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Zero density estimates of L-functions associated with cusp forms

by

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1. Introduction. Let k be a positive even integer, and $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ a holomorphic cusp form of weight k with respect to $\Gamma = SL_2(Z)$. We denote by $S_k(\Gamma)$ the space of those functions. Let q be a positive integer, and χ a Dirichlet character mod q. Let $s = \sigma + it$ be a complex variable. We define the L-function by

$$L_f(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^s}$$

for $\sigma > (k+1)/2$. Denote by χ^* the primitive character mod q_1 inducing χ. It is known that the function $L_f(s,\chi^*)$ has an analytic continuation to the whole complex plane and satisfies the functional equation (see [5])

$$\left(\frac{2\pi}{q_1}\right)^{-s} \Gamma(s) L_f(s,\chi^*) = i^k \left(\frac{W(\chi^*)}{|W(\chi^*)|}\right)^2 \left(\frac{2\pi}{q_1}\right)^{s-k} \Gamma(k-s) L_f(k-s,\overline{\chi^*}),$$

where $W(\chi^*)$ is Gaussian sum and $\Gamma(s)$ is the gamma function. Moreover, if the cusp form f is the normalized eigenform, that is, the eigenfunction of all Hecke operators with $a(1) = 1$, then $a(n)$'s are real numbers and $L_f(s,\chi)$ has the Euler product expansion

$$L_f(s,\chi) = \prod_p (1 - \chi(p)a(p)p^{-s} + \chi(p)^2 p^{k-1-2s})^{-1}$$

for $\sigma > (k+1)/2$, where the product runs over all prime numbers. Therefore, $L_f(s,\chi)$ has the representation

$$L_f(s,\chi) = L_f(s,\chi^*) \prod_{p|q} (1 - \chi^*(p)a(p)p^{-s} + \chi^*(p)^2 p^{k-1-2s}), \tag{1}$$

and (1) gives the analytic continuation of $L_f(s,\chi)$ to the whole complex plane for every χ. We can also see that $L_f(s,\chi)$ has no zeros for $\sigma > (k+1)/2$,

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has simple zeros at non-positive integers, and has no zeros for $\sigma < (k-1)/2$ except non-positive integers. We call zeros at non-positive integers *trivial*, and those lying in $(k-1)/2 \le \sigma \le (k+1)/2$ *non-trivial*. Since $a(n)$'s are real, we have the relation $\overline{L_f(\overline{s}, \overline{\chi})} = L_f(s, \chi)$ for any s. If χ is a primitive character, from this relation and the functional equation, non-trivial zeros of $L_f(s, \chi)$ are distributed symmetrically with respect to the line $\sigma = k/2$. In case χ is an imprimitive character, non-trivial zeros of $L_f(s, \chi)$ are those of $L_f(s, \chi^*)$ and infinite zeros on $\sigma = (k-1)/2$ which are coming from the finite products in (1).

For the purpose of counting the number of non-trivial zeros, we define

$$\begin{aligned} &N_f(T, \chi) \\ &\quad = \sharp\{\varrho = \beta + i\gamma \mid L_f(\varrho, \chi) = 0, (k-1)/2 \le \beta \le (k+1)/2, -T \le \gamma \le T\}, \\ &N_f(\sigma_0, T, \chi) = \sharp\{\varrho = \beta + i\gamma \mid L_f(\varrho, \chi) = 0, \sigma_0 \le \beta \le (k+1)/2, -T \le \gamma \le T\} \end{aligned}$$

for $\sigma_0 \ge k/2$. We can show the following results by modifying the proof for the case of Dirichlet L-functions in an obvious way (see [1]). We have

$$N_f(T+1, \chi) - N_f(T-1, \chi) \le C \log(q(T+2)) \tag{2}$$

for any $T \ge 1$ and some positive constant C. We also have

$$N_f(T, \chi) = \frac{2T}{\pi} \log \frac{T}{2\pi} + O(T \log(q+1)), \quad T \to \infty,$$

uniformly in q. In particular, for a primitive character χ,

$$N_f(T, \chi) = \frac{2T}{\pi} \log \frac{qT}{2\pi} - \frac{2T}{\pi} + O(\log(qT)), \quad T \to \infty,$$

uniformly in q.

The purpose of this paper is to show the following theorem.

Theorem 1. *Let $f \in S_k(\Gamma)$ be the normalized eigenform and χ a Dirichlet character mod q. If $q \ll T$, then*

$$\sum_{\chi} N_f(\sigma_0, T, \chi) \ll (qT)^{\frac{k+1-2\sigma_0}{k/2+1-\sigma_0}} (\log(qT))^{69}, \quad T \to \infty, \tag{3}$$

uniformly in σ_0 and q for $k/2 + 1/\log(qT) \le \sigma_0 \le k/2 + 1/3$, and

$$\sum_{\chi} N_f(\sigma_0, T, \chi) \ll (qT)^{3(k+1-2\sigma_0)/2} (\log(qT))^{100}, \quad T \to \infty, \tag{4}$$

uniformly in σ_0 and q for $k/2 + 1/3 \le \sigma_0 \le (k+1)/2$, where $\sum_\chi$ means a sum running over all Dirichlet characters mod q.

Specialising $q = 1$ in Theorem 1, we have

$$N_f(\sigma_0, T, \chi_0) \ll T^{\frac{k+1-2\sigma_0}{k/2+1-\sigma_0}} (\log T)^{69}, \quad T \to \infty,$$

uniformly for $k/2+1/\log T \le \sigma_0 \le k/2+1/3$,

$$N_f(\sigma_0,T,\chi_0) \ll T^{3(k+1-2\sigma_0)/2}(\log T)^{100}, \qquad T\to\infty,$$

uniformly for $k/2+1/3 \le \sigma_0 \le (k+1)/2$, where χ_0 is the trivial character. As regards the estimate of $N_f(\sigma_0,T,\chi_0)$, Ivić has shown in [4] that

$$N_f(\sigma_0,T,\chi_0) \ll T^{\frac{k+1-2\sigma_0}{k/2+1-\sigma_0}+\varepsilon}, \qquad T\to\infty,$$

for $k/2 \le \sigma_0 \le k/2+1/4$,

$$N_f(\sigma_0,T,\chi_0) \ll T^{\frac{k+1-2\sigma_0}{\sigma_0-(k-1)/2}+\varepsilon}, \qquad T\to\infty,$$

for $k/2+1/4 \le \sigma_0 \le (k+1)/2$, and also has shown sharper bounds when σ_0 is near $(k+1)/2$. Therefore, Theorem 1 is a natural extension of Ivić's results for $k/2+1/\log T \le \sigma_0 \le k/2+1/4$.

Theorem 1 is an analogue of zero density estimates of Dirichlet L-functions by Montgomery [6]. Montgomery used the estimate of the mean fourth power of Dirichlet L-functions on the critical line for this problem. Since the corresponding fourth power result is not known in our case, we shall use the mean square of $L_f(s,\chi)$ to prove Theorem 1 (see Theorem 2 in Section 3). To estimate the mean square of $L_f(s,\chi)$, we reduce the problem to the study of the mean square of the Dirichlet polynomial by using the approximate functional equation of $L_f(s,\chi)$, which is proved by applying the method of Good [3].

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2. The approximate functional equation. Throughout this section, we suppose f is in $S_k(\Gamma)$ and χ is a primitive character mod q. We shall prove the approximate functional equation of $L_f(s,\chi)$ whose implied constant is uniform in q, following the method of Good [3].

Rankin has shown in [7] that

$$\sum_{n\le x}|a(n)|^2 = Cx^k + O(x^{k-2/5}), \qquad x\to\infty,$$

where C is a positive constant depending on k. By Cauchy's inequality,

$$\sum_{n\le x}|a(n)| \ll x^{(k+1)/2}, \qquad x\to\infty,$$

hence we obtain the following lemma by partial summation.

Lemma 1. *Let σ be a real number. Then*

$$\sum_{n\le x}|a(n)|n^{-\sigma} \ll x^{(k+1)/2-\sigma}, \qquad x\to\infty, \tag{5}$$

uniformly for $\sigma \le \sigma_1 < (k+1)/2$, *and*

$$\sum_{n\le x} |a(n)|^2 n^{-2\sigma} \ll \begin{cases} x^{k-2\sigma} & \text{uniformly for } \sigma \le \sigma_2 < k/2, \\ \log x & \text{uniformly for } k/2 - 1/\log x \le \sigma \le k/2 + 1/\log x, \end{cases} \tag{6}$$

where σ_1 *and* σ_2 *are constants.*

Following the notation in [3], let $\varphi(\varrho)$ be a real-valued function in $[0,\infty)$ which is infinitely differentiable and satisfies $\varphi(\varrho) = 1$ for $0 \le \varrho \le 1/2$ and $\varphi(\varrho) = 0$ for $\varrho \ge 2$. We denote by Φ the set of those functions. The function $\varphi_0(\varrho) = 1 - \varphi(1/\varrho)$ is also an element of Φ. For φ in Φ and for a complex variable $w = u + iv$ with $u > 0$, let

$$K_\varphi(w) = w \int_0^\infty \varphi(\varrho)\varrho^{w-1}\, d\varrho.$$

The function $K_\varphi(w)$ has an analytic continuation to the whole complex w-plane, because the relation

$$K_\varphi(w) = -\int_{1/2}^{2} \varphi'(\varrho)\varrho^{w}\, d\varrho$$

can be verified by integration by parts. Let $\varphi^{(j)}$ denote the jth derivative of φ and define

$$\|\varphi^{(j)}\|_1 = \int_0^\infty |\varphi^{(j)}(\varrho)|\, d\varrho.$$

For $\tau > 0$, $t \ne 0$, and $j = 0, 1, \ldots$, let

$$\gamma_j(s,\tau) = \frac{1}{2\pi i \Gamma(s)} \int_{\mathcal{F}} \Gamma(s+w) \frac{\left(\tau \exp\left(-i\frac{\pi}{2}\operatorname{sgn}(t)\right)\right)^w}{w(w+1)\ldots(w+j)}\, dw,$$

where $\operatorname{sgn}(t) = t/|t|$ and $\int_{\mathcal{F}}$ means that integration is taken over the curve $\mathcal{F}$ which encircles $w = 0, -1, \ldots, -j$. If $j = 0$, it is easy to see that $\gamma_0(s,\tau) = 1$ for any s. In case $j \ne 0$, it was shown in [3] that

$$\gamma_j(s, |t|^{-1}) \ll \begin{cases} |t|^{-(j+1)/2} & \text{for odd } j, \\ |t|^{-j/2} & \text{for even } j, \end{cases} \tag{7}$$

uniformly for σ which is in a fixed strip. For $x > 0$ and φ in Φ, let

$$G_f(s,x;\varphi,\chi) = \frac{1}{2\pi i \Gamma(s)} \int_{(k/2+1-\sigma)} \Gamma(s+w) L_f(s+w,\chi) \frac{K_\varphi(w)}{w} \times \left(\frac{qx}{2\pi} \exp\left(-i\frac{\pi}{2}\operatorname{sgn}(t)\right)\right)^w dw,$$

where $\int_{(k/2+1-\sigma)}$ means that integration is taken over the vertical line $u = k/2+1-\sigma$.

We can derive the following lemma by modifying Satz of [3].

LEMMA 2. *Let $x>0$, $\varphi \in \Phi$, $f \in S_k(\Gamma)$, and χ a primitive character mod q. Then the following properties hold.*

(a) *For $(k-1)/2 \le \sigma \le (k+1)/2$,*

$$\begin{aligned}\left(\frac{2\pi}{q}\right)^{-s}\Gamma(s)L_f(s,\chi) = \left(\frac{2\pi}{q}\right)^{-s}&\Gamma(s)G_f(s,x;\varphi,\chi)\\ &+ i^k\left(\frac{W(\chi)}{|W(\chi)|}\right)^2\left(\frac{2\pi}{q}\right)^{s-k}\Gamma(k-s)\\ &\times G_f(k-s,x^{-1};\varphi_0,\overline{\chi}).\end{aligned}$$

(b) *Let $y = qx|t|/(2\pi)$ and l an integer with $l > (k+1)/2$. For $|t| > l^2$,*

$$\begin{aligned}G_f(s,x;\varphi,\chi) = &\sum_{n=1}^{\infty}\frac{\chi(n)a(n)}{n^s}\sum_{j=0}^{l}\varphi^{(j)}\left(\frac{n}{y}\right)\left(-\frac{n}{y}\right)^j\gamma_j(s,|t|^{-1})\\ &+ O(\|\varphi^{(l+1)}\|_1 y^{(k+1)/2-\sigma}|t|^{-l/2}),\end{aligned}$$

where the implied constant is uniform in σ, φ, and q for $(k-1)/2 \le \sigma \le (k+1)/2$.

Put $x=1$ and $y = q|t|/(2\pi)$ in Lemma 2. Then we have

$$\begin{aligned}L_f(s,\chi) = &\sum_{n=1}^{\infty}\frac{\chi(n)a(n)}{n^s}\varphi\left(\frac{n}{y}\right)\\ &+ i^k\left(\frac{W(\chi)}{|W(\chi)|}\right)^2\left(\frac{2\pi}{q}\right)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)a(n)}{n^{k-s}}\varphi_0\left(\frac{n}{y}\right)\\ &+ R(s),\end{aligned}$$

where

$$\begin{aligned}R(s) = &\sum_{n=1}^{\infty}\frac{\chi(n)a(n)}{n^s}\sum_{j=1}^{l}\varphi^{(j)}\left(\frac{n}{y}\right)\left(-\frac{n}{y}\right)^j\gamma_j(s,|t|^{-1})\\ &+ i^k\left(\frac{W(\chi)}{|W(\chi)|}\right)^2\left(\frac{2\pi}{q}\right)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}\\ &\times\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)a(n)}{n^{k-s}}\sum_{j=1}^{l}\varphi_0^{(j)}\left(\frac{n}{y}\right)\left(-\frac{n}{y}\right)^j\gamma_j(k-s,|t|^{-1})\\ &+ O(\|\varphi^{(l+1)}\|_1 y^{(k+1)/2-\sigma}|t|^{-l/2}) + O(\|\varphi_0^{(l+1)}\|_1 y^{(k+1)/2-\sigma}|t|^{-l/2}).\end{aligned}$$

Now we fix a φ. By (5) and (7), we have

$$\begin{aligned}R(s) &\ll \sum_{j=1}^{l}|\gamma_j(s,|t|^{-1})|\sum_{n\le 2y}\frac{|a(n)|}{n^{\sigma}}\bigg(\frac{n}{q|t|}\bigg)^{j}\\&\quad+\bigg|\bigg(\frac{2\pi}{q}\bigg)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}\bigg|\sum_{j=1}^{l}|\gamma_j(k-s,|t|^{-1})|\sum_{n\le 2y}\frac{|a(n)|}{n^{k-\sigma}}\bigg(\frac{n}{q|t|}\bigg)^{j}\\&\quad+(q|t|)^{(k+1)/2-\sigma}|t|^{-l/2}\\&\ll (q|t|)^{(k+1)/2-\sigma}|t|^{-1}.\end{aligned}$$

Therefore we have

Lemma 3. *Let $\varphi\in\Phi$, $f\in S_k(\Gamma)$, χ a primitive character mod q, and $\kappa=2\pi/q$. Then*

$$\begin{aligned}L_f(s,\chi)&=\sum_{n=1}^{\infty}\frac{\chi(n)a(n)}{n^s}\varphi\bigg(\frac{\kappa n}{|t|}\bigg)\\&\quad+i^k\bigg(\frac{W(\chi)}{|W(\chi)|}\bigg)^2\kappa^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)a(n)}{n^{k-s}}\varphi_0\bigg(\frac{\kappa n}{|t|}\bigg)\\&\quad+O((q|t|)^{(k+1)/2-\sigma}|t|^{-1}),\end{aligned}$$

where the implied constant is uniform in σ and q for $(k-1)/2\le\sigma\le(k+1)/2$.

3. The mean square of $L_f(s,\chi)$. Throughout this section, we suppose f is in $S_k(\Gamma)$ and χ is a Dirichlet character mod q. The aim of this section is to estimate the mean square

$$\sum_{\chi}{}^{*}\int_{-T}^{T}|L_f(\sigma+it,\chi)|^2\,dt$$

uniformly in σ and q for $k/2-1/\log(qT)\le\sigma\le k/2+1/\log(qT)$, where $\sum_{\chi}^{*}$ means a sum running over all primitive characters mod q.

We need the following lemmas.

Lemma 4. *Let $0<\delta<\delta_1$, and let $\varphi(\varrho)$ be a real-valued function in $[0,\infty)$ which is twice continuously differentiable and satisfies $\varphi(\varrho)=1$ for $0\le\varrho\le\delta$ and $\varphi(\varrho)=0$ for $\varrho\ge\delta_1$. Let m and n be positive integers, κ and T positive real numbers, and β a real number which satisfies $-1<A\le\beta\le B<1$ for some constants A,B. Then*

$$\int_0^T \varphi\left(\frac{\kappa n}{t}\right)\varphi\left(\frac{\kappa m}{t}\right)t^{-\beta}\cos\left(t\log\frac{n}{m}\right)dt$$
$$= \begin{cases} 0 & \text{for } m \geq T\delta_1/\kappa \text{ or } n \geq T\delta_1/\kappa, \\ O(T^{1-\beta}) & \text{for } m = n < T\delta/\kappa, \\ O((\kappa n)^{1-\beta}) & \text{for } m = n \geq T\delta/\kappa, \\ \dfrac{1}{\log\frac{n}{m}}\sin\left(T\log\dfrac{n}{m}\right)\varphi\left(\dfrac{\kappa n}{T}\right)\varphi\left(\dfrac{\kappa m}{T}\right)T^{-\beta} + O\left(\dfrac{(\kappa\max(n,m))^{-\beta-1}}{(\log(n/m))^2}\right) & \\ & \text{for } m \neq n, \end{cases}$$

where the implied constants are uniform in m, n, κ, and β.

It is easy to prove Lemma 4 by modifying the proof of Lemma 7 of [3].

LEMMA 5. *Let $f \in S_k(\Gamma)$ and χ a Dirichlet character mod q. Let ε be a positive real number and assume $(k-\varepsilon)/2 < \sigma < (k+\varepsilon)/2$. If $|t| \leq C$ for some positive constant C, then*

$$\sum_{\chi}{}^{*} |L_f(s,\chi)|^2 \ll_{\varepsilon,C} \phi(q)q^{k-2\sigma+2\varepsilon}\Big(\int_1^\infty u^{2\sigma-k-1-\varepsilon}\,du + \int_1^\infty u^{k-2\sigma-1-\varepsilon}\,du\Big)$$

uniformly in σ and q, where ϕ is the Euler function.

Proof. By the automorphic property of $\sum_{n=1}^\infty \chi(n)a(n)e^{2\pi inz}$, which is the twist of f by the primitive character χ,

$$\left(\frac{2\pi}{q}\right)^{-s}\Gamma(s)L_f(s,\chi)$$
$$= \int_0^\infty u^{s-1}\sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q}\,du$$
$$= \int_1^\infty u^{s-1}\sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q}\,du$$
$$+ i^k\left(\frac{W(\chi)}{|W(\chi)|}\right)^2\int_1^\infty u^{k-s-1}\sum_{n=1}^\infty \overline{\chi}(n)a(n)e^{-2\pi nu/q}\,du.$$

Hence

$$\left(\frac{2\pi}{q}\right)^{-\sigma}|\Gamma(s)|\cdot|L_f(s,\chi)| \leq \int_1^\infty u^{\sigma-1}\Big|\sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q}\Big|\,du$$
$$+ \int_1^\infty u^{k-\sigma-1}\Big|\sum_{n=1}^\infty \overline{\chi}(n)a(n)e^{-2\pi nu/q}\Big|\,du.$$

By squaring both sides above and taking $\sum_\chi^*$, we have

$$\tag{8} \frac{1}{2}\left(\frac{2\pi}{q}\right)^{-2\sigma}|\Gamma(s)|^2 \sum_{\chi}{}^{*} |L_f(s,\chi)|^2 \le \sum_{\chi}{}^{*} \Big(\int_1^\infty u^{\sigma-1}\Big|\sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q}\Big|\,du\Big)^2 + \sum_{\chi}{}^{*} \Big(\int_1^\infty u^{k-\sigma-1}\Big|\sum_{n=1}^\infty \overline{\chi}(n)a(n)e^{-2\pi nu/q}\Big|\,du\Big)^2.$$

Let α be real. By Cauchy's inequality,

$$\sum_{\chi}{}^{*} \Big(\int_1^\infty u^{\alpha-1}\Big|\sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q}\Big|\,du\Big)^2 \le \sum_{\chi}{}^{*} \int_1^\infty u^{2\alpha-1+\varepsilon}\Big|\sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q}\Big|^2\,du \int_1^\infty u^{-1-\varepsilon}\,du \ll_\varepsilon \int_1^\infty u^{2\alpha-1+\varepsilon}\sum_{\chi}\Big|\sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q}\Big|^2\,du.$$

Here,

$$\sum_{\chi}\Big|\sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q}\Big|^2 = \phi(q) \sum_{\substack{n=1\\(n,q)=1}}^\infty \sum_{\substack{m=1\\(m,q)=1\\ n\equiv m\,(q)}}^\infty \overline{a}(n)a(m)e^{-2\pi(n+m)u/q} \le \frac{\phi(q)}{2}\sum_{n=1}^\infty \sum_{\substack{m=1\\ n\equiv m\,(q)}}^\infty (|a(n)|^2+|a(m)|^2)e^{-2\pi(n+m)u/q} \le \phi(q)\sum_{n=1}^\infty |a(n)|^2 e^{-2\pi nu/q}\sum_{r=0}^\infty e^{-2\pi ru} \ll \phi(q)\sum_{n=1}^\infty |a(n)|^2 e^{-2\pi nu/q}.$$

By using partial summation, the right-hand side is

$$\ll \phi(q)\frac{u}{q}\int_1^\infty x^k e^{-2\pi xu/q}\,dx \ll_\varepsilon \phi(q)\frac{u}{q}\int_1^\infty x^k\left(\frac{xu}{q}\right)^{-k-1-2\varepsilon}dx \ll_\varepsilon \phi(q)\left(\frac{u}{q}\right)^{-k-2\varepsilon}.$$

Hence we have

$$\sum_{\chi}{}^{*}\Big(\int_{1}^{\infty} u^{\alpha-1}\Big|\sum_{n=1}^{\infty}\chi(n)a(n)e^{-2\pi nu/q}\Big|\,du\Big)^2 \ll_{\varepsilon} \phi(q)q^{k+2\varepsilon}\int_{1}^{\infty} u^{2\alpha-k-1-\varepsilon}\,du.$$

Substituting this into (8), we obtain the assertion of Lemma 5.

THEOREM 2. *Let $f \in S_k(\Gamma)$ and χ a Dirichlet character mod q. If $q \ll T$, then*

$$\sum_{\chi}{}^{*}\int_{-T}^{T}|L_f(\sigma+it,\chi)|^2\,dt \ll \phi(q)T\log(qT), \qquad T\to\infty,$$

uniformly in σ and q for $k/2-1/\log(qT)\le\sigma\le k/2+1/\log(qT)$.

Proof. Denote the right-hand side of the formula in the statement of Lemma 3 by $f_1+f_2+f_3$, say. Let C_0 be a positive constant for which

$$f_3(\sigma+it) \ll (q|t|)^{(k+1)/2-\sigma}|t|^{-1} \tag{9}$$

for $|t|\ge C_0$. Put

$$\Lambda_{\mu\nu}(\sigma,C_0) = \int_{[-T,T]-[-C_0,C_0]}\overline{f_\mu(\sigma+it)}f_\nu(\sigma+it)\,dt, \quad \mu,\nu=1,2,3.$$

By Cauchy's inequality,

$$\begin{aligned}\Big|\sum_{\chi}{}^{*}\Lambda_{\mu\nu}(\sigma,C_0)\Big| &\le \Big(\sum_{\chi}{}^{*}\Lambda_{\mu\mu}(\sigma,C_0)\Big)^{1/2}\Big(\sum_{\chi}{}^{*}\Lambda_{\nu\nu}(\sigma,C_0)\Big)^{1/2}\\ &\le \frac{1}{2}\sum_{\chi}{}^{*}\Lambda_{\mu\mu}(\sigma,C_0)+\frac{1}{2}\sum_{\chi}{}^{*}\Lambda_{\nu\nu}(\sigma,C_0).\end{aligned}$$

Hence we have

$$\begin{aligned}&\sum_{\chi}{}^{*}\int_{-T}^{T}|L_f(\sigma+it,\chi)|^2\,dt \\ &\quad= \Big|\sum_{\mu,\nu=1}^{3}\sum_{\chi}{}^{*}\Lambda_{\mu\nu}(\sigma,C_0)\Big| + \sum_{\chi}{}^{*}\int_{-C_0}^{C_0}|L_f(\sigma+it,\chi)|^2\,dt \\ &\quad\ll \sum_{\nu=1}^{3}\sum_{\chi}{}^{*}\Lambda_{\nu\nu}(\sigma,C_0) + \sum_{\chi}{}^{*}\int_{-C_0}^{C_0}|L_f(\sigma+it,\chi)|^2\,dt.\end{aligned} \tag{10}$$

We use Lemma 5 with $\varepsilon=1/2$ for $k/2-1/\log(qT)\le\sigma\le k/2+1/\log(qT)$ and $|t|\le C_0$ to obtain

$$\sum_{\chi}{}^{*}\int_{-C_0}^{C_0}|L_f(\sigma+it,\chi)|^2\,dt \ll_{C_0} \phi(q)q. \tag{11}$$

By (9), we have

$$\sum_{\chi}^{*} \Lambda_{33}(\sigma, C_0) \ll \phi(q) q^{2/\log(qT)+1} \int_{C_0}^{T} t^{2/\log(qT)-1}\, dt \ll \phi(q) q \log T. \tag{12}$$

Substituting (11) and (12) into (10), gives

$$\sum_{\chi}^{*} \int_{-T}^{T} |L_f(\sigma+it,\chi)|^2\, dt \ll \sum_{\chi}^{*} \Lambda_{11}(\sigma) + \sum_{\chi}^{*} \Lambda_{22}(\sigma) + \phi(q) q \log T, \tag{13}$$

where

$$\Lambda_{\nu\nu}(\sigma) = \int_{-T}^{T} |f_\nu(\sigma+it)|^2\, dt, \qquad \nu = 1, 2.$$

First, we estimate $\sum_{\chi}^{*} \Lambda_{11}(\sigma)$. We have

$$\begin{aligned}
\sum_{\chi}^{*} \Lambda_{11}(\sigma) &\le \sum_{\chi} \int_{-T}^{T} |f_1(\sigma+it)|^2\, dt \\
&= 2\phi(q) \sum_{\substack{n<2T/\kappa \\ (n,q)=1}} \sum_{\substack{m<2T/\kappa \\ (m,q)=1 \\ n\equiv m\,(q)}} \frac{\overline{a}(n) a(m)}{(nm)^{\sigma}} \\
&\quad \times \int_{0}^{T} \varphi\left(\frac{\kappa n}{t}\right) \varphi\left(\frac{\kappa m}{t}\right) \cos\left(t \log \frac{n}{m}\right) dt \\
&= 2\phi(q) \Bigg\{ \sum_{\substack{n<T/(2\kappa) \\ (n,q)=1}} \frac{|a(n)|^2}{n^{2\sigma}} \int_{0}^{T} \varphi\left(\frac{\kappa n}{t}\right)^2 dt \\
&\quad + \sum_{\substack{T/(2\kappa)\le n<2T/\kappa \\ (n,q)=1}} \frac{|a(n)|^2}{n^{2\sigma}} \int_{0}^{T} \varphi\left(\frac{\kappa n}{t}\right)^2 dt \\
&\quad + \sum\nolimits_0 \frac{\overline{a}(n) a(m)}{(nm)^{\sigma}} \int_{0}^{T} \varphi\left(\frac{\kappa n}{t}\right) \varphi\left(\frac{\kappa m}{t}\right) \cos\left(t \log \frac{n}{m}\right) dt \Bigg\},
\end{aligned}$$

where we set

$$\sum\nolimits_0 = \sum_{\substack{n<2T/\kappa \\ (n,q)=1}} \sum_{\substack{m<2T/\kappa \\ (m,q)=1 \\ n\equiv m\,(q) \\ n\ne m}}.$$

Applying Lemma 4, we have

$$\sum_{\chi}{}^{*} \Lambda_{11}(\sigma) \ll \phi(q)\Big\{T \sum_{n<T/(2\kappa)} \frac{|a(n)|^2}{n^{2\sigma}} + \frac{1}{q} \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma-1}} + \sum\nolimits_0 \frac{|a(n)a(m)|}{(nm)^\sigma \left|\log \frac{n}{m}\right|} + q\sum\nolimits_0 \frac{|a(n)a(m)|}{(nm)^\sigma \max(n,m)\left(\log \frac{n}{m}\right)^2}\Big\}. \tag{14}$$

The third sum on the right-hand side is

$$\begin{aligned} &\le \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} \sum_{\substack{m<2T/\kappa \\ m\equiv n\,(q) \\ m\ne n}} \frac{1}{\left|\log \frac{n}{m}\right|} \\ &= \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} \sum_{\substack{m<n \\ m\equiv n\,(q)}} \frac{1}{\left|\log \frac{n}{m}\right|} + \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} \sum_{\substack{n<m<2T/\kappa \\ m\equiv n\,(q)}} \frac{1}{\left|\log \frac{n}{m}\right|}. \end{aligned}$$

In the first term we put $m = n - qr$ to get

$$\sum_{\substack{m<n \\ m\equiv n\,(q)}} \frac{1}{\left|\log \frac{n}{m}\right|} < \frac{n}{q} \sum_{1\le r<2T/(q\kappa)} \frac{1}{r} \ll \frac{n}{q}\log T,$$

and in the second term we put $m = n + qr$ to get

$$\sum_{\substack{n<m<2T/\kappa \\ m\equiv n\,(q)}} \frac{1}{\left|\log \frac{n}{m}\right|} < \sum_{1\le r<2T/(q\kappa)} \frac{n+qr}{qr} \ll T + \frac{n}{q}\log T.$$

Therefore we have

$$\sum\nolimits_0 \frac{|a(n)a(m)|}{(nm)^\sigma \left|\log \frac{n}{m}\right|} \ll T \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} + \frac{\log T}{q} \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma-1}}. \tag{15}$$

Next, the fourth sum on the right-hand side of (14) is

$$\begin{aligned} &\le \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} \sum_{\substack{m<2T/\kappa \\ m\equiv n\,(q) \\ m\ne n}} \frac{1}{\max(n,m)\left(\log \frac{n}{m}\right)^2} \\ &= \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma+1}} \sum_{\substack{m<n \\ m\equiv n\,(q)}} \frac{1}{\left(\log \frac{n}{m}\right)^2} + \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} \sum_{\substack{n<m<2T/\kappa \\ m\equiv n\,(q)}} \frac{1}{m\left(\log \frac{n}{m}\right)^2}. \end{aligned}$$

In the first term we put $m=n-qr$ to get

$$\sum_{\substack{m<n\\ m\equiv n\,(q)}}\frac{1}{\left(\log\frac{n}{m}\right)^2}<\frac{n^2}{q^2}\sum_{1\le r<n/q}\frac{1}{r^2}\ll\frac{n^2}{q^2},$$

and in the second term we put $m=n+qr$ to get

$$\sum_{\substack{n<m<2T/\kappa\\ m\equiv n\,(q)}}\frac{1}{m\left(\log\frac{n}{m}\right)^2}<\sum_{1\le r<2T/(q\kappa)}\frac{1}{n+qr}\left(\frac{n+qr}{qr}\right)^2\ll\frac{n}{q^2}+\frac{1}{q}\log T.$$

Therefore we have

$$q{\sum}_0\frac{|a(n)a(m)|}{(nm)^\sigma\max(n,m)\left(\log\frac{n}{m}\right)^2}\ll(\log T)\sum_{n<2T/\kappa}\frac{|a(n)|^2}{n^{2\sigma}}+\frac{1}{q}\sum_{n<2T/\kappa}\frac{|a(n)|^2}{n^{2\sigma-1}}.\tag{16}$$

Substituting (15) and (16) to (14), we obtain

$${\sum_\chi}^*\Lambda_{11}(\sigma)\ll\phi(q)\left(T\sum_{n<2T/\kappa}\frac{|a(n)|^2}{n^{2\sigma}}+\frac{\log T}{q}\sum_{n<2T/\kappa}\frac{|a(n)|^2}{n^{2\sigma-1}}\right).$$

Combining this estimate with (6), we obtain

$${\sum_\chi}^*\Lambda_{11}(\sigma)\ll\phi(q)T\log(qT),\qquad T\to\infty,\tag{17}$$

uniformly in σ and q for $k/2-1/\log(qT)\le\sigma\le k/2+1/\log(qT)$.

Second, we estimate $\sum_\chi^*\Lambda_{22}(\sigma)$. We have

$$\begin{aligned}{\sum_\chi}^*\Lambda_{22}(\sigma)&\le\sum_\chi\int_{-T}^{T}|f_2(\sigma+it)|^2\,dt\\&=2\phi(q)\kappa^{2(2\sigma-k)}\sum_{\substack{n<2T/\kappa\\(n,q)=1}}\sum_{\substack{m<2T/\kappa\\(m,q)=1\\ n\equiv m\,(q)}}\frac{\overline{a}(n)a(m)}{(nm)^{k-\sigma}}\\&\quad\times\int_0^T\varphi_0\left(\frac{\kappa n}{t}\right)\varphi_0\left(\frac{\kappa m}{t}\right)\left|\frac{\Gamma(k-s)}{\Gamma(s)}\right|^2\cos\left(t\log\frac{n}{m}\right)dt.\end{aligned}$$

Note that the interval $[0,T]$ of integration can be replaced by an interval $[(\kappa/2)\max(n,m),T]$, because $\varphi_0(\kappa n/t)\varphi_0(\kappa m/t)=0$ for $0\le t\le(\kappa/2)\max(n,m)$. By Stirling's formula, we have

$$\left|\frac{\Gamma(k-s)}{\Gamma(s)}\right|^2=|t|^{2(k-2\sigma)}\left(1+O\left(\frac{1}{t^2}\right)\right)$$

for $0<\sigma<k$ and $|t|\ge C_1$, where C_1 is some positive constant. In case n and m satisfy $C_1\le(\kappa/2)\max(n,m)$, we have

$$\int\limits_{(\kappa/2)\max(n,m)}^{T}\varphi_0\left(\frac{\kappa n}{t}\right)\varphi_0\left(\frac{\kappa m}{t}\right)\left|\frac{\Gamma(k-s)}{\Gamma(s)}\right|^2\cos\left(t\log\frac{n}{m}\right)dt$$
$$=\int\limits_{(\kappa/2)\max(n,m)}^{T}\varphi_0\left(\frac{\kappa n}{t}\right)\varphi_0\left(\frac{\kappa m}{t}\right)t^{2(k-2\sigma)}\cos\left(t\log\frac{n}{m}\right)dt+O(1)$$

uniformly in σ and q for $k/2-1/\log(qT)\le\sigma\le k/2+1/\log(qT)$. The same result also holds in case $C_1>(\kappa/2)\max(n,m)$, because in this case

$$\int\limits_{(\kappa/2)\max(n,m)}^{C_1}\varphi_0\left(\frac{\kappa n}{t}\right)\varphi_0\left(\frac{\kappa m}{t}\right)\left|\frac{\Gamma(k-s)}{\Gamma(s)}\right|^2\cos\left(t\log\frac{n}{m}\right)dt=O(1)$$

and

$$\int\limits_{(\kappa/2)\max(n,m)}^{C_1}\varphi_0\left(\frac{\kappa n}{t}\right)\varphi_0\left(\frac{\kappa m}{t}\right)t^{2(k-2\sigma)}\cos\left(t\log\frac{n}{m}\right)dt=O(1).$$

Let us denote

$$\sum\nolimits_1=\sum_{\substack{n<2T/\kappa\\(n,q)=1}}\sum_{\substack{m<2T/\kappa\\(m,q)=1\\ n\equiv m\,(q)}}$$

and $\sum_0$ is as before. From the above result, it follows that

$$\begin{aligned}\sum\nolimits_1&\frac{\overline{a}(n)a(m)}{(nm)^{k-\sigma}}\\&\times\int\limits_{(\kappa/2)\max(n,m)}^{T}\varphi_0\left(\frac{\kappa n}{t}\right)\varphi_0\left(\frac{\kappa m}{t}\right)\left|\frac{\Gamma(k-s)}{\Gamma(s)}\right|^2\cos\left(t\log\frac{n}{m}\right)dt\\&=\sum\nolimits_1\frac{\overline{a}(n)a(m)}{(nm)^{k-\sigma}}\\&\quad\times\int\limits_{(\kappa/2)\max(n,m)}^{T}\varphi_0\left(\frac{\kappa n}{t}\right)\varphi_0\left(\frac{\kappa m}{t}\right)t^{2(k-2\sigma)}\cos\left(t\log\frac{n}{m}\right)dt\\&\quad+O\left(\sum\nolimits_1\frac{|a(n)a(m)|}{(nm)^{k-\sigma}}\right)\\&=\sum_{\substack{n<T/(2\kappa)\\(n,q)=1}}\frac{|a(n)|^2}{n^{2(k-\sigma)}}\int\limits_0^T\varphi_0\left(\frac{\kappa n}{t}\right)^2t^{2(k-2\sigma)}\,dt\end{aligned}$$

$$+\sum_{\substack{T/(2\kappa)\le n<2T/\kappa\\(n,q)=1}}\frac{|a(n)|^2}{n^{2(k-\sigma)}}\int_0^T\varphi_0\Big(\frac{\kappa n}{t}\Big)^2 t^{2(k-2\sigma)}\,dt$$

$$+\sum\nolimits_0\frac{\overline{a}(n)a(m)}{(nm)^{k-\sigma}}\int_0^T\varphi_0\Big(\frac{\kappa n}{t}\Big)\varphi_0\Big(\frac{\kappa m}{t}\Big)t^{2(k-2\sigma)}\cos\Big(t\log\frac{n}{m}\Big)\,dt$$

$$+O\Big(\sum\nolimits_1\frac{|a(n)a(m)|}{(nm)^{k-\sigma}}\Big).$$

Since $-4/\log(qT)\le -2(k-2\sigma)\le 4/\log(qT)$, by using Lemma 4, we see that the right-hand side of the above is

$$\ll T\sum_{n<T/(2\kappa)}\frac{|a(n)|^2}{n^{2(k-\sigma)}}+\frac{1}{q}\sum_{n<2T/\kappa}\frac{|a(n)|^2}{n^{2(k-\sigma)-1}}+\sum\nolimits_0\frac{|a(n)a(m)|}{(nm)^{k-\sigma}\big|\log\frac{n}{m}\big|}$$

$$+q\sum\nolimits_0\frac{|a(n)a(m)|}{(nm)^{k-\sigma}\max(n,m)\big(\log\frac{n}{m}\big)^2}+\sum\nolimits_1\frac{|a(n)a(m)|}{(nm)^{k-\sigma}}$$

uniformly in σ and q for $k/2-1/\log(qT)\le\sigma\le k/2+1/\log(qT)$. By (15), (16), and the estimate

$$\sum\nolimits_1\frac{|a(n)a(m)|}{(nm)^{k-\sigma}}\le\sum_{n<2T/\kappa}\frac{|a(n)|^2}{n^{2(k-\sigma)}}\sum_{\substack{m<2T/\kappa\\m\equiv n\ (q)}}1\ll T\sum_{n<2T/\kappa}\frac{|a(n)|^2}{n^{2(k-\sigma)}},$$

we have

$$\sum_{\chi}{}^{*}\varLambda_{22}(\sigma)\ll\phi(q)\Big(T\sum_{n<2T/\kappa}\frac{|a(n)|^2}{n^{2(k-\sigma)}}+\frac{\log T}{q}\sum_{n<2T/\kappa}\frac{|a(n)|^2}{n^{2(k-\sigma)-1}}\Big),$$

hence, by (6), we obtain

$$\sum_{\chi}{}^{*}\varLambda_{22}(\sigma)\ll\phi(q)T\log(qT),\qquad T\to\infty,\tag{18}$$

uniformly in σ and q for $k/2-1/\log(qT)\le\sigma\le k/2+1/\log(qT)$.

Combining (13), (17), (18), and the assumption $q\ll T$, we obtain the assertion of Theorem 2.

Corollary 1. *Under the same notation as in Theorem 2, we have*

$$\sum_{\chi}{}^{*}\int_{-T}^{T}|L_f'(k/2+it,\chi)|^2\,dt\ll\phi(q)T(\log(qT))^3,\qquad T\to\infty,$$

uniformly in q.

Proof. Put $r=(\log(qT))^{-1}$. Since

$$|L'_f(k/2+it,\chi)|^2 \ll r^{-3}\int\limits_{|z-k/2-it|=r} |L_f(z,\chi)|^2|dz|,$$

we have

$$\sideset{}{^*}\sum_{\chi}\int\limits_{-T}^{T}|L'_f(k/2+it,\chi)|^2\,dt \ll r^{-3}\sideset{}{^*}\sum_{\chi}\int\limits_{-T}^{T}\int\limits_{|z-k/2-it|=r}|L_f(z,\chi)|^2|dz|\,dt.$$

From Theorem 2, it follows that

$$\begin{aligned}
&\sideset{}{^*}\sum_{\chi}\int\limits_{-T}^{T}\int\limits_{|z-k/2-it|=r}|L_f(z,\chi)|^2|dz|\,dt\\
&\quad\le 2\int\limits_{k/2-r}^{k/2+r}\sideset{}{^*}\sum_{\chi}\int\limits_{-T-1}^{T+1}|L_f(\sigma+it,\chi)|^2\,dt\biggl(1-\biggl(\frac{\sigma-k/2}{r}\biggr)^2\biggr)^{-1/2}d\sigma\\
&\quad\le 2\biggl\{\int\limits_{k/2-r}^{k/2+r}\biggl(\sideset{}{^*}\sum_{\chi}\int\limits_{-T-1}^{T+1}|L_f(\sigma+it,\chi)|^2\,dt\biggr)^3d\sigma\biggr\}^{1/3}\\
&\qquad\times\biggl\{\int\limits_{k/2-r}^{k/2+r}\biggl(1-\biggl(\frac{\sigma-k/2}{r}\biggr)^2\biggr)^{-3/4}d\sigma\biggr\}^{2/3}\\
&\quad\ll r\phi(q)T\log(qT)\ll\phi(q)T.
\end{aligned}$$

This proves the corollary.

Corollary 2. *Let χ be a Dirichlet character mod q, and χ^* the primitive character inducing χ. Let δ be a positive real number such that $\delta\ll T$, and $\mathcal{T}_{\chi^*}$ a finite subset of $[-T,T]$ with $|t-t'|\ge\delta$ for any distinct t and t' in $\mathcal{T}_{\chi^*}$. If $q\ll T$, then*

$$\sum_{\chi}\sum_{t\in\mathcal{T}_{\chi^*}}|L_f(k/2+it,\chi^*)|^2\ll\biggl(\frac{1}{\delta}+\log(qT)\biggr)qT\log(qT),\qquad T\to\infty,$$

uniformly in q.

Corollary 2 can be derived from Theorem 2 and Corollary 1 by the same argument as the proof of Corollary 10.4 of [6].

4. Proof of Theorem 1. Our argument is a modification of the proof of the zero density estimates of Dirichlet L-functions in [6], so we give only a sketch.

Let $L_f(s)=\sum_{n=1}^{\infty}a(n)n^{-s}$ for $\sigma>(k+1)/2$. We define $\mu_f(n)$ by

$$\frac{1}{L_f(s)}=\sum_{n=1}^{\infty}\frac{\mu_f(n)}{n^s}$$

for $\sigma>(k+1)/2$. By the Euler product expansion of $L_f(s)$ and the estimate $|a(n)|\le n^{(k-1)/2}d(n)$ (see [2]), where $d(n)$ is the divisor function, it is easy to see that the following properties hold:

$$|\mu_f(n)|\le n^{(k-1)/2}d(n),$$
$$\sum_{\substack{d\mid n\\ d>0}}\mu_f(d)a\left(\frac{n}{d}\right)=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{otherwise.}\end{cases}$$

Since $L_f(s,\chi)$ and $L_f(s,\chi^*)$ have the same zeros for $\sigma\ge k/2$, it is enough to consider $N_f(\sigma_0,T,\chi^*)$ instead of $N_f(\sigma_0,T,\chi)$. Let A_1 be a positive real number, and let X and Y be parameters satisfying $2\le X\le Y\le (qT)^{A_1}$. We define

$$M(s,\chi^*)=\sum_{n\le X}\frac{\mu_f(n)\chi^*(n)}{n^s}.$$

Then it follows that, for $\sigma>(k+1)/2$,

$$L_f(s,\chi^*)M(s,\chi^*)=\sum_{n=1}^{\infty}\frac{h(n)\chi^*(n)}{n^s},$$

where $h(n)=\sum_{d\mid n, 0<d\le X}\mu_f(d)a(n/d)$ has the following properties: $h(1)=1$, $h(n)=0$ for $2\le n\le X$, and $|h(n)|\le n^{(k-1)/2}d(n)^3$ for $n>X$. By using the Mellin integral formula, we have

$$\begin{aligned}e^{-1/Y}&+\sum_{n>X}h(n)\chi^*(n)n^{-s}e^{-n/Y}\\ &=\frac{1}{2\pi i}\int_{(k+1)/2+1-i\infty}^{(k+1)/2+1+i\infty}L_f(s+w,\chi^*)M(s+w,\chi^*)Y^w\Gamma(w)\,dw\end{aligned}$$

for $\sigma>-1$. Let $\varrho=\beta+i\gamma$ be a zero of $L_f(s,\chi^*)$ such that $\sigma_0\le\beta\le (k+1)/2$ and $-T\le\gamma\le T$, and take $s=\varrho$ in the equation above. Since $L_f(\varrho+w,\chi^*)M(\varrho+w,\chi^*)Y^w\Gamma(w)$ is holomorphic for $-1/2\le\Re w$, the path of integration in the above can be moved to the line $\Re w=k/2-\beta$. Therefore, if Y is large, every ϱ counted by $N_f(\sigma_0,T,\chi^*)$ has at least one of the following properties:

(a) $$\Big|\sum_{X<n\le Y^2}h(n)\chi^*(n)n^{-\varrho}e^{-n/Y}\Big|\ge\frac{1}{5},$$

$$\text{(b)} \qquad \Big| \int_{k/2-\beta-iz}^{k/2-\beta+iz} L_f(\varrho + w, \chi^*) M(\varrho + w, \chi^*) Y^w \Gamma(w)\, dw \Big| \geq \frac{2\pi}{5},$$

where $z = A_2 \log(qT)$ for a large absolute constant A_2. Let $\mathcal{R}(\chi^*)$ be a set of ϱ's which are well-spaced, that is, $3z \leq |\gamma - \gamma'|$ for any distinct $\varrho = \beta + i\gamma$ and $\varrho' = \beta' + i\gamma'$. We denote by $R(\chi^*)$ the number of elements of $\mathcal{R}(\chi^*)$. From (2) and the definition of $\mathcal{R}(\chi^*)$, it follows that

$$N_f(\sigma_0, T, \chi^*) \ll R(\chi^*)(\log(qT))^2,$$

hence

$$\sum_{\chi} N_f(\sigma_0, T, \chi) = \sum_{\chi} N_f(\sigma_0, T, \chi^*) \ll R(\log(qT))^2, \tag{19}$$

where $R = \sum_{\chi} R(\chi^*)$. The sets $\mathcal{R}_1(\chi^*)$ and $\mathcal{R}_2(\chi^*)$ are defined to be the subsets of $\mathcal{R}(\chi^*)$ such that every element of $\mathcal{R}_1(\chi^*)$ satisfies the condition (a), and every element of $\mathcal{R}_2(\chi^*)$ satisfies the condition (b). Denote by $R_j(\chi^*)$ the number of elements of $\mathcal{R}_j(\chi^*)$, $j = 1, 2$. Put

$$\mathcal{R}_j = \bigcup_{\chi} \mathcal{R}_j(\chi^*) \quad \text{and} \quad R_j = \sum_{\chi} R_j(\chi^*), \quad j = 1, 2,$$

and we shall estimate R_1 and R_2.

First, we estimate R_1. For every ϱ in $\mathcal{R}_1$,

$$\max_{1 \leq l \leq l_0 + 1} \Big\{ \Big| \sum_{\substack{2^{l-1}X < n \leq 2^l X \\ n \leq Y^2}} h(n)\chi^*(n) n^{-\varrho} e^{-n/Y} \Big| \Big\} \geq \frac{1}{15 \log Y}$$

for large Y, where $l_0 = [(\log 2)^{-1} \log(X^{-1}Y^2)]$. Hence, there exists U such that $X < U \leq Y^2$ and the inequality

$$\Big| \sum_{\substack{U < n \leq 2U \\ n \leq Y^2}} h(n)\chi^*(n) n^{-\varrho} e^{-n/Y} \Big| \geq \frac{1}{15 \log Y}$$

holds for more than $R_1/(4 \log Y)$ zeros of $\mathcal{R}_1$. Therefore, by Theorem 7.6 of [6],

$$\begin{aligned} R_1 &\ll (\log Y)^3 \sum_{\chi} \sum_{\varrho \in \mathcal{R}_1(\chi^*)} \Big| \sum_{\substack{U < n \leq 2U \\ n \leq Y^2}} h(n)\chi^*(n) n^{-\varrho} e^{-n/Y} \Big|^2 \\ &\ll (qTX^{k-2\sigma_0} + Y^{k+1-2\sigma_0})(\log(qT))^{67}. \end{aligned} \tag{20}$$

Second, we estimate R_2. For every ϱ in $\mathcal{R}_2$,

$$\int_{-z}^{z} |L_f(k/2+i(\gamma+v),\chi^*)M(k/2+i(\gamma+v),\chi^*)| \\ \times Y^{k/2-\beta}|\Gamma(k/2-\beta+iv)|\,dv \ge \frac{2\pi}{5}.$$

Let $t_\varrho=\gamma+v$ be a value for which $|L_f(k/2+i(\gamma+v),\chi^*)M(k/2+i(\gamma+v),\chi^*)|$ is maximal. Since

$$\int_{-z}^{z} |\Gamma(k/2-\beta+iv)|\,dv \ll \int_{-1}^{1} \frac{1}{\beta-k/2}\,dv \ll \log(qT),$$

we have

$$|L_f(k/2+it_\varrho,\chi^*)M(k/2+it_\varrho,\chi^*)| \gg Y^{\sigma_0-k/2}(\log(qT))^{-1}.$$

Hence,

$$\begin{aligned} Y^{\sigma_0-k/2}(\log(qT))^{-1}R_2 &\ll \sum_{\chi}\sum_{\varrho\in\mathcal{R}_2(\chi^*)} |L_f(k/2+it_\varrho,\chi^*)M(k/2+it_\varrho,\chi^*)| \\ &\le \Big(\sum_{\chi}\sum_{\varrho\in\mathcal{R}_2(\chi^*)} |L_f(k/2+it_\varrho,\chi^*)|^2\Big)^{1/2} \\ &\quad\times\Big(\sum_{\chi}\sum_{\varrho\in\mathcal{R}_2(\chi^*)} |M(k/2+it_\varrho,\chi^*)|^2\Big)^{1/2}. \end{aligned}$$

Since $|t_\varrho-t_{\varrho'}|\ge z$, we can use Corollary 2 under the assumption $q\ll T$:

$$\sum_{\chi}\sum_{\varrho\in\mathcal{R}_2(\chi^*)} |L_f(k/2+it_\varrho,\chi^*)|^2 \ll qT(\log(qT))^2.$$

From Theorem 7.6 of [6], if $X\le qT$, then

$$\sum_{\chi}\sum_{\varrho\in\mathcal{R}_2(\chi^*)} |M(k/2+it_\varrho,\chi^*)|^2 \ll qT(\log(qT))^6.$$

Therefore, if $q\ll T$ and $X\le qT$, we obtain

$$R_2 \ll Y^{k/2-\sigma_0}qT(\log(qT))^5. \tag{21}$$

Substituting (20) and (21) into (19) gives

$$\sum_{\chi} N_f(\sigma_0,T,\chi) \ll (qTX^{k-2\sigma_0}+Y^{k+1-2\sigma_0}+qTY^{k/2-\sigma_0})(\log(qT))^{69},$$

and putting $X=qT$, $Y=(qT)^{1/(k/2+1-\sigma_0)}$, we now obtain (3) uniformly in σ_0 and q for $k/2+1/\log(qT)\le\sigma_0\le(k+1)/2$.

Finally, the estimate (4) can be derived by a different treatment of R_1 and R_2. This is almost identical to the proof of Theorem 12.1 of [6], so we omit the details.

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