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Perturbations of Schauder bases in the spaces $C(K)$ and $L^p, p > 1$

by

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Abstract. Let K be an uncountable compact metric space and let $B, \delta > 0$. Let $\{x_n\}$ be a Schauder basis for $X = C(K)$ or $X = L^p(0,1)$, $p > 1$, and let $T: X \to X$ be a bounded operator such that $\|Tx_n - x_n\| < B - \delta \ \forall n$. We investigate the question of when the space TX contains an isomorph of X, and answer it in the affirmative (with $B = 1$) for some useful bases.

1. Introduction. In the study of Banach spaces, it is sometimes useful to study perturbations of natural Schauder bases in these spaces. In this paper we investigate the type of perturbation defined by

DEFINITION 1.1. Let $B > 0$. A Schauder basis $\{x_n\}$ for a Banach space X is B-*perturbable* if for each $\delta > 0$ and each bounded linear operator $T: X \to X$ satisfying $\|Tx_n - x_n\| < B - \delta$ for all n, the space TX contains an isomorph of X. The largest B for which $\{x_n\}$ is B-perturbable is called the *perturbation constant* of the basis $\{x_n\}$.

In Section 2 we demonstrate that the Haar and Walsh systems in L_p, $p > 1$, have perturbation constant 1. In Section 3 we study perturbations of bases in $C(K)$ spaces, where K is an uncountable compact metric space. We give a criterion for the perturbability of a basis for $C(K)$, and show that the Haar system in $C(\Delta)$ (Δ denotes the Cantor set) and the Schauder system in $C([0,1])$ are both 1-perturbable. There exist bases for $C([0,1])$ which are not B-perturbable for any $B > 0$.

In both the cases $X = L^p$ and $X = C(K)$, we show that TX contains a complemented isomorph of X. Hence, if TX is itself complemented, it follows from the Pełczyński decomposition method that TX and X are isomorphic.

Our notation is basically that of Lindenstrauss and Tzafriri [3]. If $\{y_n\}$ is a sequence in a Banach space X, we denote by $[\{y_n\}]$ the smallest closed subspace of X containing $\{y_n\}$. If $\{x_n\}$ is a basic sequence in X,

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we denote by $\{x_n^*\}$ the biorthogonal sequence, and by P_N the natural projection onto $[\{x_j\}_{i=1}^N]$.

A well-known perturbation result we shall use is [3]

Lemma 1.1. (a) *If $\{x_n\}$ is a basis for X with basis constant M, and if $\sum\|x_n-y_n\|<1/2M$, then $\{y_n\}$ is a basis for X.*

(b) *If $\{x_n\}$ is a basic sequence in X with basis constant M, and if $[\{x_n\}]$ is complemented by a projection P, then $\sum\|x_n-y_n\|<1/8M\|P\|$ implies that $\{y_n\}$ is a basic sequence and $[\{y_n\}]$ is complemented in X.*

The sequences $\{x_n\}$ and $\{y_n\}$ are equivalent in the sense that $\sum a_n x_n$ converges if and only if $\Sigma a_n y_n$ converges.

2. Perturbations in $L^p(0,1)$, $1<p<\infty$. In this section we show that the Haar and Walsh systems in L^p, $p>1$, have perturbation constant 1.

The Haar system $\{\varphi_n\}_{n=0}^\infty$ is defined by

$$\varphi_0\equiv 1, \qquad \varphi_1=\chi_{[0,1/2)}-\chi_{[1/2,1]},$$

and

$$\varphi_{2^n+i}(t)=2^{n/p}\varphi_1(2^n(t-i/2^n)) \quad \text{for} \quad n=1,2,\ldots;\ i=0,\ldots,2^n-1.$$

Biorthogonal functionals to the Haar system are defined by

$$\varphi_{2^n+i}^*(f)=2^{n/q}\int 2^{-n/p}\varphi_{2^n+i}(x)f(x)\,dx,$$

where q denotes the exponent conjugate to p. It is well known that the Haar system is an unconditional basis for L^p, $p>1$. We denote its unconditional constant in L^p by M_p. Recall

$$M_p=\sup_{\substack{\|x\|\leqslant 1\\ \{a_n:|a_n|\leqslant 1\}}}\Big\|\sum a_n x_n^*(x)x_n\Big\|.$$

To define the Walsh system, let $\{\tilde{\varphi}_n\}$ denote the Haar system normalized in the sup norm, and define Rademacher functions $\{r_n\}_{n=0}^\infty$ by

$$r_n(t)=\sum_{i=0}^{2^n-1}\tilde{\varphi}_{2^n+i}(t).$$

Now each $n>0$ has a unique decomposition $n=2^{n_1}+2^{n_2}+\ldots+2^{n_j}$ with $n_1<n_2<\ldots<n_j$. The Walsh functions $\{w_n\}_{n=0}^\infty$ are defined by

$$w_0\equiv 1 \quad\text{and}\quad w_n(t)=r_{n_1}(t)r_{n_2}(t)\ldots r_{n_j}(t).$$

It is known [4] that the sequence $\{w_n\}$ is a basis for L_p, $p>1$.

We shall use the following two lemmas. Details of the first may be found in [3], and the second is a special case of a lemma in [1]. Lebesgue measure is denoted by m, and $[\cdot]$ denotes the greatest integer function.

Lemma 2.1. *For any σ-algebra B of measurable sets, the conditional expectation operator E_B is a projection of norm one from L^p onto the subspace consisting of the B-measurable functions.*

Lemma 2.2. *Let $\{\psi_n\}_{n=0}^\infty\subset L^p$, $1\leqslant p<\infty$, be a sequence of $\{-1,0,1\}$-valued functions, and let $A_{0,0}=\psi_0^{-1}(1)$, $A_{n,i}=\psi^{-1}_{2^n-1+[i/2]}(-1)^i$ for $n=1,2,\ldots;\ i=0,\ldots,2^n-1$. Suppose*

(i) *$m(A_{n,i})=2^{-n}$, $n>0$,*

(ii) *$A_{0,0}=(0,1)$,*

(iii) *$A_{n,i}\cap A_{n,j}=\emptyset$, $i\neq j$, and*

(iv) *$A_{n+1,2i}\cup A_{n+1,2i+1}=A_{n,i}$.*

Then $\{\psi_n/\|\psi_n\|\}$ is isometrically equivalent to the Haar system.

Theorem 2.3. *The Haar system in L^p, $p>1$, has perturbation constant 1. In fact, if δ and T are as in Definition 1.1, the space TL^p contains a complemented isomorph of L^p.*

Proof. Let $\delta>0$ and suppose $T\colon L^p\to L^p$ is a bounded linear operator which satisfies $\|T\varphi_n-\varphi_n\|<1-\delta$ for all n. We construct a sequence $\{\psi_n\}$, isometrically equivalent to the original Haar system such that $T|[\{\psi_n\}]$ has a bounded inverse, and such that $T([\{\psi_n\}])$ is complemented in L^p.

We write $T\varphi_n=a_n\varphi_n+r_n$, with $\varphi_n^*(r_n)=0$. Since $\|\varphi_n^*\|=1$, we have $\delta<a_n<2-\delta$ and $\|r_n\|<2$ for all n.

Let $\eta>0$ and $\{\eta_n\}$ be a positive sequence such that $\sum\limits_{n=0}^{\infty}\eta_n=\eta/3$. We define the sequence $\{\psi_n\}$ inductively as follows. Let $\psi_0=\varphi_0$, and choose $N_0>0$ such that $\|(I-P_{N_0})(T\psi_0)\|<\eta_0$. For $n>0$ and $0\leqslant i<2^n$, assume $\psi_0,\ldots,\psi_{2^n+i-1}$ have been defined and that an increasing sequence of integers $\{N_j\}_{j=0}^{2^n+i-1}$ has been selected such that for each j

$$(P_{N_j}-P_{N_{j-1}})(\psi_j)=\psi_j, \tag{1}$$

$$\|P_{N_{j-1}}(T\psi_j)\|<\eta_j, \tag{2}$$

and

$$\|(I-P_{N_j})(T\psi_j)\|<\eta_j. \tag{3}$$

For each k such that $2^k>N_{2^n+i-1}$ and $k>n$, let $\{j_i\}_{i=1}^{2^{k-n}}$ be an enumeration of those indices j such that $\operatorname{supp}\varphi_j\subset\psi^{-1}_{2^n-1+[i/2]}\big((-1)^i2^{(n-1)/p}\big)$ and $2^k\leqslant j<2^{k+1}$. For $h=1,\ldots,2^{2^{k-n}}$, let $\big\{\{\varepsilon_i^{(h)}\}_{i=1}^{2^{k-n}}\big\}$ be an enumeration of all possible sequences of ±1's, and define the sequence $\{f_m\}$ by

$$f_{M_k+h}=2^{(n-k)/p}\sum_{i=1}^{2^{k-n}}\varepsilon_i^{(h)}\varphi_{j_i}. \qquad M_k=\sum_{l=n}^{k-l}2^{2^{l-n}}. \tag{4}$$

Recall that n is fixed. Each f_m is a candidate for ψ_{2^n+i}. Now $\|f_m\| = 1$ for each m, and $\{f_m\}$ converges weakly to zero. Hence, $\{Tf_m\}$ converges weakly to zero, and there exists M such that $m > M$ implies

$$\|P_{N_{2^n+i-1}}(Tf_m)\| < \eta_{2^n+i}. \tag{5}$$

Now for each m, $M_k < m \leqslant M_{k+1}$, there exists h such that

$$Tf_m = 2^{(n-k)/p}\sum_{i=1}^{2^{k-n}} a_i\varepsilon_i^{(h)}\varphi_{j_i} + 2^{(n-k)/p}\sum_{i=1}^{2^{k-n}} \varepsilon_i^{(h)} r_{j_i}.$$

Let

$$f'_m = 2^{(n-k)/p}\sum_i a_i\varepsilon_i^{(h)}\varphi_{j_i},$$

$$r'_m = 2^{(n-k)/k}\sum_i \varepsilon_i^{(h)} r_{j_i},$$

and

$$f^*_m = 2^{(n-k)/q}\sum_i \varepsilon_i^{(h)}\varphi^*_{j_i}.$$

We wish to select m (and k) so that $f^*_m(r'_m)$ is small. Now $\|f^*_m\| = 1$, and since $\varphi^*_j(r_j) = 0$ for all j, we have

$$f^*_m(r'_m) = \Big(2^{(n-k)/q}\sum_i \varepsilon_i^{(h)}\varphi^*_{j_i}\Big)\Big(\sum_i \varepsilon_i^{(h)} r_{j_i}\Big)$$

$$= 2^{(n-k)/q}\sum_{i\neq s} \varepsilon_i^{(h)}\varepsilon_s^{(h)}\varphi^*_{j_i}(r_{j_s}).$$

It follows that averaged over choices of signs, $\{\varepsilon_i^{(h)}\}$, $f^*_m(r'_m)$ is zero. Hence there are m, m', $M_k < m, m' \leqslant M_{k+1}$, corresponding to choices of signs $\{\varepsilon_i^{(h)}\}$ and $\{\varepsilon_i^{(h')}\}$ such that $f^*_m(r'_m) \geqslant 0$ and $f^*_{m'}(r'_{m'}) \leqslant 0$. Since we may move from the choice $\{\varepsilon_j^{(h)}\}$ to $\{\varepsilon_j^{(h')}\}$ by changing one sign at a time, we may select m, m' (reordering the sequence $\{j_i\}$ if necessary), such that

$$f^*_m(r'_m) \leqslant 0,$$

$$f^*_{m'}(r'_{m'}) \geqslant 0,$$

$$\varepsilon_1^{(h)} \neq \varepsilon_1^{(h')} \quad\text{and}\quad \varepsilon_i^{(h)} = \varepsilon_i^{(h')}, \quad i > 1.$$

Then

$$\min_{M_k<s\leqslant M_{k+1}} |f^*_s(r'_s)| \tag{6}$$

$$\leqslant \min(|f^*_m(r'_m)|, |f^*_{m'}(r'_{m'})|) \leqslant \tfrac{1}{2}|f^*_m(r'_m) - f^*_{m'}(r'_{m'})|$$

$$= \tfrac{1}{2}\Big|2^{(n-k)/p}2^{(n-k)/q}\Big[\Big(\varepsilon_1\varphi^*_{j_1} + \sum_{i=2}^{2^{k-n}}\varepsilon_i\varphi^*_{j_i}\Big)\Big(\varepsilon_1 r_{j_1} + \sum_{i=2}^{2^{k-n}}\varepsilon_i r_{j_i}\Big) -$$

$$-\Big(-\varepsilon_1\varphi^*_{j_1} + \sum_{i=2}^{2^{k-n}}\varepsilon_i\varphi^*_{j_i}\Big)\Big(-\varepsilon_1 r_{j_1} + \sum_{i=2}^{2^{k-n}}\varepsilon_i r_{j_i}\Big)\Big]\Big|$$

$$= 2^{(n-k)/p}2^{(n-k)/q}\Big|(\varepsilon_1\varphi^*_{j_1})\Big(\sum_{i=2}^{2^{k-n}}\varepsilon_i r_{j_i}\Big) + \Big(\sum_{i=2}^{2^{k-n}}\varepsilon_i\varphi^*_{j_i}\Big)(\varepsilon_1 r_{j_1})\Big|.$$

We make the following estimates on the terms of (6):

$$\|\varepsilon_1\varphi^*_{j_1}\| = 1, \tag{7}$$

$$\|\varepsilon_1 r_{j_1}\| \leqslant 2, \tag{8}$$

and

$$\Big\|\sum_{i=2}^{2^{k-n}}\varepsilon_i\varphi^*_{j_i}\Big\| \leqslant \Big\|\sum_{i=1}^{2^{k-n}}\varepsilon_i\varphi^*_{j_i}\Big\| = 2^{(k-n)/q}. \tag{9}$$

Since

$$\sum_{i=2}^{2^{k-n}}\varepsilon_i r_{j_i} = T\Big(\sum_{i=2}^{2^{k-n}}\varepsilon_i\varphi_{j_i}\Big) - \sum_{i=2}^{2^{k-n}} a_i\varepsilon_i\varphi_{j_i},$$

$$\Big\|\sum_{i=2}^{2^{k-n}}\varepsilon_i r_{j_i}\Big\| \leqslant \|T\|\Big\|\sum_{i=2}^{2^{k-n}}\varepsilon_i\varphi_{j_i}\Big\| + \Big\|\sum_{i=2}^{2^{k-n}} a_i\varepsilon_i\varphi_{j_i}\Big\| \leqslant (\|T\|+2)2^{(k-n)/p}. \tag{10}$$

Substituting (7)–(10) into (6) yields

$$\min_{M_k<s\leqslant M_{k+1}} |f^*_s(r'_s)| \leqslant 2^{(n-k)/p}2^{(n-k)/q}[(\|T\|+2)2^{(k-n)/p} + 2^{(k-n)/q}\cdot 2] \tag{11}$$

$$= (\|T\|+2)2^{(n-k)/q} + 2\cdot 2^{(n-k)/p}.$$

Therefore

$$\lim_{k\to\infty}\Big[\min_{M_k<s\leqslant M_{k+1}} |f^*_s(r'_s)|\Big] = 0,$$

and we may choose m such that

$$\|f^*_m(r'_m)f_m\| < \eta_{2^n+i}. \tag{12}$$

Now let $\psi_{2^n+i} = f_m$, and choose $N_{2^n+i} > j_{2^{k-n}}$ such that $\|(I-P_{N_{2^n+i}})(T\psi_{2^n+i})\| < \eta_{2^n+i}$. Hence (3) holds, and by the restriction $2^k > N_{2^n+i-1}$, and since $N_{2^n+i} > j_{2^{k-n}}$, (1) holds with $j = 2^n+i$. In this case, (2) follows from (5).

It is easily verified, using Lemma 2.2, that $\{\psi_n\}_{n=0}^{\infty}$ is isometrically equivalent to the Haar system. We now show that for η sufficiently small, $T|[\{\psi_n\}]$ is invertible, and that $T[\{\psi_n\}]$ is complemented.

Recall that for each $N = 2^n+i$, there exists k, and a choice of signs $\{\varepsilon_i\}$ such that

$$T\psi_{2^n+i} = 2^{(n-k)/p}\sum_{s=1}^{2^{k-n}} a_s\varepsilon_s\varphi_{j_s} + 2^{(n-k)/p}\sum_{s=1}^{2^{k-n}}\varepsilon_s r_{j_s}.$$

We denote

$$\psi'_{2^n+i} = 2^{(n-k)/p}\sum_s a_s\varepsilon_s\varphi_{j_s},$$

$$R_{2^n+i} = 2^{(n-k)/p}\sum_s \varepsilon_s r_{j_s},$$

and

$$\psi^*_{2^n+i} = 2^{(n-k)/q}\sum_s \varepsilon_s \varphi^*_{j_s}.$$

Setting

$$R'_{2^n+i} = (P_{N_{2^n+i}} - P_{N_{2^n+i-1}})\big(R_{2^n+i} - \psi^*_{2^n+i}(R_{2^n+i})\psi_{2^n+i}\big),$$

it follows from (1) that $\{\psi'_j + R'_j\}_{j=0}^{\infty}$ is a block basic sequence with respect to the Haar system and satisfies

(13) $$\psi^*_j(R'_j) = 0 \quad \forall j.$$

Furthermore, for $\eta < 1/8M_p$, it follows from (2), (3), (12) and Lemma 1.1 that the sequences $\{T\psi_n\}$ and $\{\psi'_n + R'_n\}$ are equivalent, and that $T[\{\psi_n\}]$ is complemented in L^p. That $[\{\psi'_n + R'_n\}]$ is complemented is a consequence of the argument that $T[\{\psi_n\}]$ is isomorphic to L^p. Let $S\colon T\psi_n \to \psi'_n + R'_n$ be the equivalence operator.

For $f \in L^p$, let f^+ and f^- denote the positive and negative parts of f, and for each $N \geqslant 0$ let B_N be the algebra generated by the sets $\operatorname{supp}\psi_n^+$ and $\operatorname{supp}\psi_n^-$, $n \leqslant N$. We consider the conditional expectation operator E_{B_N}. For each n, we will denote by $\bar{a}_n$ the average of the a_i's occurring in the Haar expansion of ψ'_n. It is then clear that

(14) $$E_{B_n}(\psi'_n) = \bar{a}_n\psi_n,$$

and

(15) $$E_{B_m}(\psi'_n) = E_{B_n}(\psi'_n), \qquad m > n.$$

We will show

(16) $$E_{B_n}(R'_m) = 0 \quad \forall n, m.$$

The case $m > n$ is obvious, and we present the case $m = n$ in some detail. The case $m < n$ follows from similar considerations.

It sufficies to show

(17) $$\int\limits_{\operatorname{supp}\psi_n^+} R'_n = \int\limits_{\operatorname{supp}\psi_n^-} R'_n = 0.$$

Let $j(l)$ be the first (last) m such that $\varphi^*_m(\psi_n) \neq 0$, and $R^{(1)} = P_j(R'_n)$, $R^{(2)} = (P_l - P_j)(R'_n)$, $R^{(3)} = (I - P_l)(R'_n)$. Since the Haar functions in the expansions of $R^{(3)}$ occur later than those in ψ_n,

$$\int\limits_{\operatorname{supp}\psi_n^+} R^{(3)} = \int\limits_{\operatorname{supp}\psi_n^-} R^{(3)} = 0.$$

Next,

$$\psi^*_n(R^{(2)}_n) = \psi^*_n(R') = 0,$$

by (13). But

$$\psi^*_n(R^{(2)}) = 2^{(n-k)/q}\Big[\int\limits_{\operatorname{supp}\psi_n^+} R^{(2)} + \int\limits_{\operatorname{supp}\psi_n^-} R^{(2)}\Big].$$

Since these integrals are obviously equal, they are both zero.

Now suppose φ_k occurs in the expansion of $R^{(1)}$. If $\operatorname{supp}\varphi_k \cap \operatorname{supp}\psi_n = \emptyset$, there is nothing to prove. Otherwise, let m be the largest index such that $\operatorname{supp}\varphi_k \subset \operatorname{supp}\psi_m$. Since $\{\psi_j + R'_j\}$ is a block basis with respect to the Haar system, ψ_m is in the level previous to n, and either $\operatorname{supp}\varphi_k \subset \operatorname{supp}\psi_m^+$ or $\operatorname{supp}\varphi_k \subset \operatorname{supp}\psi_m^-$. Since $\operatorname{supp}\psi_n$ exhausts one of these sets, and is disjoint from the other, and since the Haar functions in the expansion of ψ_n have finer support than φ_k does, it follows that $E_{B_n}(\varphi_k) = 0$. Hence $E_{B_n}(R^{(1)}) = 0$.

Define $S_1\colon [\{\psi'_n + R'_n\}] \to [\{\psi_n\}]$ by $S_1(\psi'_n + R'_n) = \bar{a}_n\psi_n$. For any N and any scalar sequence $\{b_n\}_{n=0}^N$, we have by virtue of (14), (15) and (16), that

$$S_1\Big(\sum_1^N b_n(\psi'_n + R'_n)\Big) = E_{B_{N+1}}\Big(\sum_1^N b_n(\psi'_n + R'_n)\Big).$$

Hence $\|S_1\| \leqslant 1$. Also, by unconditionality, the operator $U\colon [\{\psi_n\}] \to [\{\psi_n\}]$ defined by $U\psi_n = \bar{a}_n^{-1}\psi_n$ is bounded, and $\|U\| \leqslant \delta^{-1}M_p$. Hence $\|U\circ S_1 \circ S\| \leqslant \|S\|\,\delta^{-1}M_p$, so $U\circ S_1\circ S$ is bounded. Since $U\circ S_1 \circ S$ is inverse to $T|[\{\psi_n\}]$, the proof is complete.

We make the following comments concerning Theorem 2.3.

(1) The construction of the sequence $\{\psi_n\}$ using the weak convergence of $\{f_m\}$ to zero yields an alternate proof of the theorem of Lindenstrauss and Pełczyński that the Haar system in L^p, $p \geqslant 1$, is a precisely reproducible basis.

(2) A rearrangement invariant (r.i.) Banach space X is a space of measurable functions on $(0, 1)$ such that whenever T is a measure-automorphism of $(0, 1)$ and $f \in X$, $\|f\| = \|f(Tx)\|$. Lemmas 2.1 and 2.2 remain valid in all r.i. spaces X. Thus, Theorem 2.3 is valid in all r.i. spaces X in which the Haar system is an unconditional basis and the sequence $\{f_m\}$ defined in (4) converges weakly to zero. In particular, the Haar system may be 1-perturbed in any reflexive Orlicz space L_M.

Theorem 2.4. *The Walsh system $\{w_n\}$ has perturbation constant* 1.

Proof. Let $\delta > 0$, and let $T\colon L^p \to L^p$ be a bounded linear operator satisfying $\|Tw_n - w_n\| < 1 - \delta$ for all n.

For $\beta > 0$, let $N_\beta = \{n\colon \varphi^*_n(\varphi_n) \geqslant \beta\}$ and let $A_\beta = \bigcap\limits_n \bigcup\limits_{j \in N_\beta \cap [n,\infty)} \operatorname{supp}\varphi_j$.

By using the techniques of [1] and Theorem 2.3, it suffices to show the existence of $\beta > 0$ such that $mA_\beta > 0$. With

$$A_{n,\beta} = \bigcap_{N_\beta \cap [2^n, 2^{n+1})} \operatorname{supp} \varphi_j, \quad A_\beta = \bigcap_j \bigcup_{n=j}^{\infty} A_{n,\beta},$$

so the perturbability of $\{w_n\}$ follows from the existence of $\alpha, \beta > 0$ such that $mA_{n,\beta} > \alpha$ for all n.

Again, $\delta < w_j^*(Tw_j) < 2-\delta$, and we write $Tw_j = a_j w_j + r_j$, with $w_j^*(r_j) = 0$. For each $n \geqslant 1$ and each l, $2^n \leqslant l < 2^{n+1}$, there is a choice of signs $\{\varepsilon_i^{(l)}\}$ such that

$$\varphi_l = 2^{-n/q} \sum_{i=2^n}^{2^{n+1}-1} \varepsilon_i^{(l)} w_i,$$

and hence

$$\begin{aligned} \varphi_l^*(T\varphi_l) &= 2^{-n/q} \sum_{i=2^n}^{2^{n+1}-1} \varepsilon_i^{(l)} \varphi_l^*(Tw_i) \\ &= 2^{-n/q} \sum_i \varepsilon_i^{(l)} \sum_j 2^{-n/p} \varepsilon_j^{(l)} w_j^*(Tw_i) \\ &= 2^{-n} \sum_i w_i^*(Tw_i) + 2^{-n} \sum_{i \neq j} \varepsilon_i^{(l)} \varepsilon_j^{(l)} w_j^*(Tw_i) = \bar{a}_n + b_l, \end{aligned} \tag{18}$$

where

$$\bar{a}_n = 2^{-n} \sum_{i=2^n}^{2^{n+1}-1} a_n, \quad \text{and} \quad b_l = 2^{-n} \sum_{i \neq j} \varepsilon_i^{(l)} \varepsilon_j^{(l)} w_j^*(Tw_i).$$

It is clear that $\delta < \bar{a}_n < 2-\delta$ and

$$-(\|T\| + \bar{a}_n) \leqslant b_l \leqslant \|T\| - \bar{a}_n. \tag{19}$$

Now select $0 < \alpha < 1$ such that

$$\frac{\alpha}{1-\alpha} [\|T\| - \delta] < \delta/8,$$

and let $N_+ = \{l\colon 2^n \leqslant l < 2^{n+1},\ b_l \geqslant 0\}$ and $N_- = \{l\colon 2^n \leqslant l < 2^{n+1},\ b_l < 0\}$. We consider two cases, and in each show that $mA_{n,3\delta/4} > \alpha/2$.

Case 1. $\#N_+ \geqslant 2^n \alpha$. If $l \in N_+$, then $\varphi_l^*(T\varphi_l) \geqslant \bar{a}_n \geqslant \delta$, so (18) implies that $mA_{n,\delta} > \alpha$.

Case 2. $\#N_+ < 2^n \alpha$. In this case we show that sufficiently many of the b_l, $l \in N_-$, are small in absolute value as compared to $\bar{a}_n$. It is clear that $\sum_{l=2^n}^{2^{n+1}-1} b_l = 0$, and from (19), that $\sum_{i \in N_+} b_l \leqslant 2^n \alpha [\|T\| - \bar{a}_n]$. Hence

$$\sum_{l \in N_-} (-b_l) \leqslant 2^n \alpha [\|T\| - \bar{a}_n].$$

Now, $\#N_- > 2^n(1-\alpha)$, and there is a set $N'_- \subset N_-$, $\#N'_- \geqslant \left(\frac{\alpha}{1-\alpha}\right)\#N_-$ such that

$$\sum_{l \in N'_-} (-b_l) \leqslant \frac{\alpha}{(1-\alpha)} 2^n \alpha [\|T\| - \bar{a}_n].$$

Averaging over N'_- yields

$$\frac{1}{\#N'_-} \sum (-b_l) \leqslant \frac{1-\alpha}{\alpha \# N_-} \cdot \frac{\alpha}{1-\alpha} 2^n \alpha [\|T\| - \bar{a}_n] < \delta/8,$$

by the choice of α.

It now follows that there is a set $M \subset N'_-$, $\#M \geqslant (\alpha/2)\cdot 2^n$, such that $l \in M$ implies $b_l > -\delta/4$, and hence $l \in M$ implies $\varphi_l^*(T\varphi_l) = \bar{a}_n + b_l > 3\delta/4$. Hence $mA_{n,3\delta/4} > \alpha/2$, and the theorem follows.

3. Perturbations of $C(K)$ bases. In this section we present a criterion for the perturbability of a basis for $C(K)$, where K is an uncountable compact metric space. By Milutin's theorem, all such Banach spaces are isomorphic, so in studying isomorphic properties of $C(K)$ spaces, one may choose whichever space K is convenient. Two usual choices are the Cantor set Δ and the unit interval. We show that the Haar system for $C(\Delta)$ and the Schauder system for $C([0,1])$ both have perturbation constant 1.

By a theorem of Pełczyński [5], a subspace Y of $C(K)$ containing an isomorph of $C(K)$ also contains a complemented isomorph of $C(K)$. Therefore, whenever $\{x_n\}$ is a perturbable basis for $C(K)$ and δ and T are as in Definition 1.1, the space $TC(K)$ contains a complemented isomorph of $C(K)$.

Recall that the Haar system may be defined as follows. Let $\{\Delta_{n,i}\}_{n=0,i=0}^{\infty,2^n-1}$ be a basis for the topology of Δ consisting of open and closed sets satisfying

(i) $\Delta_{0,0} = \Delta$,

(ii) $\Delta_{n,i} \cap \Delta_{n,j} = \emptyset$, $i \neq j$,

(iii) $\bigcup_{i=0}^{2^n-1} \Delta_{n,i} = \Delta$, and

(iv) $\Delta_{n+1,2i} \cup \Delta_{n+1,2i+1} = \Delta_{n,i}$.

Define $\varphi_0 = \chi_{\Delta_{0,0}}$, and for $n = 0, 1, 2, \ldots;$ $0 \leqslant i < 2^n$, define $\varphi_{2^n+i} = \chi_{\Delta_{n+1,2i}} - \chi_{\Delta_{n+1,2i+1}}$. It is clear that the Haar system forms a monotone basis for $C(\Delta)$.

The Schauder system $\{p_n\}$ is a monotone basis for $C([0,1])$ defined by $p_0(t) = t$, $p_1(t) = 1-t$,

$$p_2(t) = \begin{cases} 2t, & 0 \leqslant t \leqslant 1/2, \\ 2(1-t), & 1/2 < t \leqslant 1, \end{cases}$$

and for $n \geqslant 1$; $1 \leqslant k < 2^n$.

$$p_{2^n+k}(t) = \begin{cases} p_2\big(2^n(t-(k-1)/2^n)\big), & (k-1)/2^n < t < k/2^n, \\ 0 & \text{otherwise}. \end{cases}$$

We say the function p_{2^n+k} occurs on the rth level of the Schauder system.

Our results follow from the following theorem of Rosenthal [6].

THEOREM. *Let K be an uncountable compact metric space, X a Banach space, and $T\colon C(K) \to X$ a bounded linear operator. If $T^* X^*$ is nonseparable, there exists a subspace Y of $C(K)$, isometric to $C(\Delta)$, such that $T|Y$ is an isomorphism.*

Our perturbation criterion is

THEOREM 3.1. *Let K be an uncountable compact metric space and $\{b_i\}$ a basis for $C(K)$. If there exists a constant $A > 0$ and an uncountable set $M \subset B\big(C(K)^*\big)$ such that $\mu, \nu \in M$ implies $|\mu(b_i) - \nu(b_i)| \geqslant A$ for some i, then $\{b_i\}$ may be $A/2$-perturbed.*

Proof. Let $\varepsilon > 0$ and suppose $T\colon C(K) \to C(K)$ satisfies $\|Tb_i - b_i\| < A/2 - \varepsilon$ for all i. By Rosenthal's theorem it suffices to show that $T^*\big(C(K)^*\big)$ is nonseparable. But for $\mu, \nu \in M$, we have

$$\begin{aligned}\|T^*\mu - T^*\nu\| &= \sup_{\|b\|=1} |(T^*\mu - T^*\nu)(b)| \geqslant \sup_i |(T^*\mu - T^*\nu)(b_i)| \\ &= \sup_i |(\mu-\nu)(Tb_i)| = \sup_i |(\mu-\nu)(Tb_i - b_i) + (\mu-\nu)(b_i)| \\ &\geqslant \sup_i [|(\mu-\nu)(b_i)| - |(\mu-\nu)(Tb_i - b_i)|] \\ &\geqslant A - 2\sup_i \|Tb_i - b_i\| > 2\varepsilon > 0.\end{aligned}$$

Since M is uncountable, this implies that $T^*\big(C(K)^*\big)$ is nonseparable, and the theorem follows.

In the applications which follow, M will be taken to be a collection of point masses. Thus, the condition of Theorem 3.1 is that the basis (b_i) should uniformly separate uncountably many points. Of course, since $C(K)$ separates points, the basis $\{b_i\}$ separates points.

THEOREM 3.2. *The Haar system $\{\varphi_n\}$ in $C(\Delta)$ may be 1-perturbed.*

Proof. Take $M = \{\delta_x \colon x \in \Delta\}$, and let δ_x, δ_y be distinct elements of M. Since the sets $\Delta_{n,i}$ form a basis for the topology of Δ, there is a set $\Delta_{n,i}$ such that

$$x \in \Delta_{n,i}; \quad y \notin \Delta_{n,i}. \tag{1}$$

Assuming n to be the smallest integer such that (1) holds, we have

$$x \in \Delta_{n-1,[i/2]} \quad \text{and} \quad y \in \Delta_{n-1,[i/2]}.$$

Hence

$$|(\delta_x - \delta_y)(\varphi_{2^{n-1}+[i/2]})| = |\varphi_{2^{n-1}+[i/2]}(x) - \varphi_{2^{n-1}+[i/2]}(y)| = 2.$$

Since Δ, and hence M, is uncountable, Theorem 3.2 follows from Theorem 3.1.

As for the Schauder system $\{p_n\}$, taking $M = \{\delta_x \colon x \in [0,1]\}$, it is easily seen that $\mu, \nu \in M$ implies $|\mu(p_n) - \nu(p_n)| \geqslant 1/2$ for some n, so that the Schauder system may be $1/4$-perturbed. By constructing a subset of the point masses in a fashion analogous to the classical construction of the Cantor set, it follows from Theorem 3.1 that the Schauder system may be $1/2$-perturbed. In fact, this technique can be used for other bases. In the case of the Schauder system this result may be improved.

THEOREM 3.3. *The Schauder system $\{p_n\}$ has perturbation constant 1.*

Proof. Let $\delta > 0$ and suppose $T\colon C([0,1]) \to C([0,1])$ satisfies $\|Tp_n - p_n\| < 1 - \delta$ for all n. We show directly that T^* has nonseparable range.

Let $p_n' = Tp_n$, and for each interval $I \subset [0,1]$, define

$$o(I) = \sup_{\substack{x,y \in I \\ n}} |p_n'(x) - p_n'(y)|.$$

We first show that $\inf_I o(I) > 0$. Suppose to the contrary that $\inf_I o(I) = 0$. Then there exists an interval I with $o(I) < \delta/2$. Since $\|p_n - p_n'\| < 1 - \delta$, we have that $\sup_x p_n'(x) > \delta$, and whenever $\operatorname{supp} p_n \subset I$, this implies $\inf_{y \in I} p_n'(y) > \delta/2$. Now choose $N > 2\|T\|\delta^{-1}$, and disjointly supported Schauder functions $p_{n_1}, \ldots, p_{n_N}$ with $\operatorname{supp} p_{n_i} \subset I$ for each i. Then with $f = \sum_{i=1}^N p_{n_i}$, we have $\|f\| = 1$, yet for $x \in I$, $Tf(x) = \sum_{i=1}^N p'_{n_i}(x) > N(\delta/2) > \|T\|$. This contradiction establishes that $\inf_I o(I) \geqslant \delta/2$.

Thus, for any $I \subset [0,1]$, there exists points $x, y \in I$, and n such that $|p_n'(x) - p_n'(y)| > \delta/3$, so by a construction analogous to the classical construction of the Cantor set, there is an uncountable set of points M such that for each pair of distinct $x, y \in M$, there exists n with $|(\delta_x - \delta_y)(p_n')| > \delta/3$. It follows that $T^*\big(C([0,1])^*\big)$ is nonseparable, and hence that $\{p_n\}$ is 1-perturbable.

Bases for $C(K)$ spaces are not in general B-perturbable for any B. Warren [7] and Wojtaszczyk [8] have shown the existence of a normalized basis $\{f_n\}$ for $C([0,1])$ which is weakly convergent to 0. Warren's construction provides an example of a basis for $C([0,1])$ which is not B-perturbable for any $B > 0$.

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Subspaces of smooth sequence spaces

by

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Abstract. This work is concerned with subspaces of nuclear Fréchet smooth sequence spaces. Particular attention is paid to those subspaces which are isomorphic to power series spaces.

The investigation of all infinite-dimensional subspaces of nuclear power series spaces of finite and infinite types is the subject of two important papers of Dubinsky [6], [7]. The earlier works of Rolewicz [12] and Zahariuta [16] were concerned, to some extent, with subspaces of power series spaces. The concepts of smooth sequence spaces of finite and infinite types were introduced in [13] as a generalization of the notion of power series spaces and nuclearities based on such spaces were briefly studied in [4]. The present paper is basically concerned with subspaces of nuclear Fréchet smooth sequence spaces.

In Section 1 we collect the necessary definitions and in Section 2 obtain some properties of block basic sequences with respect to the canonical basis of nuclear Köthe spaces. Section 3 is on basic sequences in $\Lambda_1(\alpha)$ and G_∞-subspaces of $\Lambda_1(\alpha)$ and G_1-subspaces of $\Lambda_1(\alpha)$. In particular, it is proved that if a G_1-space is isomorphic to a subspace of $\Lambda_1(\alpha)$, then it is isomorphic to a power series space of finite type (Theorem 7). In Section 4 we study subspaces of G_∞-spaces; the subspaces considered are power series spaces of infinite type or $L_f(b, \infty)$ spaces of Dragilev [3] or G_∞-spaces. Zahariuta [16] showed earlier that an $L_f(b, \infty)$ space is either isomorphic to a power series space of infinite type or has no subspace isomorphic to a power series space. We show that this result does not extend to general G_∞-spaces. We also give examples of G_∞-spaces which do not contain subspaces isomorphic to power series spaces while these G_∞-spaces are themselves isomorphic to subspaces of each nuclear power series space $\Lambda_\infty(\beta)$ which is stable.

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1. Preliminaries. We refer the reader to [4], [9], [10], and [11] for terms that are not defined here.