STUDIA MATHEMATICA, T. LXIV (1979)

# The weak Radon-Nikodym property in Banach spaces

by

KAZIMIERZ MUSIAŁ (Wrocław)

**Abstract.** The notion of the weak Radon–Nikodym property of a Banach space is introduced. The main result is that if a Banach space $X$ is separably complementable (i.e. each separable subspace of $X$ is contained in a separable and complemented subspace of $X$) then $X^*$ possesses the weak Radon–Nikodym property if and only if $X$ does not contain any isomorphic copy of $l_1$. This theorem yields some new results in the theory of Banach spaces and in the theory of Banach space valued functions (e.g., if a separable $X$ is weak* $\omega_1$-sequentially dense in $X^{**}$, then it is weak* sequentially dense in $X^{**}$; if $X$ is separable, $l_1 \not\subset X$ and $X^*$ is non-separable, then there exists an $X^*$-valued weakly measurable function which is not weakly equivalent to any strongly measurable function).

**1. Introduction.** Let $(S, \Sigma, \mu)$ be a finite complete and non-negative measure space, and let $\nu$ be a $\mu$-continuous measure taking values in a Banach space $X$. We present a treatment of the following question: when does $X$ possess the *weak Radon–Nikodym property*, i.e. when, for each $(S, \Sigma, \mu)$ as above, and each $\mu$-continuous measure $\nu\colon \Sigma \to X$ of $\sigma$-finite variation, does there exist a Pettis integrable function $f\colon S \to X$ such that $\nu(E) = \text{Pettis-}\int_E f\,d\mu$, for all $E \in \Sigma$? If there exists a strongly measurable function $f$ with the above property, then $X$ is said to have the *Radon–Nikodym property*. The problem of characterization of Banach spaces posessing the Radon-Nikodym property has been treated by many authors (cf. [7]).

Using the constructions of James [15] and that of Davis, Figiel Johnson and Pełczyński [5], we show that there are Banach spaces which possess the weak Radon–Nikodym property and fail to possess the Radon–Nikodym property.

In order to prove the main theorems of this paper we use methods which are quite different from those used in the theory of Banach spaces possessing the Radon–Nikodym property. In particular, we use some properties of Banach spaces which do not contain isomorphic copies of $l_1$.

Using our main theorem (Theorem 3), we show that there exists a Banach space not containing any isomorphic copy of $l_\infty$ and a bounded Pettis integrable function with values in this space which is not weakly equivalent to any strongly measurable function (Corollary 7).

icm©



We also prove that if the set of extreme points of the unit ball of $X^*$ is norm separable, then $X^*$ is norm separable (this was a problem posed by Stegall [32] and first solved by Rybakov).

We also prove that a separable $X$ is weak* $\omega_1$-sequentially dense in $X^{**}$ (see Section 5 for the definition) if and only if $X$ is weak* sequentially dense in $X^{**}$.

An abstract of this paper was presented at the conference in Measure Theory at Štefanova (Czechoslovakia) in January 1976.

**2. Preliminaries.** Throughout the paper the letters $X$, $Y$ and $Z$ denote infinite-dimensional Banach spaces (real or complex).

If $X$ is a Banach space, then $X^*$ denotes the topological dual or conjugate space to $X$, and, if $U: X \to Y$ is an operator, then $U^*: Y^* \to X^*$ is the adjoint operator to $U$. If $x \in X$ and $x^* \in X^*$, then $\langle x^*, x\rangle = x^*(x)$.

The topological sum of $X$ and $Y$ is denoted by $X \oplus Y$. We write $X \subset Y$ whenever $X$ is a closed subspace of $Y$, with the induced topology.

$\Gamma$ denotes a linear subset of $X^*$ total on $X$.

Let $(S, \Sigma, \mu)$ be a finite positive measure space. A function $f: S \to X$ is $\Gamma$-measurable, whenever for every $x^* \in \Gamma$, there exists a $\mu$-null set $N \in \Sigma$ such that $\langle x^*, f\rangle$ is a measurable function on $(S \setminus N, \Sigma | S\setminus N)$, where $S \setminus N$ denotes the complement of $N$ in $S$, $\Sigma | S\setminus N = \{E \in \Sigma: E \subset S \setminus N\}$, and $\langle x^*, f\rangle(s) = x^*[f(s)]$, $s \in S$.

If $X = Y^*$ for a Banach space $Y$ and $\Gamma = Y$, we say also that $f$ is *weak* measurable*. If $\Gamma = X^*$, then $f$ is said also to be *weakly measurable*. A weakly measurable and almost separable $X$-valued function is called *strongly measurable*.

$f(S)$ denotes the range of the function $f: S \to X$.

A $\Gamma$-measurable function $f: S \to X$ is said to be *$\Gamma$-uniformly bounded* provided there is a number $M < \infty$ such that, for every $x^* \in \Gamma$, we have, $|\langle x^*, f\rangle| \leqslant M\|x^*\|$ $\mu$-a.e.

Using the lattice properties of $L_1(S, \Sigma, \mu)$, we can easily see that if $f: S \to X$ is a $\Gamma$-measurable function, then there exists a non-negative measurable function $\psi_f^\Gamma$ with the following properties:

(i) for every $x^* \in \Gamma$ $|\langle x^*, f(s)\rangle| \leqslant \psi_f^\Gamma(s)\|x^*\|$ $\mu$-a.e.;

(ii) $\psi_f^\Gamma(s) \leqslant \|f(s)\|_\Gamma$ $\mu$-a.e.;

(iii) if $\varphi: S \to [0, \infty)$ is a measurable function satisfying conditions (i) and (ii) (with $\psi_f^\Gamma$ replaced by $\varphi$), then $\psi_f^\Gamma(s) \leqslant \varphi(s)$ $\mu$-a.e.

If $\|f(\cdot)\|$ is a measurable function, then $\psi_f^\Gamma(s) = \|f(s)\|_\Gamma$ $\mu$-a.e.

Moreover, for every $\Gamma$-measurable function $f: S \to X$ there exists a sequence of $\Gamma$-uniformly bounded functions $f_n: S \to X$, $n = 1, 2, \ldots$ such that $f = \sum_n f_n$ and the supports of $f_n$'s are pairwise disjoint.



A $\Gamma$-measurable function $f: S \to X$ is *$\Gamma$-scalarly integrable* if $\langle x^*, f\rangle \in L_1(S, \Sigma, \mu)$ for every $x^* \in \Gamma$. A $\Gamma$-scalarly integrable function $f: S \to X$ is *$\Gamma$-integrable* if for every $E \in \Sigma$ there exists an element $f_E \in X$ such that

$$\langle x^*, f_E\rangle = \int_E \langle x^*, f\rangle\, d\mu,$$

for every $x^* \in \Gamma$. In such a case we say that $f_E$ is a *$\Gamma$-integral of $f$ on the set* $E \in \Sigma$, and we denote it by

$$f_E = \Gamma - \int_E f\, d\mu.$$

If $X$ is a conjugate of a Banach space $Y$ and $\Gamma = Y$, then we say also that $f$ is *weak* integrable* and $f_E$ is the *weak* integral of $f$ on $E$*.

An $X^*$-integrable $f: S \to X$ is said to be *weakly* (or *Pettis*) *integrable*, and $f_E$ is the *weak* (or *Pettis*) *integral of $f$ on $E$*: $f_E = P - \int_E f\, d\mu$.

Two $\Gamma$-measurable functions $f: S \to X$ and $g: S \to X$ are called *$\Gamma$-equivalent* provided, for every $x^* \in \Gamma$, $\langle x^*, f\rangle = \langle x^*, g\rangle$ $\mu$-a.e. As above, we can speak about weak* and weak equivalence of functions.

A mapping $\nu: \Sigma \to X$ is an *$X$-valued $\Gamma$-measure* provided $\langle x^*, \nu\rangle$ is countably additive for every $x^* \in \Gamma$. If $\nu$ is countably additive in the norm topology of $X$, then we say simply that $\nu$ is an *$X$-valued measure* (by the Orlicz–Pettis theorem, $\nu$ is an $X$-valued measure if and only if it is an $X$-valued $X^*$-measure). If $X = Y^*$ and $\Gamma = Y$, then we speak about *weak* measures*. It is easy to see that if $f: S \to X$ is a $\Gamma$-integrable function, then the $\Gamma$-integral of $f$ is an $X$-valued $\Gamma$-measure.

$\nu(\Sigma)$ is the range of the $\Gamma$-measure $\nu: \Sigma \to X$.

If $\nu: \Sigma \to X$ is a $\Gamma$-measure, then by the *$\Gamma$-variation* of $\nu$ we mean a mapping $|\nu|_\Gamma: \Sigma \to [0, \infty]$ given by

$$|\nu|_\Gamma(E) = \sup \sum_{j=1}^{n} \|\nu(A_j)\|_\Gamma,$$

where $E \in \Sigma$, sup is taken over all finite sequences of pairwise disjoint elements $A_j$ of $\Sigma$ such that $A_j \subset E$ and for every $x \in X$

$$\|x\|_\Gamma = \sup\{|\langle x^*, x\rangle|: \|x^*\| \leqslant 1, x^* \in \Gamma\}.$$

If $\Gamma$ is a norming set, then

$$|\nu|_\Gamma(E) = \sup \sum_{j=1}^{n} \|\nu(A_j)\|,$$

where sup and $A_j$ are as above. In that case $|\nu|_\Gamma$ is called a *variation* of $\nu$, and it is denoted by $|\nu|$.

icm©



A standard calculation shows that $|\nu|_\Gamma$ is always a non-negative measure.

If $|\nu|_\Gamma$ is ($\sigma$-)finite, then $\nu$ is said to be *of ($\sigma$-)finite $\Gamma$-variation.*

If $\mu$ is a finite positive measure on $(S, \Sigma)$ and $\nu\colon \Sigma \to X$ is a $\Gamma$-measure, then we say that $\nu$ is *$\mu$-continuous* if $|\nu|_\Gamma$ is $\mu$-continuous, i.e., for every positive number $\varepsilon$, there is a positive number $\delta$ such that $|\nu|_\Gamma(E) < \varepsilon$ for every $E \in \Sigma$ for which $\mu(E) < \delta$.

The following theorem obtained by Rybakov [28], will be frequently used in our considerations. For measures of finite variation it is a direct consequence of a representation theorem of A. and C. Ionescu Tulcea [41] (see also Dinculeanu [9], §13, Theorem 5).

THEOREM 0. *Let $(S, \Sigma, \mu)$ be a finite complete measure space, let $X$ be a Banach space and let $\nu\colon \Sigma \to X^*$ be a $\mu$-continuous measure of $\sigma$-finite variation. Then there exists a weak* measurable function $f\colon S \to X^*$ such that*

$$\nu(E) = X - \int_E f\,d\mu, \quad E \in \Sigma.$$

**3. The weak Radon-Nikodym property. Connections with the Radon-Nikodym property.** We begin with a theorem which shows that from the integral point of view the only interesting $\Gamma$-measures are those of $\sigma$-finite $\Gamma$-variation.

PROPOSITION 1. *If $f\colon S \to X$ is a $\Gamma$-integrable function and*

$$\nu(E) = \Gamma - \int_E f\,d\mu, \quad E \in \Sigma,$$

*then the $\Gamma$-variation $|\nu|_\Gamma$ of $\nu$ is a $\sigma$-finite measure and*

$$|\nu|_\Gamma(E) = \int_E \psi_f^\Gamma\,d\mu, \quad E \in \Sigma.$$

Proof. If $E \in \Sigma$, then we have for every $x^* \in \Gamma$

$$|\langle x^*, \nu(E)\rangle| \leqslant \int_E |\langle x^*, f\rangle|\,d\mu \leqslant \|x^*\| \int_E \psi_f^\Gamma\,d\mu.$$

Hence,

$$|\nu|_\Gamma(E) \leqslant \int_E \psi_f^\Gamma\,d\mu,$$

and the first assertion follows since $\psi_f^\Gamma$ is finite everywhere.

By the classical Radon–Nikodym theorem there exists a non-negative measurable function $h$ on $S$ such that

$$|\nu|_\Gamma(E) = \int_E h\,d\mu, \quad E \in \Sigma.$$



Now, the inequality $|\nu|_\Gamma(E) \leqslant \int_E \psi_f^\Gamma\,d\mu$ yields the relation

$$h(s) \leqslant \psi_f^\Gamma(s) \quad \mu\text{-a.e.}$$

If $x^* \in \Gamma$ and $\|x^*\| \leqslant 1$, then

$$|\langle x^*, \nu\rangle|(E) = \int_E |\langle x^*, f\rangle|\,d\mu \leqslant |\nu|_\Gamma(E) = \int_E h\,d\mu.$$

Hence $|\langle x^*, f(s)\rangle| \leqslant h(s)$ $\mu$-a.e. Condition (iii) now yields $\psi_f^\Gamma(s) \leqslant h(s)$ $\mu$-a.e., and this completes the proof.

COROLLARY 1 (Rybakov [29]). *If $f\colon S \to X$ is Pettis integrable and $\nu(E) = P - \int_E f\,d\mu$, then the variation of $\nu$ is $\sigma$-finite.*

COROLLARY 2. *If $X = Y^*$ for a Banach space $Y$ and $f\colon S \to X$ is weak* integrable, then the (weak*) variation of the weak* measure $\nu(E) = Y - \int_E f\,d\mu$, $E \in \Sigma$, is $\sigma$-finite.*

A Banach space $X$ is said to have the *Radon–Nikodym property* (RNP) if and only if, for each finite measure space $(S, \Sigma, \mu)$ and each $\mu$-continuous $X$-valued measure $\nu\colon \Sigma \to X$ of finite variation, there exists a strongly measurable function $f\colon S \to X$ such that

$$\nu(E) = \text{Bochner} - \int_E f\,d\mu.$$

It is easy to see that $X$ has the RNP if and only if each $X$-valued measure of $\sigma$-finite variation is the Pettis integral of a strongly measurable function.

If we consider weakly measurable functions, then the only natural generalization of these two integrals is that of Pettis. So, inspired by Proposition 1, we introduce the following generalization of the RNP:

DEFINITION 1. $X$ has the *weak Radon–Nikodym property* (WRNP) if and only if every $X$-valued measure $\nu$ on a finite complete measure space $(S, \Sigma, \mu)$ which is $\mu$-continuous and of $\sigma$-finite variation has a Pettis-integrable derivative $f\colon S \to X$ (i.e. $\nu(E) = P - \int_E f\,d\mu$ for each $E$ in $\Sigma$.)

It can be shown that $X$ has the weak Radon–Nikodym property if and only if every $X$-valued measure $\nu$ on a finite complete measure space $(S, \Sigma, \mu)$ which is $\mu$-continuous and of finite variation has a Pettis-integrable derivative $f\colon S \to X$.

It is quite clear that for a separable Banach space the WRNP and the RNP are equivalent.

It is also clear that a Banach space $X$ possessing the WRNP has the RNP if and only if, given any complete measure space $(S, \Sigma, \mu)$ and any Pettis integrable function $f\colon S \to X$, there exists a strongly measurable function $g\colon S \to X$ which is weakly equivalent to $f$.

In particular, if $X$ has the WRNP and each weakly measurable $X$-valued function is weakly equivalent to a strongly measurable function, then $X$ has the RNP.

PROBLEM 1. Let $X$ be a Banach space with the RNP. Is each $X$-valued weakly measurable function weakly equivalent to an $X$-valued strongly measurable function?

As is shown in Corollary 7, without the assumption that $X$ has the RNP the answer may be a negative one.

Problem 1 is connected with the following one (cf. [8]).

PROBLEM 2. Assume that $X$ is a Banach space not containing any isomorphic copy of $c_0$ (or stronger: $X$ has the RNP). Is every weakly scalarly integrable $X$-valued function Pettis integrable?

It is easy to see that the positive answer (at least for $X$ with the RNP) yields a positive answer to Problem 1.

It is shown in Section 5 that there are Banach spaces with the WRNP and without the RNP. In particular, it is shown that the WRNP is not hereditary with respect to subspaces. However, it is easy to prove that the following theorem holds:

PROPOSITION 2. *If $X = Y \oplus Z$, then $X$ has weak Radon–Nikodym property if and only if both $Y$ and $Z$ do.*

The next proposition is due to Professor C. Ryll-Nardzewski.

PROPOSITION 3. *$l_\infty$ does not possess the weak Radon–Nikodym property.*

Proof. Let $(S, \Sigma, \mu) = ([0,1], \mathscr{L}, \mu)$, where $\mu$ is the Lebesgue measure and $\mathscr{L}$ is the $\sigma$-algebra of $\mu$-measurable sets.

For each $s \in [0,1]$ denote by $0, s_1 s_0 \ldots$ its dyadic expansion. Setting $f(s) = \{s_n\}_{n=1}^{\infty}$, we get an $l_\infty$-valued $l_1$-measurable function.

Now, for every $E \in \Sigma$, put $\nu(E) = \{\int_E s_n \, d\mu\}_{n=1}^{\infty}$. Since $|\nu|(E) \leqslant \mu(E)$, $E \in \Sigma$, $\nu$ is an $l_\infty$-valued measure of finite variation. Clearly, we have

$$\nu(E) = l_1 - \int_E f \, d\mu, \qquad E \in \Sigma.$$

However, by a theorem of Sierpiński [31] there exists a purely additive set function $\eta$ on the $\sigma$-algebra of all subsets of a countable set, such that $\langle \eta, f(\cdot) \rangle$ is not $\mu$-measurable (it is known that $l_\infty^* = l_1 \oplus c_0^\perp$ and $\eta \in c_0^\perp$).

Thus, $f$ is not weakly measurable and this implies that $\nu$ cannot be represented as a Pettis integral.

Taking into account the above two propositions, we get

PROPOSITION 4. *If $X$ has the weak Radon-Nikodym property, then $X$ does not contain any isomorphic copy of $l_\infty$.*

Proof. Suppose that $X$ contains an isomorphic copy of $l_\infty$. Since $l_\infty$ is a $P_1$-space, there exists a Banach space $Y$ such that $X = l_\infty \oplus Y$. In view of Propositions 2 and 3, $X$ does not possess the WRNP.



It follows from the above Proposition 4 and the theorem of Bessaga and Pełczyński [3] that $c_0$ cannot be isomorphically embedded into any conjugate Banach space with the WRNP.

PROBLEM 3. Can $c_0$ be isomorphically embedded into a Banach space possessing the WRNP?

It is well known (cf. Lewis [17]) that Banach spaces with the RNP cannot contain $c_0$.

We conclude this section by indicating a class of Banach spaces for which the WRNP and the RNP are equivalent.

We shall say that $Y$ is *separably complementable* if for every separable space $X \subset Y$ there exists a separable space $Z \subset Y$ complemented in $Y$ and containing $X$.

THEOREM 1. *If $X$ is a subspace of a separably complementable space, then $X$ possesses the weak Radon-Nikodym property if and only if it possesses the Radon-Nikodym property.*

Proof. Let $(S, \Sigma, \mu)$ be the unit interval $[0,1]$ endowed with the Lebesgue measure and the Lebesgue measurable sets, and let $\nu \colon \Sigma \to X$ be a $\mu$-continuous measure of finite variation.

If $X$ possesses the WRNP, then there exists a function $f\colon [0,1] \to X$ such that

$$\nu(E) = P - \int_E f \, d\mu, \qquad E \in \Sigma.$$

Since $\nu(\Sigma)$ is separable, there exists a separable space $Z \subset Y$ (where $Y$ is separably complementable and contains $X$) which is complemented in $Y$ and contains $\nu(\Sigma)$.

If $Q\colon Y \to Y$ is the projection of $Y$ onto $Z$, then the equality

$$\nu(E) = P - \int_E Qf \, d\mu$$

holds for every $E \in \Sigma$ and $Qf$ is a strongly measurable function.

In virtue of a result of Rieffel ([25], Proposition 1.10) we have

$$Qf(S) \subset \overline{\operatorname{conv} \mathscr{A}_\nu(S)} \quad \mu\text{-a.e.}$$

where

$$\mathscr{A}_\nu(S) = \left\{ \frac{\nu(E)}{\mu(E)} \colon \; E \in \Sigma, \; \mu(E) > 0 \right\}$$

and the closure is taken in the norm topology of $Y$. Thus we have $Qf(s) \in X$ $\mu$-a.e., and this shows that $X$ possesses the RNP with respect to $(S, \Sigma, \mu)$. In view of a result of Chatterji ([4], Theorem 2) $X$ possesses the RNP. This completes the proof.

Corollary 3. *A weakly compactly generated Banach space possesses the* WRNP *if and only if it possesses the* RNP. *An L-space possesses the* WRNP *if and only if it is isomorphic to a Banach space $l_1(T)$ for some set $T$.*

Proof. If $X$ is WCG, then in virtue of a result of Amir and Lindenstrauss ([1], Lemma 4) $X$ is separably complementable. The conclusion follows from Theorem 1.

It is also known (cf. Grothendieck [12], p. 325, Exercise 2) that an $L$-space is separably complementable.

Assume now that $Y = L_1(S, \Sigma, \mu)$ is isomorphic to $X$ (cf. Semadeni [30], Theorem 26.3.1).

If $\mu$ has a non-atomic part $\mu_1$, then owing to the decomposition of $\mu$ into non-atomic and purely atomic parts, $Y$ contains a (complemented) copy of $L_1(S_1, \Sigma_1, \mu_1)$ which fails to have the RNP.

If $\mu$ is purely atomic, then $Y$ is isomorphic to some $l_1(T)$ which is known to have the RNP. This proves the assertion.

I was informed by the rewiewer that a result similar to that contained in Corollary 3 was proved earlier by D. R. Lewis (Stegall [33]).

The following proposition will be needed in Section 5:

Proposition 5. *Let the unit interval $[0,1]$ be endowed with the Lebesgue (or Borel) measurable sets and the Lebesgue measure $\mu$. Then there exists a weak* scalarly integrable function $f\colon [0,1] \to C^*[0,1]$ which is not weakly measurable.*

Proof(¹). Let $E$ be a not $\mu$-measurable subset of $[0,1]$. Define $x^{**} \in C^{**}[0,1]$ by

$$x^{**}(\mu) = \mu_a(E),$$

where $\mu_a$ is the atomic part of $\mu \in C^*[0,1]$ and put

$$f(s) = \delta_s$$

for every $s \in [0,1]$.

$f$ is weak* scalarly integrable but it is not weakly measurable because $x^{**}(f) = \chi_E$ is a non-measurable function.

**4. A function characterization of conjugate Banach spaces possessing the weak Radon-Nikodym property.** We give here a characterization of the conjugate Banach spaces possessing the weak Radon–Nikodym property in terms of weak* scalarly integrable functions.

We begin with a lemma.

Lemma 1. *Let $X$ be a Banach space and let $Y$ be a separable subspace of $X^*$. If $X$ is separably complementable, then there exists a separable complemented space $Z \subset X$ such that $Y$ embeds isometrically into $Z^*$.*

(¹) The proof presented here was proposed by Professor C. Ryll-Nardzewski. Our previous proof was based on the fact that each weakly $\mu$-measurable $C^*[0,1]$-valued function is strongly measurable.



Proof. In order to prove the assertion it is sufficient to use the same arguments (and the separable complementability of course) as in the classical case (cf. [11], Lemma VI. 8.8).

Theorem 2. *The following statements concerning $X^*$ are equivalent:*

(i) *$X^*$ has the weak Radon-Nikodym property;*

(ii) *given any complete measure space $(S, \Sigma, \mu)$ and any weak* scalarly integrable function $f\colon S \to X^*$; then there exists a Pettis integrable function $g\colon S \to X^*$, which is weak* equivalent to $f$.*

*If $X$ is separably complementable and each $X^*$-valued measure of $\sigma$-finite variation has a norm-separable range (e.g. $X$ is separable), then the completeness of $(S, \Sigma, \mu)$ is superfluous. For separable $X$ the equality $f = g$ $\mu$-a.e. holds.*

Proof. (i) → (ii). Let $f\colon S \to X^*$ be a weak* measurable function such that $\langle f, x\rangle \in L_1(S, \Sigma, \mu)$ for each $x \in X$.

An easy application of the closed graph theorem implies that for each $E \in \Sigma$ there exists an $f_E \in X^*$ such that

$$\langle f_E, x\rangle = \int_E \langle f, x\rangle\, d\mu.$$

The mapping $\nu\colon \Sigma \to X^*$ given by $\nu(E) = f_E$ is a weak* measure. By a result of Diestel and Faires ([6], Corollary 1.2), $\nu$ is norm countably additive since, in view of Proposition 4, $X^*$ does not contain any isomorphic copy of $l_\infty$. Moreover, in virtue of Proposition 1, the variation of $\nu$ is $\sigma$-finite.

Hence, by the assumption, there exists a weakly measurable function $g\colon S \to X^*$ such that

$$\nu(E) = P\text{-}\int_E g\, d\mu, \qquad E \in \Sigma.$$

Clearly, $g$ is weak* equivalent to $f$.

If $X$ is separable and $(S, \Sigma, \mu)$ is not necessarily complete, then let $(S, \check{\Sigma}, \check{\mu})$ be the completion of $(S, \Sigma, \mu)$. If $f$ is weak* scalarly integrable with respect to $(S, \Sigma, \mu)$, then it is also weak* scalarly integrable with respect to $(S, \check{\Sigma}, \check{\mu})$. In view of the first part of the proof there exists a Pettis integrable $g\colon S \to X^*$ on $(S, \check{\Sigma}, \check{\mu})$ such that

$$\langle f(s), x\rangle = \langle g(s), x\rangle \ \check{\mu}\text{-a.e. for every } x \in X.$$

If $\{x_n\}_{n=1}^{\infty}$ is a countable dense subset of $X$, then for every $n$ there exists a set $N_n \in \Sigma$ such that $\mu(N_n) = 0$ and

$$\langle f(s), x_n\rangle = \langle g(s), x_n\rangle, \qquad \text{whenever } s \notin N_n.$$

Put $\bar{g}(s) = g(s)$ if $s \notin \bigcup_{n=1}^{\infty} N_n$ and $\bar{g}(s) = 0 \in X^*$, otherwise. Clearly, $\bar{g}\colon S \to X^*$ is Pettis integrable on $(S, \Sigma, \mu)$ and $f(s) = \bar{g}(s)$ $\mu$-a.e.

If $X$ is separably complementable then there exists a complemented separable space $Y \subset X$ such that $\nu(\Sigma) \subset Y^*$.

Let $P$ be a projection of $X^*$ onto $Y^*$ and let $f$ and $\nu$ be as in the first part of the proof. Clearly, $Pf$ is $Y$-measurable and

$$\nu(E) = Y - \int_E Pf\,d\mu$$

for all $E \in \Sigma$.

Since $Y$ is separable and $Y^*$ has the WRNP (it follows from Proposition 2), there exists a $Y^{**}$-integrable function $g\colon S \to Y^* \subset X^*$ such that $Pf = g$ $\mu$-a.e., and

$$\nu(E) = Y^{**} - \int_E g\,d\mu, \qquad E \in \Sigma.$$

Clearly, $g$ is $X^{**}$-integrable and it is $X$-equivalent to $f$.

(ii) $\to$ (i) Let $\nu\colon \Sigma \to X^*$ be a measure of $\sigma$-finite variation. In virtue of Theorem 0 there exists a weak* measurable function $f\colon S \to X^*$ such that

$$\nu(E) = X - \int_E f\,d\mu, \qquad E \in \Sigma.$$

By the assumption there exists a Pettis integrable function $g\colon S \to X^*$ such that for every $x \in X$

$$\langle f(s), x\rangle = \langle g(s), x\rangle \ \mu\text{-a.e. for every } x \in X.$$

Let

$$\check{\nu}(E) = P - \int_E g\,d\mu, \qquad E \in \Sigma.$$

Since $X$ is total on $X^*$ and $\langle \nu(E), x\rangle = \langle \check{\nu}(E), x\rangle$ for every $E \in \Sigma$ and $x \in X$, we have $\check{\nu}(E) = \nu(E)$ for every $E \in \Sigma$. This yields the WRNP of $X^*$.

COROLLARY 4. *If $X^*$ has the WRNP, then for each finite complete measure space $(S, \Sigma, \mu)$ and each weak* measurable function $f\colon S \to X^*$, there exists a weakly measurable function $g\colon S \to X^*$ which is weak* equivalent to $f$. If $X$ is separable, then the completeness of $(S, \Sigma, \mu)$ is superfluous and $f$ is simply weakly measurable.*

Proof. Since we can decompose $f\colon S \to X$ into a series of weak* uniformly bounded functions, the assertion is a consequence of Theorem 2.

PROBLEM 4. Is the converse of the statement of Corollary 4 true, i.e. does $X^*$ have the WRNP if for each finite and complete measure space $(S, \Sigma, \mu)$ each weak* measurable function $f\colon S \to X^*$ is weak* equivalent to a weakly measurable $g\colon S \to X^*$?



As Theorem 5 shows, in the case of separably complementable $X$ the answer is affirmative.

Remark 1. Assume that real measurable cardinals do not exist. Then it follows from the properties of $C^*[0,1]$ that it is impossible to replace in Theorem 2 weak* scalarly integrable functions by weakly scalarly integrable functions. Namely, if $f\colon S \to C^*[0,1]$ is a weakly scalarly integrable function, then in view of Grothendieck's theorem ([12], p. 327) $f$ is strongly measurable. Since $c_0 \not\subset C^*[0,1]$, it follows from the theorem of Dimitrov [8] and Diestel, Faires ([6], Corollary 1.3) that $f$ is Pettis integrable. On the other hand, $C^*[0,1]$ does not possess the WRNP (Corollary 3).

**5. The weak Radon-Nikodym property in a space possessing a separable predual.** It is the aim of this section to give a complete characterization of Banach spaces possessing the WRNP which are conjugate to separable Banach spaces.

Moreover, using our main theorem, we give examples (in fact we adapt the examples constructed by others to our purposes) of Banach spaces which have the WRNP but fail to have the RNP. As is known (cf. Musiał [22]), if $X$ is separable then $X^*$ has the RNP if and only if each $X^*$-valued weak* measurable function is strongly measurable. We give here a similar characterization of the WRNP of $X^*$.

The main theorem is the following

THEOREM 3. *If $X$ is separable, then the following statements concerning $X$ are equivalent:*

(i) *$X^*$ possesses the weak Radon–Nikodym property;*

(ii) *given any (complete) measure space $(S, \Sigma, \mu)$ and any weak* scalarly integrable function $f\colon S \to X^*$, $f$ is Pettis integrable;*

(iii) *given any (complete) measure space $(S, \Sigma, \mu)$ and any weak* measurable function $f\colon S \to X^*$, $f$ is weakly measurable;*

(iv) *$X$ does not contain any isomorphic copy of $l_1$.*

Proof. We only prove the version with complete measures. The equivalence of (i) and (ii) is proved in Theorem 2 and (iii) can be derived from (ii) by using the decomposition property of weak* measurable functions.

(iii) $\to$ (iv). Assume that $X$ contains a copy of $l_1$. In virtue of a result of Pełczyński ([24], Theorem 3.4) $C[0,1]$ is a quotient of $X$; let $V\colon X \to C[0,1]$ be the quotient map. Then $U = V^*\colon C^*[0,1] \to X^*$ is an isomorphic embedding of $C^*[0,1]$ into $X^*$ such that $U^*(X) \subset C[0,1]$.

Now, in virtue of Proposition 5, there exists a weak* scalarly integrable function $f\colon [0,1] \to C^*[0,1]$ which is not weakly measurable.

Let us consider the function $Uf\colon [0,1] \to X^*$. It follows from the Hahn–Banach theorem that $Uf$ is not weakly measurable.

On the other hand, we have $U^*(X) \subset C[0,1]$ and this proves the weak* scalar integrability of $Uf$. Thus (iii) does not hold, and this completes the proof of the implication.

(iv) $\to$ (i). Assume that $l_1 \not\subset X$ and take a finite complete measure space $(S, \Sigma, \mu)$ and a $\mu$-continuous measure $\nu: \Sigma \to X^*$ of finite variation. In virtue of a result of Odell and Rosenthal [23] $X$ is weak* sequentially dense in $X^{**}$ and in virtue of Theorem 0 there exists a weak* measurable function $f: S \to X^*$ such that

$$\nu(E) = X - \int_E f\,d\mu, \quad E \in \Sigma.$$

Let

$$Q = \left\{x^{**} \in X^{**}: \langle x^{**}, \nu(E)\rangle = \int_E \langle x^{**}, f\rangle\,d\mu \quad \text{for all } E \in \Sigma\right\}.$$

It can easily be shown that $Q$ is weak* sequentially closed in $X^{**}$ (cf. Lipecki and Musiał [20], Lemma). Hence, $Q = X^{**}$ and

$$\nu(E) = P - \int_E f\,d\mu \quad \text{for} \quad E \in \Sigma.$$

This proves that $X^*$ has the WRNP. (Compare the proof of this implication with a Theorem of Wilhelm [20].)

Having proved the above theorem, we are able to give examples of Banach spaces with the weak Radon–Nikodym property but without the Radon–Nikodym property.

For instance, the space $Y$ constructed in [5], p. 325, does not contain $l_1$ and $Y^*$ is non-separable. In view of Theorem 3 and the result of Stegall [32], $Y^*$ possesses the WRNP but does not possess the RNP.

Also the space $JT^*$, where $JT$ is the space constructed by James [15] (cf. also [19]), has the WRNP but does not have the RNP.

Let $Z$ be one of the spaces $Y$ or $JT$ and let $W$ be a subspace of $l_1$ which is not isomorphic to any conjugate Banach space (such a space was constructed by Lindenstrauss [18]). Since $W$ has the RNP (as a subspace of a space possessing the RNP), we infer in view of Proposition 2, that $Z^* \oplus W$ possesses the WRNP, does not possess the RNP, and, is not isomorphic to any conjugate space.

If $B$ is the Banach space constructed by Lindenstrauss and Stegall [19], then all the odd conjugates of $B$ have the RNP while the even conjugates do not ([19], Corollary 4). It follows from the representations of the conjugates of $B$ and from Theorem 3 that all even conjugates of $B$ (excluding $B$) have the WRNP.

If $X = B \oplus B^*$, then all the conjugates of $X$ (excluding $X$) have the WRNP and none of them have the RNP.



Corollary 5. *There exists a separable Banach space $X$ such that $c_0$ and $L_1[0,1]$ cannot be isomorphically embedded in $X$ and $X$ fails to have the* RNP.

Proof. Let $Z$ be, as above, one of the spaces $Y$ or $JT$, and let $Z_1$ be a norm separable subspace of $Z^*$ not possessing the RNP. Since $l_1 \not\subset Z$, it follows from a result of Hagler ([13], Theorem 5) that $L_1[0,1] \not\subset Z_1$. By a result of Bessaga and Pełczyński [3] we also have $c_0 \not\subset Z_1$.

Since we obtained our results a paper of Lindenstrauss and Stegall [19] has appeared in which the authors prove ([19], Corollary 5) that there is a weakly measurable function $f$ from the Cantor set (endowed with the Haar measure) into $JT^*$ which is not equivalent to any strongly measurable function. They prove it by using the structure of the space, however, as we show, the only important thing is that $X$ does not contain any isomorphic copy of $l_1$.

Corollary 6. *Let $X$ be a separable Banach space such that $X^*$ is non-separable and $l_1 \not\subset X$. Then, given any finite non purely atomic measure space $(S, \Sigma, \mu)$, there exists a Pettis integrable function $f: S \to X^*$ which is not weak* equivalent (and hence not weakly equivalent either) to any strongly measurable function $g: S \to X^*$.*

Proof. Let $(S, \Sigma, \mu)$ be a finite non purely atomic measure space such that each $X^*$-valued and Pettis integrable $f: S \to X^*$ is weak* equivalent to a strongly measurable $X^*$-valued function $g: S \to X^*$.

Let $\nu: \Sigma \to X^*$ be a measure of finite variation. Since $X^*$ has the WRNP and $X$ is separable, we have

$$\nu(E) = P - \int_E f\,d\mu, \quad E \in \Sigma,$$

for a certain weakly measurable $f: S \to X^*$, and

$$\int_S \|f\|\,d\mu = |\nu|(S) < \infty$$

in view of Proposition 1. Hence

$$\nu(E) = P - \int_E g\,d\mu,$$

where $g: S \to X^*$ is strongly measurable.

Thus $X^*$ has the RNP with respect to $(S, \Sigma, \mu)$. In virtue of Chatterji's theorem [4], $X^*$ has the RNP, which contradicts Stegall's theorem [32], since $X^*$ is nonseparable. This proves the corollary.

Remark 2. Using the decomposition of weakly scalarly measurable functions into a series of weakly uniformly bounded functions, we can easily see that the function $f$ in Corollary 6 can be taken to be weakly uniformly bounded.

It is not known whether every separable Banach space with the RNP can be isomorphically embedded in a separable conjugate Banach space (Uhl [35]). The next corollary shows that a large class of separable Banach spaces can be isometrically embedded in conjugate Banach spaces with the WRNP.

COROLLARY 7. *If $X$ is separable and $l_1 \not\subset X^*$, then there exists a separable Banach space $Z$ such that $Z^*$ has the* WRNP *and $X$ can be isometrically embedded in $Z^*$.*

Proof. Assume that $X$ is canonically embedded in $X^{**}$. Since $X$ is separable, there exists a separable Banach space $Z \subset X^*$ such that $X$ is isometrically embedded in $Z^*$ ([11], VI. 8.8). In virtue of Theorem 3, $Z^*$ possesses the WRNP.

In order to formulate the next theorem we need a definition.

Let $T$ be an arbitrary subset of $X^*$. We denote by $T_0$ the set $T$ itself, and if $\alpha < \omega_1$ is a non-limit ordinal, then we let $T_\alpha$ be the weak* sequential closure of the set $\bigcup_{\beta<\alpha} T_\beta$. If $\alpha < \omega_1$ is a limit ordinal we put $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$.

We say that $T \subset X^*$ is *weak* $\omega_1$-sequentially dense in* $X^*$ if $X^* = \bigcup_{\alpha<\omega_1} T_\alpha$.

Having the above notion of weak* $\omega_1$-sequential density, we can formulate a generalization of one implication of Theorem 3 in the case of a non-separable $X$ (the proof is essentially the same as the proof of the appropriate implication of Theorem 3). The theorem can also be treated as a generalization of Theorem 2 and of Corollary 1.3 of Diestel and Faires [6] in the special case of $X$ being weak* $\omega_1$-sequentially dense in $X^{**}$.

THEOREM 4. *If $X$ is weak* $\omega_1$-sequentially dense in $X^{**}$, then $X^*$ has the weak Radon-Nikodym property and each $X^*$-valued weak* scalarly integrable function is Pettis integrable.*

As is shown by the example of the Banach space $X = c_0(T)$ with an uncountable set $T$, there exists a Banach space $X$ such that $X^*$ has the WRNP and $X$ fails to be weak* $\omega_1$-sequentially dense in $X^{**}$. Thus, the converse to Theorem 4 in false.

COROLLARY 9. *If $X$ is separable and weak* $\omega_1$-sequentially dense in $X^{**}$, then $X$ is weak* sequentially dense in $X^{**}$.*

Proof. In virtue of Theorem 4 $X^*$ has the WRNP and hence $X$ does not contain any isomorphic copy of $l_1$ (Theorem 3). Applying the theorem of Odell and Rosenthal [23], we get the weak* sequential density of $X$ in $X^{**}$.

As a simple consequence of Theorem 3 we get the following characterization of Banach spaces not containing any isomorphic copy of $l_1$:

COROLLARY 9. *The following statements concerning a Banach space $X$ are equivalent:*



(i) *$X$ does not contain any isomorphic copy of $l_1$;*

(ii) *every separable subspace of $X$ has a dual possessing the* WRNP;

(iii) *every separable subspace of $X^*$ embeds into a Banach space which is conjugate to a separable Banach space and possesses the* WRNP.

Proof. The equivalence of (i) and (ii) is a simple consequence of Theorem 3. Similarly, (i) yields (iii). In order to complete the proof it is enough to show that (i) is a consequence of (iii).

So let $Y$ be a separable subspace of $X^*$ and let $Z$ (e.g. $Z \subset X$) be a separable Banach space such that $Y$ embeds isometrically in $Z^*$ and $Z^*$ has the weak Radon–Nikodym property.

In virtue of Theorem 3, $l_1 \not\subset Z$ and hence it follows from the result of Hagler ([13], Theorem 5) that $Z^*$ does not contain any isomorphic copy of $L_1[0,1]$. It follows that $Y$ does not contain $L_1[0,1]$ either. Since $Y$ has been arbitrary, it follows that $L_1[0,1]$ cannot be isomorphically embedded in $X^*$. Hence, $l_1 \not\subset X$ (Pełczyński [24], Proposition 3.3).

We conclude this section by indicating a solution of a problem posed by Stegall [32]: Is $X^*$ separable provided the set of extreme points of the unit ball in $X^*$ is separable?

This problem was earlier solved (by a different method) by Rybakov (unpublished). After this paper had already been submitted for publication the solution of Kadets and Phonph [16] appeared.

PROPOSITION 6. *If the set of the extreme points of the unit ball of $X^*$ is norm separable, then $X^*$ is separable.*

Proof. Clearly, we can assume that $X$ is separable. Moreover, it follows from the properties of $C^*[0,1]$ (or $l_\infty$) that $l_1 \not\subset X$. Now, the required result follows from a result of Odell and Rosenthal [23] stating that the unit ball of $X^*$ is the norm closed convex hull of its extreme points.

**6. The weak Radon-Nikodym property in a conjugate Banach space possessing a separably complementable predual.** It is the aim of this section to generalize Theorem 3 to the case of arbitrary conjugate Banach spaces possessing separably complementable preduals (by a result of Amir and Lindenstrauss ([1], Lemma 4) each weakly compactly generated Banach space is separably complementable).

LEMMA 2. *Let $(S, \Sigma, \mu)$ be a finite positive measure space and let $X$ be a Banach space not containing any isomorphic copy of $l_1$. If $U\colon X \to L_1(S, \Sigma, \mu)$ is a bounded linear operator whose representing weak* measure $\nu\colon \Sigma \to X^*$* (cf. [11], Theorem VI.8.1) *is an $X^*$-valued measure of $\sigma$-finite variation, then $U$ is compact.*

Proof. In virtue of Theorem 0 there exists a weak* scalarly integrable function $f\colon S \to X^*$ such that

icm©



$$\nu(E) = X - \int_E f\,d\mu$$

for every $E \in \Sigma$. Hence we have

$$Ux = \langle x, f\rangle$$

for every $x \in X$.

In order to show the compactness of $U$ take a bounded sequence $x_n \in X$, $n = 1, 2, \ldots$

Since $l_1 \not\subset X$, there is a weak Cauchy subsequence of $\{x_n\}$ (Rosenthal [27] in the case of real $X$ and Dor [10] in the case of complex $X$). For the simplicity of notation we assume that $\{x_n\}$ itself is weakly Cauchy. Let $x_0$ be its weak* limit in $X^{**}$.

Since $L_1(S, \Sigma, \mu)$ is weakly sequentially complete (cf. Dunford and Schwartz [11], Theorem IV.8.6), $\{Ux_n\}$ is weakly convergent.

Since simultaneously $\langle x_n, f(s)\rangle \to \langle x_0, f(s)\rangle$ for all $s \in S$, we can apply Theorem IV.8.12. of [11] to get the norm convergence of $\{Ux_n\}$ in $L_1(S, \Sigma, \mu)$.

This proves the compactness of $U$.

COROLLARY 10 (Rybakov, unpublished). *$X$ does not contain any isomorphic copy of $l_1$ if and only if each $X^*$-valued measure of $\sigma$-finite variation has a conditionally compact range.*

Proof. (The proof given here is different from that of Rybakov):

$\to$ Let $(S, \Sigma)$ be a measurable space and let $\nu\colon \Sigma \to X^*$ be a measure. Without loss of generality we may assume that $\nu$ is of finite variation. Denote by $\mu$ the variation of $\nu$.

Let $U\colon X \to L_1(S, \Sigma, \mu)$ be given by $Ux = d\langle x, \nu\rangle/d\mu$ (the Radon–Nikodym derivative of $\langle x, \nu\rangle$ with respect to $\mu$). In virtue of Lemma 2, $U$ is compact. Hence

$$U^*\colon L_\infty(S, \Sigma, \mu) \to X^*$$

is compact as well. In particular,

$$\nu(\Sigma) = U^*\{\chi_E\colon E \in \Sigma\}$$

is a conditionally compact set.

$\leftarrow$ Suppose that $l_1 \subset X$. Then (Pełczyński [24]) $L_1[0,1] \subset X^*$. Let $U\colon L_1[0,1] \to X^*$ be an isomorphic embedding, and let $\Sigma$ be the $\sigma$-algebra of Borel sets on the unit interval $S$.

If we define $\nu(E) = \chi_E \in L_1[0,1]$, $E \in \Sigma$, then $\nu$ is an $L_1[0,1]$-valued measure on $(S, \Sigma)$ such that $|\nu|(E) = \mu(E)$, where $\mu$ is the Lebesgue measure. Hence $\nu$ is of finite variation. It is also easily seen that $\nu(\Sigma)$ is not conditionally compact (cf. Hoffmann-Jørgensen [14], Example 6).



It follows that $U\nu\colon \Sigma \to X^*$ is an $X^*$-valued measure of finite variation and with a non-conditionally compact range.

Chatterji [4] proved that a Banach space $X$ has the RNP if and only if $X$ has the RNP with respect to the unit interval endowed with the Lebesgue measure. It appears that a similar result holds for conjugate spaces possessing a separably complementable predual. This is the reason why we formulate Theorem 5 in an extended form containing function characterizations of $X^*$.

THEOREM 5. *If $X$ is separably complementable, then the following statements concerning $X$ are equivalent:*

(i) *$X^*$ possesses the weak Radon–Nikodym property;*

(ii) *given any (complete) measure space $(S, \Sigma, \mu)$ and any weak* measurable function $f\colon S \to X^*$, $f$ is weak* equivalent to a weakly measurable function;*

(iii) *given any weak* measurable function $f\colon [0,1] \to X^*$ on the unit interval endowed with the Borel sets (the Lebesgue measurable sets) and the Lebesgue measure, $f$ is weak* equivalent to a weakly measurable function;*

(iv) *given any (complete) measure space $(S, \Sigma, \mu)$ and any weak* scalarly integrable function $f\colon S \to X^*$, $f$ is weak* equivalent to a Pettis integrable function;*

(v) *given any weak* scalarly integrable function $f\colon [0,1] \to X^*$ with the unit interval endowed with the Borel sets (the Lebesgue measurable sets) and the Lebesgue measure, $f$ is weak* equivalent to a Pettis integrable function;*

(vi) *$X$ does not contain any isomorphic copy of $l_1$.*

Proof. Using the same arguments as in the proof of Theorem 3, we see that the implications (iv) $\to$ (ii) $\to$ (iii) and (iv) $\to$ (i) $\to$ (iv) $\to$ (v) $\to$ (iii) hold.

Thus, in order to complete the proof it is sufficient to prove that (vi) implies (i) and (iii) yields (vi).

(vi) $\to$ (i) Let $(S, \Sigma, \mu)$ be a finite complete measure space and let $\nu\colon \Sigma \to X^*$ be a $\mu$-continuous measure of $\sigma$-finite variation.

In virtue of Lemma 1 and Corollary 10 there exists a separable and complemented space $Y \subset X$ such that $\nu(\Sigma) \subset Y^* \subset X^*$. Since $Y$ does not contain any isomorphic copy of $l_1$, it follows from Theorem 3 that $Y^*$ has the WRNP.

Let $f\colon S \to Y^*$ be a function such that

$$\nu(E) = Y^{**} - \int_E f\,d\mu$$

icm©



for every $E \in \Sigma$. It follows that the equality

$$\nu(E) = X^{**} - \int_E f\,d\mu, \quad E \in \Sigma,$$

holds as well.

(iii) → (vi) Suppose that $X$ contains a separable space $Y$ which is isomorphic to $l_1$. In virtue of our assumptions there exists a complemented separable subspace $Z$ of $X$ which contains $Y$.

It follows from the proof of Theorem 3 that there exists a weak* measurable function $f\colon [0,1] \to Z^*$ (the unit interval is endowed with the Borel or Lebesgue measurable sets) which is not $Z^{**}$-measurable. It can easily be seen that $f$ cannot be $X$-equivalent to any $X^*$-valued and $X^{**}$-measurable function.

Thus (iii) does not hold, and this completes the proof of the theorem.

As a corollary we get the following

THEOREM 6. *If $X$ is weakly compactly generated, then $X^*$ possesses the weak Radon–Nikodym property if and only if $X$ does not contain any isomorphic copy of $l_1$.*

Proof. In virtue of a result of Amir and Lindenstrauss ([1], Lemma 4) each weakly compactly generated Banach space is separably complementable. Thus, the assertion is a consequence of Theorem 5.

PROBLEM 5. Let $X$ be an arbitrary Banach space. Are the following conditions equivalent?

(a) $X^*$ has the weak Radon–Nikodym property;

(b) $X$ does not contain any isomorphic copy of $l_1$.

We conjecture that at least (a) implies (b).

In connection with Corollary 10 we get the following

PROBLEM 6. Let $(S, \Sigma, \mu)$ be a finite complete measure space, let $X$ be a Banach space and let $f\colon S \to X$ be a Pettis integrable function. Must the range of a measure $\nu\colon \Sigma \to X$ be conditionally compact if $\nu(E) = P - \int_E f\,d\mu$, for all $E \in \Sigma$?

We conjecture that if $X$ has the WRNP then the answer is affirmative.

Let us observe that a negative answer to implication (a) → (b) yields a negative answer to Problem 6. Indeed, if $X$ contains an isomorphic copy of $l_1$ and $X^*$ has the WRNP, then $L_1[0,1] \subset X^*$, and so we can take the Lebesgue measure on the unit interval and the measure $\nu(E) = \chi_E \in L_1[0,1]$, which is of finite variation, its range not being conditionally compact. On the other hand, there exists a function $f\colon S \to X^*$ such that

$$\nu(E) = P - \int_E f\,d\mu \quad \text{for} \quad E \in \Sigma.$$



**7. Determination of the weak Radon-Nikodym property by subspaces.** As has already been remarked, the weak Radon–Nikodym property is not separably determined, however, there are cases where it is determined by weak* separable and weak* closed subspaces.

THEOREM 7. *Let $X$ be such that, given any measurable space $(S, \Sigma)$ and any $X^*$-valued measure $\nu\colon \Sigma \to X^*$ of $\sigma$-finite variation, the range of $\nu$ is a norm separable set. Then, if each norm-separable subspace of $X^*$ is a subspace of a (weak* separable) subspace of $X^*$ possessing the weak Radon–Nikodym property, then $X^*$ possesses the weak Radon–Nikodym property.*

*In particular, $X^*$ possesses the weak Radon–Nikodym property if all weak* closed subspaces of $X^*$ which are weak* separable do so.*

Proof. Let $(S, \Sigma, \mu)$ be a finite and complete measure space and let $\nu\colon \Sigma \to X^*$ be a $\mu$-continuous $X^*$-valued measure of $\sigma$-finite variation. By the assumptions there exists a (weak* separable) subspace $Z$ of $X^*$ possessing the WRNP and containing $\nu(\Sigma)$.

Hence there exists a $Z^*$-integrable function $f\colon S \to Z$ such that

$$\langle z^*, \nu(E)\rangle = \int_E \langle z^*, f\rangle\,d\mu,$$

for all $z^* \in Z^*$ and $E \in \Sigma$. By the Hahn–Banach theorem, $f$ is $X^*$-measurable and

$$\nu(E) = X^* - \int_E f\,d\mu, \quad E \in \Sigma.$$

This completes the proof.

Remark 3. If $X$ is weakly compactly generated and possesses the Dunford–Pettis property, then every weakly compact subset of $X^*$ is norm-separable (Rosenthal [26], Proposition 4.7).

In particular, the range of each $X^*$-valued measure, being weakly conditionally compact (Bartle, Dunford, Schwartz [2]), is norm separable. It has also been shown that for $X$ not containing any isomorphic copy of $l_1$ each $X^*$-valued measure of $\sigma$-finite variation has a norm-separable range (Corollary 10). Observe that there is no correlation between the above two examples. Indeed, if $X = l_1$, then all weakly compact subsets of $l_\infty$ are separable ([26], Proposition 4.7); on the other hand, if $JT$ is the James tree (constructed by James in [15]), then $JT^*$ does not contain $l_1$ and $JT^{**}$ is a non-separable weakly compactly generated space [19].

It is true that if $X$ is a Banach space such that $X^*$ has the Radon–Nikodym property and $Z$ is a closed subspace of $X$, then $(X/Z)^*$ has the Radon–Nikodym property as well. Indeed, the dual of $X/Z$ can be isomorphically embedded into $X^*$, and hence it must have the RNP. We do not know whether a similar result for the weak Radon–Nikodym property holds.

Problem 7. Let $X$ be a Banach space such that $X^*$ possesses the weak Radon–Nikodym property. Must $Z$ have the weak Radon–Nikodym property whenever it is a weak* closed subspace of $X^*$? Must $Z$ be contained in a weak* separable subspace of $X^*$ possessing the weak Radon–Nikodym property provided it is also (weak*) separable?

Having proved Theorem 7 we can easily see that the implication (b) → (a) can be answered affirmatively if the following problem can:

Problem 8. Let $X$ be a non-separable Banach space such that $l_1 \not\subset X$ and $X^*$ is weak* separable. Does $X^*$ have the weak Radon–Nikodym property?

**Notes added in proof.** All the problems posed in this paper have been already solved.

Assuming the existence of real measurable cardinals Edgar [35] and Rybakov [43] proved the following: if $T$ is a set of measurable cardinality then there is an $l_1$-valued weakly scalarly integrable function which is not Pettis integrable and not weakly equivalent to any strongly measurable function. This solves in the negative Problems 1 and 2. A negative answer for Problem 3 was given by Jeurnink (letter communication) and Janicka [42]. Musiał and Ryll-Nardzewski [37] proved that if $X^*$ has the WRNP then $l_1 \not\subset X$. The converse implication also holds. Indeed, assume that $X$ is a Banach space not containing any isomorphic copy of $l_1$ and $\nu$ is an $X^*$-valued measure of finite variation defined on a measurable space $(S, \Sigma)$. If $\nu$ has a weak* density (with respect to $\mu = |\nu|$) $f\colon S \to X^*$ which is measurable with respect to the $\sigma$-algebra of the weak* Borel subsets of $X^*$ and a measure $\mu f^{-1}$ defined on the weak* Borel subsets of $X^*$ is regular, then a simple application of Stegall's reformulation [33] of a result of Haydon [40] yields the Pettis integrability of $f$. Moreover, the equality

$$\nu(E) = P-\int_E f d\mu \quad \text{holds for every } E \in \Sigma.$$

As it was observed by Edgar [36] and Weizsäcker [38] such a density always did exist (this fact was communicated to me by Janicka), and so, if $X \not\supset l_1$ then $X^*$ has the WRNP. As a direct consequence of this theorem we get an affirmative answer to Problem 7 and the WRNP of $(X/Z)^*$ for $X^*$ possessing the WRNP and $Z \subset X$. We can also prove now that if $X^*$ has the WRNP with respect to the unit interval endowed with the Lebesgue measure, then it has the WRNP. Indeed, if $X^*$ has the WRNP with respect to the Lebesgue measure, and $Y$ is a separable closed subspace of $X$, then, in virtue of Theorem 3 of Musiał and Ryll-Nardzewski [37], $Y^*$ has the WRNP with respect to the Lebesgue measure as well. Hence, in virtue of Theorem 5, we have $l_1 \not\subset Y$. Consequently $X$ does not contain any isomorphic copy of $l_1$. In a similar way, using Proposition 3 from [37], we can answer affirmatively Problem 4.

The above results can be summarized to the following general theorem:

Theorem 5′. *For an arbitrary Banach space $X$ the conditions* (i), (ii), (iii), (iv), (v) *and* (vi) *of Theorem* 5 *are equivalent* (*the version with complete measures*).

Fremlin and Talagrand [39] and Stegall (oral communication) proved that if $X$ was a Banach space, $(S, \Sigma, \mu)$ was perfect measure space and $f\colon S \to X$ was Pettis integrable then the range of measure given by the Pettis integral of $f$ with respec



to $\mu$ was conditionally compact. Fremlin and Talagrand proved also that in general the perfectness of $\mu$ could not be omitted. This gives an answer to Problem 6. I can prove that for $X$ possessing the WRNP Problem 6 has always affirmative answer. Moreover, I can prove that for an arbitrary $X$, if $X$ has the WRNP with respect to the Lebesgue measure, then $X$ has the WRNP. The proofs will appear elsewhere.

At least let us remark that the characterization of the WRNP for conjugate spaces, a result of Johnson ([7], p. 38) and the theorem allows us to formulate the following characterization of the conjugate Banach spaces possessing the RNP: *$X^*$ has the* RNP *if and only if $X$ does not contain any isomorphic copy of $l_1$ and $X^*$ is separably complementable.*

Rybakov's proof of Collorary 10 (exactly the same as presented here) was published in [43].

J. Diestel informed me that H. Maynard also obtained some results proved in this paper.

**Acknowledgements.** I am grateful to J. Diestel and V. I. Rybakov for their remarks concerning previous version of this paper.

**References**


[1] D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Ann. of Math. 80 (1968), pp. 34–46.

[2] R. G. Bartle, N. Dunford, and J. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. 7 (1955), pp. 289–305.

[3] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), pp. 151–174.

[4] S. D. Chatterji, *Martingale convergence and the Radon–Nikodym theorem in Banach spaces*, Math. Scand. 22 (1968), pp. 21–41.

[5] W. J. Davis, T. Figiel, W. B. Johnson, and A. Pełczyński, *Factoring weakly compact operators*, J. Functional Analysis 17 (1974), pp. 311–327.

[6] J. Diestel and B. Faires, *On vector measures*, Trans. Amer. Math. Soc. 198 (1974), pp. 253–271.

[7] — and J. J. Uhl, Jr., *The Radon–Nikodym theorem for Banach space valued measures*, Rocky Mountain J. Math. 6 (1976), pp. 1–46.

[8] D. B. Dimitrov, *A remark on the Gelfand integral*, (in Russian) Funkcional. Anal. i Priložen. 5 (1971), pp. 84–85.

[9] N. Dinculeanu, *Vector measures*, Internat. Ser. of Monographs in Pure and Appl. Math., vol. 95, Pergamon Press, New York; VEB Deutscher Verlag der Wissenschaften, Berlin 1967.

[10] L. E. Dor, *On sequences spanning a complex $l_1$ space*, Proc. Amer. Math. Soc. 47 (1975), pp. 515–516.

[11] N. Dunford and J. T. Schwartz, *Linear operators. I: General theory*, Pure and Appl. Math. vol. 7, Interscience, New York 1958.

[12] A. Grothendieck, *Espaces vectoriels topologiques*, 3. a. Edicão, Publicacão da Sociedade de Matemática de S. Paulo, São Paulo 1964.

[13] J. Hagler, *Some more Banach spaces which contain $l_1$*, Studia Math. 46 (1973), pp. 35–42.

[14] J. Hoffmann–Jørgensen, *Vector measures*, Math. Scand. 28 (1971), pp. 5–32.

[15] R. C. James, *A separable somewhat reflexive Banach space with nonseparable dual*, Bull. Amer. Math. Soc. 80 (1974), pp. 738–743.

[16] M. I. Kadets, and V. P. Phonph, *Some properties of the set of extreme points of the unit ball in conjugate Banach spaces*, (in Russian) Mat. Zametki 20 (1976), pp. 315–319.
[17] D. R. Lewis, *A vector measure with no derivative*, Proc. Amer. Math. Soc. 32 (1972), pp. 535–536.
[18] J. Lindenstrauss, *On a certain subspace of $l_1$*, Bull. Acad. Polon. Sci. 12 (1964), pp. 539–542.
[19] — and C. Stegall, *Examples of separable spaces which do not contain $l_1$ and whose duals are non-separable*, Studia Math. 54 (1975), pp. 81–105.
[20] Z. Lipecki, and K. Musiał, *On the Radon–Nikodym derivative of a measure taking values in a Banach space with basis*, Rev. Roumaine Math. Pures Appl. 28 (1978), pp. 911–915.
[21] K. Musiał, *The weak Radon–Nikodym property in Banach spaces*, Aarhus University, Preprint Series 7, 1976.
[22] — *Functions with values in a Banach space possessing the Radon–Nikodym property*, ibidem 29 (1977).
[23] E. Odell and H. P. Rosenthal, *A double-dual characterization of separable Banach spaces containing $l_1$*, Israel J. Math. 20 (1975), pp. 375–384.
[24] A. Pełczyński, *On Banach spaces containing $L_1(\mu)$*, Studia Math. 30 (1968), pp. 231–246.
[25] M. A. Rieffel, *The Radon–Nikodym theorem for the Bochner integral*, Trans. Amer. Math. Soc. 131 (1968) pp. 466–487.
[26] H. P. Rosenthal, *On injective Banach spaces $L^\infty(\mu)$ for finite measures $\mu$*, Acta Math. 124 (1970), pp. 205–248.
[27] — *A characterization of Banach spaces containing $l_1$*, Proc. Nat. Acad. Sc. USA 71 (1974), pp. 2411–2413.
[28] V. I. Rybakov, *On vector measures*, (in Russian) Izv. Vysš. Učebn. Zaved. Matiematika 79 (1968), pp. 92–101.
[29] — *On the completeness of the space of Pettis integrable functions*, (in Russian) Učen. Zap. Mosk. Gos. Ped. Inst. 277 (1971), pp. 58–64.
[30] Z. Semadeni, *Banach spaces of continuous functions*, Vol. I, PWN, Warszawa 1971.
[31] W. Sierpiński, *Fonctions additives non complètemant additives et fonctions non mesurables*, Fund. Math. 30 (1939), pp. 96–99.
[32] C. Stegall, *The Radon–Nikodym property in conjugate Banach spaces*, Trans. Amer. Math. Soc. 206 (1975), pp. 213–223.
[33] — Seminaire Maurey–Schwartz, 1975.
[34] J. J. Uhl, Jr., *A note on the Radon–Nikodym property for Banach spaces*, Rev. Roumaine Math. Pures Appl. 17 (1972), pp. 113–115.

**References added in proof**

[35] G. A. Edgar, *Measurability in a Banach space*, Indiana Univ. Math. J. 26 (1977), pp. 663–677.
[36] — *On the Radon–Nikodym property and martingale convergence*, Vector Space Measures and Applications II, Proceedings, Dublin 1977, Lecture Notes in Math. 645 (1978), pp. 62–76.
[37] K. Musiał and C. Ryll-Nardzewski, *Liftings of vector measures and their applications to RNP and WRNP*, ibidem, pp. 162–171.
[38] H. Weizäcker, *Strong measurability, liftings and the Choquet–Edgar theorem*, ibidem, pp. 209–218.



[39] D. H. Fremlin and M. Talagrand, *A decomposition theorem for additive set functions, with applications to Pettis integrals and ergodic means* (preprint).
[40] R. Haydon, *Some more characterization of Banach spaces containing $l_1$*, Math. Proc. Cambridge Phil. Soc. 80 (1976), pp. 269–276.
[41] A. Ionescu Tulcea and C. Ionescu Tulcea, *On the lifting property II. Representation of linear operators on spaces $L_E^r$, $1 \leqslant r < \infty$*, J. Math. Mech. 11 (1962), pp. 773–795.
[42] L. Janicka, *Properties of the Radon–Nikodym type* (preprint).
[43] V. I. Rybakov, *Properties of measures defined on a normed space possessing RNP* (in Russian), Mat. Zametki 21 (1977), pp. 81–92.

WROCŁAW UNIVERSITY
and
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES

*Received September 18, 1976*
*in revised version January 24, 1977* (1206)