ACTA ARITHMETICA
LXXXVIII.4 (1999)

# On smooth integers in short intervals under the Riemann Hypothesis

by

Ti Zuo Xuan (Beijing)

**1. Introduction.** We say a natural number $n$ is *$y$-smooth* if every prime factor $p$ of $n$ satisfies $p \le y$. Let $\Psi(x,y)$ denote the number of $y$-smooth integers up to $x$. The function $\Psi(x,y)$ is of great interest in number theory and has been studied by many researchers.

Let $\Psi(x,z,y) = \Psi(x+z,y) - \Psi(x,y)$. In this paper, we will give an estimate for $\Psi(x,z,y)$ under the Riemann Hypothesis (RH).

Various estimates for $\Psi(x,z,y)$ have been given by several authors. (See [1]–[9].)

In 1987, Balog [1] showed that for any $\varepsilon > 0$ and $X \ge X_0(\varepsilon)$ the interval $(X, X + X^{1/2+\varepsilon}]$ contains an integer having no prime factors exceeding $X^{\varepsilon}$.

Harman [6] improved this result, and he proved that the bound $X^{\varepsilon}$ on the size of the prime factors can be replaced by $\exp\{(\log x)^{2/3+\varepsilon}\}$.

Recently, Friedlander and Granville [3] improved the "almost all" results of Hildebrand and Tenenbaum [9] and proved the following result:

Fix $\varepsilon > 0$. The estimate

$$\Psi(x,z,y) = \frac{z}{x}\Psi(x,y)\left(1 + O\left(\frac{(\log\log y)^2}{\log y}\right)\right) \tag{1.1}$$

holds uniformly for

$$x \ge y \ge \exp\{(\log x)^{5/6+\varepsilon}\} \tag{1.2}$$

with

$$x \ge z \ge x^{1/2} y^2 \exp\{(\log x)^{1/6}\}. \tag{1.3}$$

The authors of [3] also point out that up to now there is no indication of how to break the "$\sqrt{x}$ barrier", that is, to prove that $\Psi(x+\sqrt{x},y) -$

1991 *Mathematics Subject Classification*: Primary 11N25.
Project supported by the National Natural Science Foundation of People's Republic of China.

$\Psi(x,y)>0$ when $y$ is an arbitrarily small power of $x$; this is evidently the most challenging open problem in this area.

The problem is very difficult indeed. In this paper, we only prove that

$$\Psi(X+\sqrt{X}(\log X)^{1+\varepsilon}, X^{\delta}) - \Psi(X, X^{\delta}) > 0,$$

even if the RH is true, and we state it formally as a theorem.

THEOREM. *If the RH is true, then for any $\varepsilon>0$, $\delta>0$ and $X \ge X_0(\varepsilon,\delta)$, the interval $(X, X+Y]$, where $\sqrt{X}(\log X)^{1+\varepsilon} \le Y \le X$, contains an integer having no prime factors exceeding $X^{\delta}$.*

**2. Proof of the Theorem.** To prove the Theorem, we need the following lemmas.

LEMMA 1. *For $N, T \ge 1$ and any sequence $b_n$ of complex numbers, we have*

$$\int_0^T \Big|\sum_{n\le N} b_n n^{it}\Big|^2\,dt \ll (T+N)\sum_{n\le N}|b_n|^2.$$

Proof. See Theorem 6.1 of [10].

LEMMA 2. *If the RH is true then for $1/2+\varepsilon \le \sigma \le 2$ we have uniformly*

$$\frac{\zeta'}{\zeta}(s) \ll \log(|t|+2).$$

Proof. See [12, p. 340].

Let $0<\varepsilon<1/8$ be fixed. We put

$$M = X^{1/2}(\log X)^{-1-\varepsilon}, \qquad N = (\log X)^{2+2\varepsilon},$$
$$Y \ge \frac{X}{M} = X^{1/2}(\log X)^{1+\varepsilon}, \qquad y = X^{\delta},$$
$$a(m) = \begin{cases} 1 & \text{if } p\,|\,n \Rightarrow p \le y,\\ 0 & \text{otherwise,}\end{cases} \qquad M(s) = \sum_{M<m\le 2M} \frac{a(m)}{m^s}.$$

As in [6] we will show that

$$\int_X^{X+Y} \Big(\sum\nolimits^{*} a(m_1)a(m_2)\Lambda(r)\Big)\,dx = Y^2M^2(1) + O(Y^2(\log X)^{-\varepsilon/4}), \tag{2.1}$$

where $*$ represents the summation conditions

$$m_1m_2r \in (x, x+Y], \qquad X \le x \le X+Y,$$
$$M < m_i \le 2M, \qquad i = 1,2.$$

By the Perron formula (see Lemma 3.19 of [12]) we have for $x \notin \mathbb{Z}$, $x+Y \notin \mathbb{Z}$,

$$(2.2)\qquad \sum\nolimits^{*} a(m_1)a(m_2)\Lambda(r)$$
$$= \frac{-1}{2\pi i}\int\limits_{c-iT}^{c+iT} \frac{\zeta'}{\zeta}(s)M^2(s)\frac{(x+Y)^s - x^s}{s}\,ds + O\biggl(\frac{x\log^2 x}{T}\biggr) + O(\log x),$$

where $c = 1 + 1/\log X,\ T = X^4$, and the $O$ constants are absolute.

We now integrate (2.2) with respect to $x$ between $X$ and $X+Y$, and obtain that

$$(2.3)\qquad \int\limits_{X}^{X+Y} \Bigl(\sum\nolimits^{*} a(m_1)a(m_2)\Lambda(r)\Bigr)\,dx$$
$$= \frac{-1}{2\pi i}\int\limits_{c-iT}^{c+iT} \frac{\zeta'}{\zeta}(s)M^2(s)A(s)\,ds + O\biggl(\frac{XY\log^2 X}{T}\biggr) + O(Y\log X),$$

where

$$A(s) = \frac{(X+2Y)^{s+1} - 2(X+Y)^{s+1} + X^{s+1}}{s(s+1)}.$$

We note that $A(1) = Y^2$, and

$$(2.4)\qquad A(s) \ll \min(Y^2X^{\sigma-1}, X^{\sigma+1}|t|^{-2}).$$

From the definitions of $T$ and $Y$, it follows that the two error terms in (2.3) are $\ll Y^2\exp\{-(\log X)^{1/2}\}$.

By the theorem of residues, the integral on the right side of (2.3) is

$$(2.5)\qquad Y^2M^2(1) + \frac{1}{2\pi i}\biggl(\int\limits_{c-iT}^{\eta-iT} + \int\limits_{\eta-iT}^{\eta+iT} + \int\limits_{\eta+iT}^{c+iT}\biggr),$$

where $\eta = 1/2 + \varepsilon/3$.

Here we estimate $|M(s)|$ trivially as

$$(2.6)\qquad |M(s)| \le M^{1-\sigma}.$$

From this, (2.4) and Lemma 2, the integrals along the lines $[c-iT, \eta-iT]$ and $[\eta+iT, c+iT]$ are

$$(2.7)\qquad \ll \int\limits_{\eta}^{c} M^{2-2\sigma}X^{\sigma+1}T^{-2}\log T\,d\sigma$$
$$\ll X^2T^{-2}\log T \ll Y^2\exp\{-(\log X)^{1/2}\}.$$

Also,

$$\int_{\eta-iT}^{\eta+iT} \frac{\zeta'}{\zeta}(s)M^2(s)A(s)\,ds \ll Y^2X^{\eta-1}\log X \int_0^{X/Y} |M(\eta+it)|^2\,dt + X^{\eta+1}\log X \int_{X/Y}^{T} |M(\eta+it)|^2 t^{-2}\,dt = I_1 + I_2. \tag{2.8}$$

By Lemma 1, we have

$$I_1 \ll Y^2X^{\eta-1}\log X\left(\frac{X}{Y}+M\right)M^{1-2\eta} \ll Y^2\log X\cdot N^{-1+\eta} \ll Y^2(\log X)^{-\varepsilon/4}. \tag{2.9}$$

From Lemma 1 and (2.6) together with integration by parts we have

$$I_2 \ll X^{\eta+1}\log X\left(\frac{X}{Y}\right)^{-2}\left(\frac{X}{Y}+M\right)M^{1-2\eta} \ll Y^2\log X\cdot N^{-1+\eta} \ll Y^2(\log X)^{-\varepsilon/4}. \tag{2.10}$$

So from (2.3), (2.5) and (2.7)–(2.10) we get (2.1). By Theorem 1 of [7], we have

$$M(1) = \sum_{M<m\le 2M} \frac{a(m)}{m} \gg_\delta 1.$$

The Theorem follows from (2.1) and the above estimate.

REMARKS. Using the methods of this paper, we can prove the following results.

*For any* $X \ge X_0(\varepsilon)$, *the interval* $(X, X+Y]$ *contains an integer having no prime factors exceeding* $y$, *where*

(i) $X \ge Y \ge X^{1/2}\exp\{(\log X)^{5/6+\varepsilon}\}$ *and* $X \ge y \ge \exp\{(\log X)^{5/6+\varepsilon}\}$,

*or*

(ii) $X \ge Y \ge X^{1/2}\exp\left\{\frac{\log X}{(\log\log X)^b}\right\}$ *and*
$X \ge y \ge \exp\{C(\log X)^{2/3}(\log\log X)^{4/3+b}\}$,

*where* $b$ *is any fixed positive number and* $C$ *is a sufficiently large absolute constant.*

The result suitable for the ranges (ii) is stronger than one of Harman [6] and the ranges (i) are wider than the ranges (1.2) and (1.3) of the asymptotic estimate of Friedlander and Granville [3] since the bound for $Y$ in (i) is independent of $y$.

The proofs of the results are similar to that of the Theorem, but for the ranges (i) with

$$M = X^{1/2}\exp\{(-\log X)^{5/6+\varepsilon}\}, \qquad N = \exp\{2(\log X)^{5/6+\varepsilon}\},$$

and

$$\eta = 1 - \frac{c_1}{(\log X)^{2/3+\varepsilon}};$$

and for the ranges (ii) with

$$M = X^{1/2}\exp\left\{-\frac{\log X}{(\log\log X)^b}\right\}, \qquad N = \exp\left\{\frac{2\log X}{(\log\log X)^b}\right\},$$

and

$$\eta = 1 - \frac{c_1}{(\log X)^{2/3}(\log\log X)^{1/3}}.$$

Moreover, in the proof we also need the following result: the estimate

$$\frac{\zeta'}{\zeta}(s) \ll \log(|t|+2)$$

holds uniformly in the ranges $\sigma \geq 1 - c_1/((\log X)^{2/3}(\log\log X)^{1/3})$ and $|t| \leq X^4$. This estimate follows from an estimate of [11] and Theorems 3.10 and 3.11 of [12] with $\varphi(t) = \frac{302}{3}\log\log t$ and $\theta(t) = (\log\log t)^{2/3}/(\log t)^{2/3}$.

**Acknowledgements.** I would like to thank the referee for his comments and suggestions.

## References

[1] A. Balog, *On the distribution of integers having no large prime factors*, Astérisque 147–148 (1987), 27–31.

[2] A. Balog and A. Sárközy, *On sums of integers having small prime factors: II*, Studia Sci. Math. Hungar. 19 (1984), 81–88.

[3] J. B. Friedlander and A. Granville, *Smoothing "smooth" numbers*, Philos. Trans. Roy. Soc. London Ser. A 345 (1993), 339–347.

[4] J. B. Friedlander and J. C. Lagarias, *On the distribution in short intervals of integers having no large prime factors*, J. Number Theory 25 (1987), 249–273.

[5] A. Granville, *Integers, without large prime factors, in arithmetic progressions. II*, Philos. Trans. Roy. Soc. London Ser. A 345 (1993), 349–362.

[6] G. Harman, *Short intervals containing numbers without large prime factors*, Math. Proc. Cambridge Philos. Soc. 109 (1991), 1–5.

[7] A. Hildebrand, *On the number of positive integers $\leq x$ and free of prime factors $> y$*, J. Number Theory 22 (1986), 289–307.

[8] A. Hildebrand and G. Tenenbaum, *On integers free of large prime factors*, Trans. Amer. Math. Soc. 296 (1986), 265–290.

[9] —, —, *Integers without large prime factors*, J. Théor. Nombres Bordeaux 5 (1993), 411–484.

[10] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Springer, 1971.

[11] H. E. Richert, *Zur Abschätzung der Riemannschen Zetafunktion in der Nähe der Vertikalen* $\sigma = 1$, Math. Ann. 169 (1967), 97–101.
[12] E. C. Titchmarsh, *The Theory of the Riemann Zeta Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford Univ. Press, 1986.

Department of Mathematics
Beijing Normal University
Beijing 100875
People's Republic of China

*Received on 28.4.1997*
*and in revised form on 22.12.1997* (3175)