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Compact sets of tight measures

by

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Abstract. Topsøe has proved generalisations of the famous Prohorov result connecting compactness and tightness in spaces of measures; his necessary and sufficient conditions were developed from the theory of extension of tight contents via inner measures. We show how these criteria can also be obtained using simple functional analytic arguments, at least when certain topological regularity properties are placed on the underlying spaces.

1. Introduction. For a very general setting, Topsøe [6] has characterised compact sets and nets in the cone $M^+(X, t)$ of tight measures on a set X, using a weak topology which generalises the usual topology of weak convergence. In this paper we show how necessary and sufficient conditions analogous to his can be derived for the latter topology by simple functional analytic arguments similar to those of Bourbaki [2]. When the underlying space X is Tychonov (completely regular and T_1) our results coincide with Topsøe's.

2. Statement of main results. Let X be a topological space and $C(X)$ the Banach space of all bounded, continuous, real-valued functions on X. To simplify slightly, we assume that X is completely Hausdorff ($C(X)$ separates the points of X), though the results could all be expressed in terms of Baire measures without this assumption. A bounded linear functional T is said to be *tight* if $T(f_\alpha)\to 0$ for every net (f_α) in $C(X)$ for which $1 \geqslant |f_\alpha|\to 0$ uniformly on compacta. It can be shown ([5] and [7]) that the tight functionals are precisely those that can be (uniquely) represented as an integral with respect to a tight signed Borel measure on X, i.e., a difference $\mu^+ - \mu^-$ of two bounded non-negative Borel measures satisfying the regularity condition: $\mu^\pm(A) = \sup\{\mu^\pm(K)\colon A \supseteq K \in \mathscr{K}\}$ for every Borel set A, where $\mathscr{K}$ is the paving of compact subsets of X.

We shall identify the functional with its representing measure, and write $M(X, t)$ for the space of all such tight functionals/measures on X. This space we equip with the topology of weak convergence which

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is the weakest topology making the map $\mu \to \mu(f)$ continuous for each fixed $f \in C(X)$. Notice that this is just the relativised weak $*$ topology when $M(X, t)$ is regarded as a subspace of the dual of $C(X)$. The basic result on compactness in $M(X, t)$ under this topology is

THEOREM 1. *$A \subseteq M(X, t)$ is relatively compact in $M(X, t)$ iff the following two conditions are satisfied:*

(i) $\sup\{\|\mu\|: \mu \in A\} < \infty$,

(ii) *for every net (f_a) in $C(X)$ for which $1 \geqslant |f_a| \to 0$ uniformly on compacta, and every $\varepsilon > 0$, there exist (finitely many) $a_1, \ldots, a_n$ for which $\sup_A \min_i |\mu(f_{a_i})| < \varepsilon$.*

In condition (i), $|\mu| = \mu^+(X) + \mu^-(X)$ denotes the total variation of μ, which is its norm in its interpretation as a linear functional on the Banach space $C(X)$; cf. [7].

This result can be re-expressed in purely measure theoretic terms when we consider only the positive cone $M^+(X, t)$. For this we need to define the paving $\mathscr{P}$ of *positive sets* of X which consists of those open sets which are complements of zero sets (= sets expressible in the form $f^{-1}(0)$ for some $f \in C(X)$). This paving has the separation property: *if $K \in \mathscr{K}$ and $G \in \mathscr{P}$ and $K \subseteq G$ then there is an $f \in C(X)$ for which $0 \leqslant f \leqslant 1$, $f = 0$ on K and $f = 1$ on $\complement G$.* Also a paving $\mathscr{G}$ is said to *dominate* $\mathscr{K}$ if each member of $\mathscr{K}$ is contained in some member of $\mathscr{G}$.

THEOREM 2. *$A \subseteq M^+(X, t)$ is relatively compact in $M^+(X, t)$ iff the following two conditions are satisfied:*

(i)′ $\sup\{\mu(X): \mu \in A\} < \infty$,

(ii)′ *for every $\mathscr{G} \subseteq \mathscr{P}$ which dominates $\mathscr{K}$, and $\varepsilon > 0$, there exist (finitely many) $G_1, \ldots, G_n \in \mathscr{G}$ for which $\sup_A \min_i \mu(\complement G_i) < \varepsilon$.*

By imposing further conditions on the topology of X we obtain the same form of the theorem as given by Topsøe [6]:

THEOREM 3. *If X is a Tychonov space, then Theorem 2 is still valid with $\mathscr{P}$ replaced by the paving of all open subsets of X.*

Notice that Topsøe's [6] w-topology is in general finer than the topology of weak convergence, but that they are the same when X is Tychonov.

3. Proof of Theorem 1. We actually prove a slightly reformulated version of this result to emphasize its functional analytic content. Suppose B is a Banach space with dual B^* (equipped with the weak $*$ topology), and let $\mathscr{D}$ be any family of nets on the unit sphere $\{b \in B: \|b\| \leqslant 1\}$ of B. Define $B^*_{\mathscr{D}}$ as the subspace of B^* consisting of those b^* for which $\langle b_a, b^* \rangle \to 0$ for every net (b_a) in $\mathscr{D}$. Then we show that $A \subseteq B^*_{\mathscr{D}}$ is relatively compact in $B^*_{\mathscr{D}}$ iff

(i) $M = \sup\{\|b^*\|: b^* \in A\} < \infty$,

(ii) for every subnet (b_a) of a net in $\mathscr{D}$ and $\varepsilon > 0$ there are (finitely many) $a_1, \ldots, a_n$ for which

$$\sup_A \min_i |\langle b_{a_i}, b^* \rangle| < \varepsilon.$$

Proof. For necessity we need only consider the case of A being a compact subset of $B^*_{\mathscr{D}}$. Then (i) is an immediate consequence of the Uniform Boundedness Theorem [3]. Also the family of sets $G_a = \{b^* \in B: |\langle b_a, b^* \rangle| < \varepsilon\}$ is an open cover of $B^*_{\mathscr{D}}$. By choosing the finite subcover $G_{a_1}, \ldots, G_{a_n}$ of the compact set A we arrive at (ii).

Conversely, if A satisfies (i) and (ii), consider its closure $\bar{A}$ in B^*. Because of (i) and the Alaoglu theorem [3] $\bar{A}$ is compact. We show that $\bar{A} \subseteq B^*_{\mathscr{D}}$. Suppose to the contrary that there is a $b_0^* \in \bar{A}$ and a net (b_β) in $\mathscr{D}$ for which $\langle b_\beta, b_0^* \rangle$ does not tend to zero. Then we can extract a subnet (b_a) for which $|\langle b_a, b_0^* \rangle| \geqslant \varepsilon > 0$ for all a. Let $f(b^*) = \min_i |\langle b_{a_i}, b^* \rangle|$ be the continuous function of b^* on B^* obtained from the a_i's given by condition (ii). Then it is elementary that $f(b^*)$ has the same supremum over A and $\bar{A}$, which contradicts (ii). The result follows. ∎

4. Proof of Theorem 2. We show that the two conditions are equivalent to those of Theorem 1. Clearly, (i) and (i)′ are equivalent.

Suppose that condition (ii) is satisfied and that $\mathscr{G} \subseteq \mathscr{P}$ dominates $\mathscr{K}$. For each $K \in \mathscr{K}$ there is a $G_K \in \mathscr{G}$ containing K. Then by the separation property of $\mathscr{P}$ and $\mathscr{K}$ we can find an $f_K \in C(X)$ with $0 \leqslant f_K \leqslant 1$ and $f_K = 0$ on K, $f_K = 1$ on $\complement G_K$. Regarding $\mathscr{K}$ as a directed set means that $(f_K)_{K \in \mathscr{K}}$ is a net tending uniformly to zero on compacta. Thus by (ii) we can find $K_1, \ldots, K_n \in \mathscr{K}$ such that $\sup_A \min_i \mu(f_{K_i}) < \varepsilon$. But $f_K \geqslant 1_{\complement G_K}$ so that $\mu(f_K) \geqslant \mu(\complement G_K)$ and (ii)′ follows.

Conversely, suppose (ii)′ is satisfied. If $1 \geqslant |f_a| \to 0$ uniformly on compacta, then the family of $\mathscr{P}$ sets of the form $G_a = \{|f_a| < \varepsilon\}$ dominates $\mathscr{K}$; so we can find $a_1, \ldots, a_n$ for which $\sup_A \min_i \mu(\complement G_{a_i}) < \varepsilon$. Then

$$\sup_A \min_i |\mu(f_{a_i})| \leqslant \sup_A \min_i \int_{G_{a_i}} + \int_{\complement G_{a_i}} |f_{a_i}| \, d\mu \leqslant \varepsilon \cdot \sup_A \mu(X) + \varepsilon,$$

and (ii) follows. ∎

5. Proof of Theorem 3. The result follows immediately from: *if open $G \supseteq K \in \mathscr{K}$, then there is a $G' \in \mathscr{P}$ with $G \supseteq G' \supseteq K$.* To see this we need only note that the topology of the Tychonov space X is induced by $C(X)$, and hence $\mathscr{P}$ is a base for the topology. Thus the family of all $\mathscr{P}$ sets contained in G filters up to G, and so the assertion follows from the compactness of K. ∎

6. Extensions to nets. Topsøe [6] has actually obtained the necessary and sufficient conditions for a net on $M^+(X, t)$ to be relatively compact (in his w-topology) in $M^+(X, t)$. We indicate how our Theorems 1, 2 and 3 can be extended to this case.

A net (y_i) on a topological space Y is said to be *relatively compact* in a subset Y_0 of Y if every subnet of (y_i) contains a further subnet convergent to a point of Y_0. Equivalently, (y_i) is relatively compact in Y_0 if every universal subnet [4] is convergent to a point of Y_0. It can also be shown that (y_i) is relatively compact in Y_0 iff *for every family of open sets covering Y_0 there is a finite subfamily in which the net eventually lies.*

Using these properties we can prove a net analogue of our functional analytic form of Theorem 1. Notice that the Uniform Boundedness Theorem is not in general true for nets [3], so we have to include norm boundedness as an assumption.

THEOREM 4. *Let (b_β^*) be a net in B^* for which $M = \limsup_\beta \|b_\beta^*\| < \infty$. Then it is relatively compact in $B^*_{\mathscr{D}}$ iff for every subnet (b_a) of a net in $\mathscr{D}$ and $\varepsilon > 0$ there exist $a_1, \ldots, a_n$ such that*

$$\limsup_\beta \min_i |\langle b_{a_i}, b_\beta^*\rangle| < \varepsilon.$$

Proof. The necessity of the condition is proved in a similar manner to the earlier result, by making use of the "open covering" characterisation of relatively compact nets.

Conversely, if (b_γ^*) is a universal subnet of (b_β^*), then for each $b \in B$ the net $\langle b, b_\gamma^*\rangle$ is universal [4] on the compact interval $[-M\|b\|, M\|b\|]$ (we may assume $\|b_\beta^*\| \leqslant M$ for all β without loss of generality). Thus $\langle b, L\rangle = \lim_\gamma \langle b, b_\gamma^*\rangle$ defines a member L of B^* with $\|L\| \leqslant M$. We show that $L \in B^*_{\mathscr{D}}$. If not, we could find a subnet (b_a) of a net in $\mathscr{D}$ for which $|\langle b_a, L\rangle| \geqslant \varepsilon > 0$ for all a. But then for any finitely many $a_1, \ldots, a_n$ we would have

$$\limsup_\beta \min_i |\langle b_{a_i}, b_\beta^*\rangle| \geqslant \limsup_\gamma \min_i |\langle b_{a_i}, b_\gamma^*\rangle| \geqslant \varepsilon$$

which is a contradiction. ∎

We omit the deduction of the analogues of Theorems 2 and 3 as there are no essential difficulties involved.

7. Remarks. By functional analytic methods it is easy to prove the existence of a cluster point of a set (or net) in a large space (B^*) and then prove that this cluster point is in fact in the required smaller space ($B^*_{\mathscr{D}}$). This contrasts with Topsøe's [6] method, where the cluster point has to be constructed by a more laborious application of the theory of tight contents. However, it should be pointed out that those techniques are

fundamental to the derivation of the representation theorem (as discussed in [5]) used to translate the functional analytic form of the results to the corresponding measure theoretic expressions.

Results for other spaces of measures such as the τ-smooth measures can be obtained by using a different class of nets $\mathscr{D}$. In the τ-smooth case, though, the proof (at least for the positive cone) reduces to the usual application of Dini's theorem, as in [1].

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