a_N-T^N a_N\|]\\&=2R(a_N, T^N a_N).\end{aligned}$$

Now we may rewrite (5.3) as

$$(a_N-1)R_0<\varepsilon. \tag{5.5}$$

Set $S=B(T^N a_N; R)\cap C$. We now show that $T^N\colon S\to S$. To see this suppose $z\in S$ and consider the two cases:

(1) $z\in G(a_N, T^N a_N;\varepsilon)$. Then, using (5.4),

$$\|T^N z - T^N a_N\| \leqslant a_N\|z-a_N\| \leqslant a_N[\|z-T^N a_N\| + \|T^N a_N - a_N\|] \leqslant a_N R_0 = R.$$

(2) $z \notin G(a_N, T^N a_N; \varepsilon)$. In this case we have (from (5.5))

$$\|T^N z - T^N a_N\| \leqslant a_N\|z-a_N\| \leqslant a_N\|T^N a_N - z\| - a_N\varepsilon \leqslant a_N R - a_N(a_N-1)R_0 = R.$$

Therefore, in either case, $T^N\colon S \to S$. By Theorem 1 of [6] the fixed point set F of T^N is nonempty (since T^N is an asymptotically nonexpansive mapping of S into S). Also Theorem 2 of [6] implies F is closed and convex, and moreover it readily follows that T is nonexpansive on F. Since $\{T^n x\}$ is bounded (i.e., finite) for $x \in F$, it follows from the Corollary in [12] that T has a fixed point in F.

6. Characterization of acute cones. In this section we provide characterizations of acute and uniformly acute cones.

We begin with some standard definitions (e.g., see Day [5], pp. 144–147. A Banach space X is said to be *smooth* provided

(6.1) $$\lim_{t\to 0} t^{-1}[\|x+th\| - \|x\|]$$

exists for each $x, h \in X$. When this is the case, the norm of X is said to be *Gâteaux differentiable*, and we denote the limit in (6.1) by $D_x(h)$. The space X is said to have *uniformly Gâteaux differentiable norm* if for each $h \in X$ the limit (6.1) is attained uniformly for x with $\|X\| = 1$, while X is said to be *uniformly smooth* if the limit (6.1) is attained uniformly for all x and h with $\|x\| = \|h\| = 1$.

It is known (cf. [5], p. 147) that X is uniformly smooth if and only if X^* is uniformly convex. In this case

(6.2) $$D_x = \|x\|^{-1} J(x)$$

where J is the duality map defined in Section 4. We note that if $D_x(h)$ exists for all $x, h \in X$, then Lemma 3.2 implies

(6.3) $$D_x(h) = \liminf_{\lambda\to\infty}[\|\lambda x + h\| - \|\lambda x\|].$$

Finally, suppose Σ^1 denotes the boundary of the unit ball in a smooth Banach space X and suppose C is a given cone in X. For each $x \in C$ set

(6.4) $$\beta(x) = \sup\{D_z(-x)\colon z \in C\cap\Sigma^1\}.$$

THEOREM 6.1. *Let C be a convex cone (with vertex 0) in a Banach space X where both X and X^* are uniformly convex. Then C is acute if and only if $\beta(x) < 0$ for each $x \in C\cap\Sigma^1$.*

Proof. Since $\|x\|^{-1}G(x, 0) = G(\|x\|^{-1}x, 0)$ for each nonzero x in C, we note that C is acute if and only if $G(x, 0)$ is bounded for each $x \in C\cap\Sigma^1$.

Now let $x \in C\cap\Sigma^1$ and suppose $\beta(x) = \gamma < 0$. Since the norm of X is uniformly Gâteaux differentiable, there exists λ_0 such that $\lambda \geqslant \lambda_0$ implies $\|\lambda z - x\| - \|\lambda z\| - D_z(-x) < -\gamma$ for each $z \in C\cap\Sigma^1$. Since $\gamma = \beta(x)$, this implies $\|\lambda z - x\| < \|\lambda z\|$. Thus if $y \in C$ and $\|y\| \geqslant \lambda_0$, it follows that $y \notin G(x, 0)$, i.e., the set $G(x, 0)$ is bounded.

Conversely, suppose $x \in C\cap\Sigma^1$ and suppose $G(x, 0)$ is bounded. If $\beta(x) \geqslant 0$, then for each n there exists $y_n \in C\cap\Sigma^1$ such that $D_{y_n}(-x) \geqslant -1/n$, and, in view of Lemma 3.2,

$$\|\lambda y_n - x\| - \|\lambda y_n\| \geqslant D_{y_n}(-x) \geqslant -1/n$$

for all $\lambda > 0$. Hence, for $\lambda > 0$, $\lambda y_n \in G(x, 0; 1/n)$. But by Lemma 3.1 (a), $\frac{1}{2}G(x, 0; 1/n) \subset G(x, 0)$ for n sufficiently large and so we have a contradiction.

Remarks. (1) We observe that the proof of sufficiency above only requires that X have a uniformly Gâteaux differentiable norm, while for necessity, it need be assumed only that X is uniformly convex.

(2) In Hilbert space the condition $\beta(x) < 0$ reduces to: $\inf\{\langle x, y\rangle\colon y \in C\cap\Sigma^1\} > 0$ for each $x \in C\cap\Sigma^1$. (Thus C is acute if and only if the angle of C is less than 90°.)

THEOREM 6.2. *Under the assumptions of Theorem 6.1, the cone C is uniformly acute if and only if $\sup\{\beta(x)\colon x \in C\cap\Sigma^1\} < 0$.*

Proof. Suppose $\sup\{\beta(x)\colon x \in C\cap\Sigma^1\} < 0$. We must show that $\sup\{R(x, 0)\colon x \in C\cap E^1\} < \infty$. Under our assumptions here, however, we may proceed as in the proof of Theorem 6.1, except that γ and λ_0 may be chosen so that $\lambda \geqslant \lambda_0$ implies $\|\lambda z - x\| - \|\lambda z\| - D_z(-x) < -\gamma$ for all $x \in C\cap\Sigma^1$. Thus $\sup\{R(x, 0)\colon x \in C\cap\Sigma^1\} < \lambda_0$.

For the necessity, suppose $\sup\{R(x, 0)\colon x \in C\cap\Sigma^1\} = \varrho < \infty$. Then for each $x \in C\cap\Sigma^1$ it follows that $\varepsilon(x, 0) \geqslant \delta(\varrho^{-1})$. Now assume $\sup\{\beta(x)\colon x \in C\cap\Sigma^1\} \geqslant 0$. Then it is possible to choose $x, y \in C\cap\Sigma^1$ so that $D_x(-y) \geqslant -\delta(\varrho^{-1})$. This leads to a contradiction in the same manner as in the proof of Theorem 6.1.

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Le groupe des isométries d'un espace de Banach

par

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Abstract. We characterize those groups G such that, for some Banach space E, G is isomorphic to the group of (norm-preserving) isometries of E: they are exactly the groups which have a normal subgroup with two elements.

Les isométries d'un espace de Banach E sont les automorphismes de E qui conservent la norme, c'est-à-dire les surjections linéaires $T\colon E\to E$ qui sont telles que

$$\forall x \ \|T(x)\| = \|x\|.$$

Le but du présent article est de caractériser les groupes d'isométries des espaces de Banach. Si G est le groupe des isométries de l'espace de Banach E, alors G a un sous-groupe normal à deux éléments formé de l'identité de la symétrie $\sigma\colon x\to -x$. Inversement, on a:

THÉORÈME 1. *Soit G un groupe qui a un sous-groupe normal à deux éléments et soit ε un nombre réel strictement positif. Il existe un espace de Banach $1+\varepsilon$-isomorphe à un espace de Hilbert et dont le groupe des isométries est isomorphe à G.*

On rappelle que deux espaces de Banach E et F sont dits $1+\varepsilon$-isomorphes s'il existe un isomorphisme $T\colon E\to F$ tel que $\|T\|\cdot\|T^{-1}\|\leqslant 1+\varepsilon$.

Remarques. 1. Soit i l'élément neutre de G et $\{i,j\}$ un sous-groupe à deux éléments. On peut alors préciser la conclusion du théorème 1 qui devient:

Il existe un espace de Banach E, $1+\varepsilon$-isomorphe à un espace de Hilbert et un isomorphisme τ de G sur le groupe des isométries de E tel que $\tau(i)$ soit l'identité de E et $\tau(j)$ la symétrie de E.

2. Si G est dénombrable, l'espace de Banach dont l'existence est affirmée par le théorème 1 peut être choisi séparable.

Pour établir le théorème 1, on prouvera au préalable le résultat suivant (suggéré à l'auteur par S. Shelah).

THÉORÈME 2. *Pour tout ensemble non vide X et pour tout $\varepsilon>0$, il existe un espace de Banach $1+\varepsilon$-isomorphe à $l^2(X)$ et qui n'admet comme isométries que l'identité et la symétrie.*