ACTA ARITHMETICA
LXXXIII.2 (1998)

An additive problem with primes and almost-primes

by

T. P. Peneva and D. I. Tolev (Plovdiv)

1. Introduction. In 1937 I. M. Vinogradov [10] proved that for every sufficiently large odd integer N the equation

$$p_1 + p_2 + p_3 = N$$

has a solution in prime numbers p_1, p_2, p_3.

Two years later van der Corput [9] used the method of Vinogradov and established that there exist infinitely many arithmetic progressions consisting of three different primes. A corresponding result for progressions of four or more primes has not been proved so far. In 1981, however, D. R. Heath-Brown [5] proved that there exist infinitely many arithmetic progressions of four different terms, three of which are primes and the fourth is P_2 (as usual, P_r denotes an integer with no more than r prime factors, counted according to multiplicity). One of the main points in [5] is a result of Bombieri–Vinogradov's type for the sum

$$\sum_{\substack{x<p_2,\, p_3 \le 2x \\ p_1+p_3=2p_2 \\ p_2-2p_3 \equiv 0 \ (\mathrm{mod}\ d)}} u(p_1)u(p_2)u(p_3),$$

where d is squarefree, $(d,6) = 1$; $u(n) = (\log n)/\log 3x$ for $n \ge 5$ and $u(n) = 0$ otherwise.

Recently Tolev [8] found an analogous result for the quantity

$$J_{k,l}(N) = \sum_{\substack{p_1+p_2+p_3=N \\ p_1 \equiv l \ (\mathrm{mod}\ k)}} \log p_1 \log p_2 \log p_3,$$

where N is a sufficiently large odd integer and $(l,k) = 1$. In [8] the Hardy–Littlewood circle method and the Bombieri–Vinogradov theorem were applied, as well as some arguments belonging to H. Mikawa.

1991 *Mathematics Subject Classification*: Primary 11N36.

It would be interesting to prove that there exist infinitely many arithmetic progressions of three different primes such that for two of them, p_1 and p_2, say, both the numbers $p_1 + 2$, $p_2 + 2$ are almost-primes. In the present paper we study this problem. Our main tool is a result of Bombieri–Vinogradov's type which we establish using the method developed in [8].

Let x be a sufficiently large real number and k_1, k_2 be odd integers. Denote by $D_{k_1,k_2}(x)$ the number of solutions of the equation

$$p_1 + p_2 = 2p_3 \tag{1}$$

in primes p_1, p_2, p_3 such that

$$x < p_1, p_2, p_3 \le 3x \tag{2}$$

and

$$p_1 + 2 \equiv 0 \pmod{k_1}, \qquad p_2 + 2 \equiv 0 \pmod{k_2}.$$

Let us also define

$$\gamma(x) = \sum_{\substack{x<m_1,m_2,m_3\le 3x\\ m_1+m_2=2m_3}} \frac{1}{\log m_1 \log m_2 \log m_3}, \qquad \sigma_0 = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right).$$

We prove the following

Theorem. *For each $A > 0$ there exists $B = B(A) > 0$ such that*

$$\sum\sum_{\substack{k_1,k_2\le \sqrt{x}/(\log x)^B\\ (k_1k_2,2)=1}} \left| D_{k_1,k_2}(x) - \frac{\sigma_0\gamma(x)}{\varphi(k_1)\varphi(k_2)} \prod_{p\mid (k_1,k_2)} \frac{p-1}{p-2}\right| \ll \frac{x^2}{(\log x)^A}.$$

For squarefree odd k we define $J_k(x)$ as the number of solutions of the equation (1) in primes satisfying (2) and such that

$$(p_1 + 2)(p_2 + 2) \equiv 0 \,(\mathrm{mod}\, k).$$

The Theorem stated above implies

Corollary 1. *For each $A > 0$ there exists $B = B(A) > 0$ such that*

$$\sum_{\substack{k\le \sqrt{x}/(\log x)^B\\ (k,2)=1}} \mu^2(k) \left| J_k(x) - \sigma_0\gamma(x) \prod_{p\mid k} \frac{2p-5}{(p-1)(p-2)}\right| \ll \frac{x^2}{(\log x)^A}.$$

Remark. We shall prove the Theorem and Corollary 1 with $B = 16A + 100$.

Using Corollary 1 we get

COROLLARY 2. *There exist infinitely many triples p_1, p_2, p_3 of distinct primes such that $p_1 + p_2 = 2p_3$ and $(p_1 + 2)(p_2 + 2) = \mathrm{P}_9$.*

2. Notations. Let x be a sufficiently large real number and A a positive constant. The constants in $\mathcal{O}$-terms and $\ll$-symbols are absolute or depend only on A. We shall denote by m, n, d, d_1, d_2, a, q, k, k_1, k_2, l, r, h, f integers, by p, p_1, p_2, p_3 prime numbers and by y, z, t, α real numbers. As usual $\mu(n)$, $\varphi(n)$ denote Möbius's function and Euler's function; $\tau_k(n)$ is the number of integer solutions of the equation $d_1 \ldots d_k = n$; $\tau(n) = \tau_2(n)$. We denote by (m, n) and $[m, n]$ the greatest common divisor and the least common multiple of m and n, respectively. For real y, z, however, (y, z) denotes the open interval on the real line with endpoints y and z. The meaning is always clear from the context. Instead of $m \equiv n \pmod{k}$ we shall write for simplicity $m \equiv n\,(k)$. We shall also use the notation $e(t) = \exp(2\pi i t)$. The letter c denotes some positive real number, not the same in all appearances. This convention allows us to write

$$(\log t)e^{-c\sqrt{\log t}} \ll e^{-c\sqrt{\log t}},$$

for example.

We define

$$\begin{gathered}
H = \frac{\sqrt{x}}{(\log x)^{16A+100}}, \qquad Q = (\log x)^{4A+20}, \qquad \tau = xQ^{-1}, \\
E_1 = \bigcup_{q \le Q} \bigcup_{\substack{a=0 \\ (a,q)=1}}^{q-1} \left(\frac{a}{q} - \frac{1}{q\tau}, \frac{a}{q} + \frac{1}{q\tau} \right), \qquad E_2 = \left(-\frac{1}{\tau}, 1 - \frac{1}{\tau} \right) \setminus E_1, \\
S_k(\alpha) = \sum_{\substack{x < p \le 3x \\ p \equiv -2\,(k)}} e(\alpha p), \qquad S(\alpha) = S_1(\alpha), \qquad V(\alpha) = \sum_{x < m \le 3x} \frac{e(\alpha m)}{\log m}, \\
\mathcal{E} = \sum\sum_{\substack{k_1, k_2 \le H \\ (k_1 k_2, 2)=1}} \left| D_{k_1,k_2}(x) - \frac{\sigma_0 \gamma(x)}{\varphi(k_1)\varphi(k_2)} \prod_{p \mid (k_1,k_2)} \frac{p-1}{p-2} \right|.
\end{gathered} \tag{3}$$

3. Proof of the corollaries. Suppose that $\mu^2(k) = 1$ and M_1, M_2 are integers. The following identity holds:

$$\mu(k) \sum_{\substack{[k_1,k_2]=k \\ k_1 \mid M_1 \\ k_2 \mid M_2}} \mu(k_1)\mu(k_2) = \begin{cases} 1 & \text{if } k \mid M_1 M_2, \\ 0 & \text{if } k \nmid M_1 M_2. \end{cases} \tag{4}$$

A similar identity has been stated in [1, Lemma 8]. For convenience we present a short proof.

If $k \nmid M_1 M_2$ the equality (4) is obvious. Suppose that $k \mid M_1 M_2$. We have

$$\mu(k) \sum_{\substack{[k_1,k_2]=k \\ k_1 | M_1 \\ k_2 | M_2}} \mu(k_1)\mu(k_2) = \mu(k) \sum_{\substack{k_1 | k \\ k_1 | M_1}} \sum_{\substack{d | k_1 \\ kd/k_1 | M_2}} \mu(k_1) \mu\left(\frac{kd}{k_1}\right)$$

$$= \sum_{\substack{k_1 | (M_1, k) \\ M_2 k_1 \equiv 0 \ (k)}} \sum_{d | (k_1, M_2 k_1 / k)} \mu(d) = \sum_{\substack{k_1 | (M_1,k) \\ M_2 k_1 \equiv 0 \ (k) \\ (k_1, M_2 k_1/k) = 1}} 1 = 1,$$

since the only integer which satisfies the conditions imposed in the last sum is $k_1 = k/(k, M_2)$. This completes the proof of (4).

Using (4) we get

$$J_k(x) = \sum_{\substack{x < p_1, p_2, p_3 \le 3x \\ p_1 + p_2 = 2p_3}} \mu(k) \sum_{\substack{[k_1,k_2]=k \\ k_1 | (p_1+2) \\ k_2 | (p_2+2)}} \mu(k_1)\mu(k_2) \tag{5}$$

$$= \mu(k) \sum_{[k_1,k_2]=k} \mu(k_1)\mu(k_2) D_{k_1,k_2}(x).$$

Suppose that $\mu^2(k) = \mu^2(l) = 1$ and $(k,2) = (l,2) = 1$. We define

$$t(l) = \prod_{p | l} \frac{p-1}{p-2}, \qquad \varrho(l) = \frac{\mu(l) t(l)}{\varphi(l)}$$

and

$$L(k) = \mu(k) \sum_{[k_1,k_2]=k} \frac{\mu(k_1)\mu(k_2)}{\varphi(k_1)\varphi(k_2)} t((k_1,k_2)). \tag{6}$$

It is clear that

$$L(k) = \frac{\mu(k)}{t(k)} \sum_{[k_1,k_2]=k} \frac{\mu(k_1)\mu(k_2) t(k_1) t(k_2)}{\varphi(k_1)\varphi(k_2)} = \frac{\mu(k)}{t(k)} \sum_{[k_1,k_2]=k} \varrho(k_1)\varrho(k_2)$$

$$= \frac{\mu(k)}{t(k)} \sum_{k_1 | k} \sum_{d | k_1} \varrho(k_1) \varrho\left(\frac{kd}{k_1}\right) = \frac{\mu(k)\varrho(k)}{t(k)} \sum_{k_1 | k} \sum_{d | k_1} \varrho(d)$$

$$= \frac{\mu(k)\varrho(k)}{t(k)} \sum_{d | k} \varrho(d) \tau\left(\frac{k}{d}\right) = \frac{\mu(k)\varrho(k)\tau(k)}{t(k)} \sum_{d | k} \frac{\varrho(d)}{\tau(d)}$$

$$= \frac{\mu(k)\varrho(k)\tau(k)}{t(k)} \prod_{p | k} \left(1 + \frac{\varrho(p)}{2}\right).$$

Using the definitions of $\varrho(l)$ and $t(l)$ we easily compute

$$L(k) = \prod_{p\mid k} \frac{2p-5}{(p-1)(p-2)}. \tag{7}$$

Now we apply (3), (5)–(7) and the Theorem to obtain

$$\begin{aligned}
&\sum_{\substack{k\le H\\ (k,2)=1}} \mu^2(k)\Big|J_k(x) - \sigma_0\gamma(x)\prod_{p\mid k}\frac{2p-5}{(p-1)(p-2)}\Big|\\
&\quad= \sum_{\substack{k\le H\\ (k,2)=1}} \mu^2(k)\\
&\qquad\times\Big|\mu(k)\sum_{[k_1,k_2]=k}\mu(k_1)\mu(k_2)\Big(D_{k_1,k_2}(x) - \frac{\sigma_0\gamma(x)}{\varphi(k_1)\varphi(k_2)}\prod_{p\mid(k_1,k_2)}\frac{p-1}{p-2}\Big)\Big|\\
&\quad\le \sum_{\substack{k_1,k_2\le H\\ (k_1k_2,2)=1}}\sum \Big|D_{k_1,k_2}(x) - \frac{\sigma_0\gamma(x)}{\varphi(k_1)\varphi(k_2)}\prod_{p\mid(k_1,k_2)}\frac{p-1}{p-2}\Big| \ll \frac{x^2}{(\log x)^A}.
\end{aligned}$$

Corollary 1 is proved.

Consider the sequence

$$\mathcal{A} = \{(p_1+2)(p_2+2) \mid x < p_1, p_2 \le 3x,\ (p_1+p_2)/2 \text{ prime}\}$$

and let $\mathcal{B}$ be the set of odd primes. Define

$$X = \sigma_0\gamma(x), \qquad \omega(k) = k\prod_{p\mid k}\frac{2p-5}{(p-1)(p-2)}.$$

We apply Theorem 10.3 of [3] choosing $\kappa = 2$, $\alpha = 1/4$, $\mu = 4.1$, $\zeta = 0.4$. It is clear that we may get rid the extra factor $3^{\nu(d)}$ in the condition $R(\kappa,\alpha)$ using, for example, the Cauchy inequality. We obtain

$$|\{\mathrm{P}_9 : \mathrm{P}_9 \in \mathcal{A}\}| \gg \frac{x^2}{\log^5 x}.$$

Since the contribution of the terms for which $p_1 = p_2$ is at most $\mathcal{O}(x)$, the last estimate proves Corollary 2.

4. Proof of the Theorem. It is clear that

$$D_{k_1,k_2}(x) = \int_{-1/\tau}^{1-1/\tau} S_{k_1}(\alpha)S_{k_2}(\alpha)S(-2\alpha)\,d\alpha = D^{(1)}_{k_1,k_2}(x) + D^{(2)}_{k_1,k_2}(x),$$

where

$$D_{k_1,k_2}^{(i)}(x) = \int_{E_i} S_{k_1}(\alpha) S_{k_2}(\alpha) S(-2\alpha)\, d\alpha, \qquad i = 1, 2.$$

Consequently,

$$\mathcal{E} \le \mathcal{E}_1 + \mathcal{E}_2, \tag{8}$$

where

$$\mathcal{E}_1 = \sum_{\substack{k_1,k_2 \le H \\ (k_1k_2,2)=1}}\sum \left| D_{k_1,k_2}^{(1)}(x) - \frac{\sigma_0 \gamma(x)}{\varphi(k_1)\varphi(k_2)} \prod_{p \mid (k_1,k_2)} \frac{p-1}{p-2} \right|, \tag{9}$$

$$\mathcal{E}_2 = \sum_{\substack{k_1,k_2 \le H \\ (k_1k_2,2)=1}}\sum |D_{k_1,k_2}^{(2)}(x)|. \tag{10}$$

The proof of the Theorem follows from (3), (8)–(10) and from the inequalities

$$\mathcal{E}_1 \ll \frac{x^2}{(\log x)^A}, \qquad \mathcal{E}_2 \ll \frac{x^2}{(\log x)^A}.$$

4.1. *The estimate of* $\mathcal{E}_1$. We have

$$D_{k_1,k_2}^{(1)}(x) = \sum_{q \le Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} I(a,q), \tag{11}$$

where

$$I(a,q) = \int_{-1/(q\tau)}^{1/(q\tau)} S_{k_1}\left(\frac{a}{q} + \alpha\right) S_{k_2}\left(\frac{a}{q} + \alpha\right) S\left(-2\left(\frac{a}{q} + \alpha\right)\right) d\alpha. \tag{12}$$

If

$$q \le Q, \qquad (a,q) = 1, \qquad |\alpha| \le \frac{1}{q\tau} \tag{13}$$

then we have

$$S\left(-2\left(\frac{a}{q} + \alpha\right)\right) = \frac{h(q)}{\varphi(q)} V(-2\alpha) + \mathcal{O}(xe^{-c\sqrt{\log x}}), \tag{14}$$

where

$$h(q) = \sum_{\substack{m=1 \\ (m,q)=1}}^{q} e\left(\frac{2m}{q}\right) = \frac{\mu\left(\frac{q}{(q,2)}\right)}{\varphi\left(\frac{q}{(q,2)}\right)} \cdot \varphi(q)$$

(the proof is similar to that of [6, Lemma 3, X]).

Consider $S_k(a/q + \alpha)$ for a, q, α satisfying (13). We are not able to find an asymptotic formula for that sum for a particular large k (unless we use some hypotheses which have not been proved yet). We shall find, however, an asymptotic formula with an error term which is small on average.

We have

$$S_k\left(\frac{a}{q}+\alpha\right)=\sum_{\substack{1\le m\le q\\(m,q)=1\\m\equiv -2\,((k,q))}} e\left(\frac{am}{q}\right)T(\alpha), \tag{15}$$

where

$$T(\alpha)=\sum_{\substack{x<p\le 3x\\p\equiv -2\,(k)\\p\equiv m\,(q)}} e(\alpha p).$$

Using the elementary theory of congruences one may easily prove that if the integers k, m, q satisfy $(k,2)=(m,q)=1$ and $m\equiv -2\,((k,q))$ then there exists an integer $f=f(k,m,q)$ such that $(f,[k,q])=1$ and such that for any integer n the congruence $n\equiv f\,([k,q])$ is equivalent to the system $n\equiv -2\,(k)$, $n\equiv m\,(q)$. Hence we have

$$T(\alpha)=\sum_{\substack{x<p\le 3x\\p\equiv f\,([k,q])}} e(\alpha p).$$

We define

$$\Delta(t,h)=\max_{y\le t}\max_{(l,h)=1}\Bigg|\sum_{\substack{p\le y\\p\equiv l\,(h)}}\log p-\frac{y}{\varphi(h)}\Bigg|.$$

Using Abel's formula we obtain

$$\begin{aligned}
T(\alpha)&=-\int_x^{3x}\Big(\sum_{\substack{x<p\le t\\p\equiv f\,([k,q])}}\log p\Big)\frac{d}{dt}\left(\frac{e(\alpha t)}{\log t}\right)dt+\Big(\sum_{\substack{x<p\le 3x\\p\equiv f\,([k,q])}}\log p\Big)\frac{e(3\alpha x)}{\log 3x}\\
&=-\int_x^{3x}\left(\frac{t-x}{\varphi([k,q])}+\mathcal{O}(\Delta(3x,[k,q]))\right)\frac{d}{dt}\left(\frac{e(\alpha t)}{\log t}\right)dt\\
&\quad+\left(\frac{2x}{\varphi([k,q])}+\mathcal{O}(\Delta(3x,[k,q]))\right)\frac{e(3\alpha x)}{\log 3x}\\
&=\frac{1}{\varphi([k,q])}\left(-\int_x^{3x}(t-x)\frac{d}{dt}\left(\frac{e(\alpha t)}{\log t}\right)dt+2x\frac{e(3\alpha x)}{\log 3x}\right)\\
&\quad+\mathcal{O}((1+|\alpha|x)\Delta(3x,[k,q])).
\end{aligned}$$

We integrate by parts, then use (13) and the well-known formula

$$\int_x^{3x} \frac{e(\alpha t)}{\log t}\, dt = V(\alpha) + \mathcal{O}(1)$$

to get

$$T(\alpha) = \frac{V(\alpha)}{\varphi([k,q])} + \mathcal{O}\biggl(\frac{Q}{q}\Delta(3x,[k,q])\biggr).$$

We substitute this expression for $T(\alpha)$ in (15) and we find that under the condition (13) we have

$$S_k\biggl(\frac{a}{q}+\alpha\biggr) = \frac{c_k(a,q)}{\varphi([k,q])}V(\alpha) + \mathcal{O}(Q\Delta(3x,[k,q])), \tag{16}$$

where

$$c_k(a,q) = \sum_{\substack{1\le m\le q\\ (m,q)=1\\ m\equiv -2\,((k,q))}} e\biggl(\frac{am}{q}\biggr). \tag{17}$$

An explicit formula for the quantity $c_k(a,q)$ is found in [7, p. 218]. It implies that

$$|c_k(a,q)| \le 1. \tag{18}$$

Furthermore, we shall use the trivial estimates

$$\biggl|S_k\biggl(\frac{a}{q}+\alpha\biggr)\biggr| \ll \frac{x}{k}, \qquad |V(\alpha)| \ll \frac{x}{\log x}, \qquad |h(q)| \ll 1. \tag{19}$$

From (14), (16), (18), (19) and the well-known estimate $\varphi(n)\gg n(\log\log n)^{-1}$ we get

$$\begin{aligned}
&S_{k_1}\biggl(\frac{a}{q}+\alpha\biggr)S_{k_2}\biggl(\frac{a}{q}+\alpha\biggr)S\biggl(-2\biggl(\frac{a}{q}+\alpha\biggr)\biggr)\\
&\quad= S_{k_1}\biggl(\frac{a}{q}+\alpha\biggr)S_{k_2}\biggl(\frac{a}{q}+\alpha\biggr)\frac{h(q)}{\varphi(q)}V(-2\alpha) + \mathcal{O}\biggl(\frac{x^3}{k_1k_2}e^{-c\sqrt{\log x}}\biggr)\\
&\quad= S_{k_1}\biggl(\frac{a}{q}+\alpha\biggr)\frac{h(q)}{\varphi(q)}\cdot\frac{c_{k_2}(a,q)}{\varphi([k_2,q])}V(\alpha)V(-2\alpha)\\
&\qquad+ \mathcal{O}\biggl(\frac{Qx^2}{qk_1}\Delta(3x,[k_2,q])\biggr) + \mathcal{O}\biggl(\frac{x^3}{k_1k_2}e^{-c\sqrt{\log x}}\biggr)\\
&\quad= \frac{h(q)c_{k_1}(a,q)c_{k_2}(a,q)}{\varphi(q)\varphi([k_1,q])\varphi([k_2,q])}V^2(\alpha)V(-2\alpha) + \mathcal{O}\biggl(\frac{x^3}{k_1k_2}e^{-c\sqrt{\log x}}\biggr)\\
&\qquad+ \mathcal{O}\biggl(\frac{Qx^2}{qk_2}\Delta(3x,[k_1,q])\biggr) + \mathcal{O}\biggl(\frac{Qx^2}{qk_1}\Delta(3x,[k_2,q])\biggr).
\end{aligned}$$

For the integral $I(a,q)$ defined by (12), we find

$$I(a,q)=\frac{h(q)c_{k_1}(a,q)c_{k_2}(a,q)}{\varphi(q)\varphi([k_1,q])\varphi([k_2,q])}\int_{-1/(q\tau)}^{1/(q\tau)}V^2(\alpha)V(-2\alpha)\,d\alpha + \mathcal{O}\Big(\frac{xQ^2}{k_2q^2}\Delta(3x,[k_1,q])\Big)+\mathcal{O}\Big(\frac{xQ^2}{k_1q^2}\Delta(3x,[k_2,q])\Big) + \mathcal{O}\Big(\frac{x^2}{k_1k_2}e^{-c\sqrt{\log x}}\Big). \tag{20}$$

We also have

$$\int_{-1/(q\tau)}^{1/(q\tau)}V^2(\alpha)V(-2\alpha)\,d\alpha=\gamma(x)+\mathcal{O}(q^2\tau^2) \tag{21}$$

(the proof is analogous to that in [6, Lemma 4, X]). Using (18)–(21) we find that

$$I(a,q)=\frac{h(q)c_{k_1}(a,q)c_{k_2}(a,q)}{\varphi(q)\varphi([k_1,q])\varphi([k_2,q])}\gamma(x)+\mathcal{O}\Big(\frac{q^2\tau^2}{\varphi(q)\varphi([k_1,q])\varphi([k_2,q])}\Big) + \mathcal{O}\Big(\frac{xQ^2}{k_2q^2}\Delta(3x,[k_1,q])\Big)+\mathcal{O}\Big(\frac{xQ^2}{k_1q^2}\Delta(3x,[k_2,q])\Big) + \mathcal{O}\Big(\frac{x^2}{k_1k_2}e^{-c\sqrt{\log x}}\Big). \tag{22}$$

Set

$$b_{k_1,k_2}(q)=\sum_{\substack{a=0\\(a,q)=1}}^{q-1}c_{k_1}(a,q)c_{k_2}(a,q), \tag{23}$$

$$\lambda_{k_1,k_2}(q)=\frac{h(q)b_{k_1,k_2}(q)\varphi((k_1,q))\varphi((k_2,q))}{\varphi^3(q)}. \tag{24}$$

From (11), (22)–(24) and the well-known formula

$$\varphi([k,q])\varphi((k,q))=\varphi(k)\varphi(q)$$

we get

$$D^{(1)}_{k_1,k_2}(x)=\frac{\gamma(x)}{\varphi(k_1)\varphi(k_2)}\sum_{q\le Q}\lambda_{k_1,k_2}(q)+\mathcal{O}\Big(\tau^2(\log x)\sum_{q\le Q}\frac{q^2}{[k_1,q][k_2,q]}\Big) + \mathcal{O}\Big(xQ^2\sum_{q\le Q}\frac{\Delta(3x,[k_1,q])}{k_2q}\Big) + \mathcal{O}\Big(xQ^2\sum_{q\le Q}\frac{\Delta(3x,[k_2,q])}{k_1q}\Big)+\mathcal{O}\Big(\frac{x^2}{k_1k_2}e^{-c\sqrt{\log x}}\Big). \tag{25}$$

Consider the function $b_{k_1,k_2}(q)$. From (18) and (23) we have

$$|b_{k_1,k_2}(q)| \le \varphi(q). \tag{26}$$

It is not difficult to see that $b_{k_1,k_2}(q)$ is multiplicative with respect to q and that for prime p we have

$$b_{k_1,k_2}(p) = \begin{cases} p-1 & \text{if } p \nmid k_1,\ p \nmid k_2, \\ 1 & \text{if } p \mid k_1,\ p \nmid k_2, \\ 1 & \text{if } p \nmid k_1,\ p \mid k_2, \\ -1 & \text{if } p \mid k_1,\ p \mid k_2. \end{cases} \tag{27}$$

We also have $b_{k_1,k_2}(4) = 0$. Therefore the function $\lambda_{k_1,k_2}(q)$ defined by (24) is multiplicative with respect to q and $\lambda_{k_1,k_2}(p^l) = 0$ if $l \ge 2$. We apply Euler's identity (see [4, Theorem 286]) and also (19), (26), (27) and the definition of σ_0. After some calculations we get

$$\sum_{q \le Q} \lambda_{k_1,k_2}(q) = \sigma_0 \prod_{p \mid (k_1,k_2)} \frac{p-1}{p-2} + \mathcal{O}\bigg(\sum_{q > Q} \frac{(k_1,q)(k_2,q)}{\varphi^2(q)} \bigg). \tag{28}$$

From (25), (28) and the trivial estimate

$$\gamma(x) \ll \frac{x^2}{\log^3 x}$$

we obtain

$$\begin{aligned} D^{(1)}_{k_1,k_2}(x) = & \frac{\sigma_0 \gamma(x)}{\varphi(k_1)\varphi(k_2)} \prod_{p \mid (k_1,k_2)} \frac{p-1}{p-2} + \mathcal{O}\bigg(x^2 \sum_{q > Q} \frac{(k_1,q)(k_2,q)\log q}{k_1 k_2 q^2} \bigg) \\ & + \mathcal{O}\bigg(xQ^2 \sum_{q \le Q} \frac{\Delta(3x,[k_1,q])}{k_2 q} \bigg) + \mathcal{O}\bigg(xQ^2 \sum_{q \le Q} \frac{\Delta(3x,[k_2,q])}{k_1 q} \bigg) \\ & + \mathcal{O}\bigg(\tau^2(\log x) \sum_{q \le Q} \frac{q^2}{[k_1,q][k_2,q]} \bigg) + \mathcal{O}\bigg(\frac{x^2}{k_1 k_2} e^{-c\sqrt{\log x}} \bigg). \end{aligned} \tag{29}$$

Using (9) and (29) we find

$$\mathcal{E}_1 \ll xQ^2 \Sigma_1 + \tau^2(\log x)\Sigma_2 + x^2 \Sigma_3 + x^2 e^{-c\sqrt{\log x}}, \tag{30}$$

where

$$\Sigma_1 = \sum_{k_1,k_2 \le H} \sum_{q \le Q} \frac{\Delta(3x,[k_2,q])}{k_1 q}, \qquad \Sigma_2 = \sum_{k_1,k_2 \le H} \sum_{q \le Q} \frac{q^2}{[k_1,q][k_2,q]},$$

$$\Sigma_3 = \sum_{k_1,k_2 \le H} \sum_{q > Q} \frac{(k_1,q)(k_2,q)\log q}{k_1 k_2 q^2}.$$

Consider Σ_1. We have

$$\Sigma_1 \ll (\log x) \sum_{k\le H} \sum_{q\le Q} \frac{\Delta(3x,[k,q])}{q} = (\log x) \sum_{h\le HQ} \Delta(3x,h)\eta(h),$$

where

$$\eta(h) = \sum_{\substack{k\le H\\ [k,q]=h}} \sum_{q\le Q} \frac{1}{q} = \sum_{d\le Q} \sum_{\substack{k\le H\\ [k,q]=h\\ (k,q)=d}} \sum_{q\le Q} \frac{1}{q} \le \sum_{d\le Q} \sum_{\substack{q\le Q\\ q\equiv 0\ (d)}} \frac{1}{q} \ll \log^2 x.$$

Hence

$$\Sigma_1 \ll (\log^3 x) \sum_{h\le HQ} \Delta(3x,h).$$

Now we use the definitions of H, Q and the Bombieri–Vinogradov theorem (see [2, Chapter 28], for example) and we find

$$\Sigma_1 \ll \frac{x}{(\log x)^{12A+72}}. \tag{31}$$

We now treat Σ_2. We have

$$\begin{aligned}\Sigma_2 &= \sum_{d_1,d_2\le Q} d_1d_2 \sum_{\substack{k_1,k_2\le H\\ (k_1,q)=d_1\\ (k_2,q)=d_2}} \sum_{q\le Q} \frac{1}{k_1k_2}\\ &\le Q \sum_{d_1,d_2\le Q} \frac{d_1d_2}{[d_1,d_2]} \sum_{\substack{k_1,k_2\le H\\ k_1\equiv 0\ (d_1)\\ k_2\equiv 0\ (d_2)}} \frac{1}{k_1k_2} \ll Q(\log^2 x)\Sigma^*,\end{aligned}$$

where

$$\begin{aligned}\Sigma^* &= \sum_{d_1,d_2\le Q} \frac{1}{[d_1,d_2]} = \sum_{d\le Q} \sum_{\substack{d_1,d_2\le Q\\ (d_1,d_2)=d}} \frac{d}{d_1d_2}\\ &\le \sum_{d\le Q} \frac{1}{d} \sum_{d_1\le Q/d} \frac{1}{d_1} \sum_{d_2\le Q/d} \frac{1}{d_2} \ll \log^3 x.\end{aligned} \tag{32}$$

Hence

$$\Sigma_2 \ll Q\log^5 x. \tag{33}$$

To complete the estimate of $\mathcal{E}_1$ we have to consider Σ_3. Obviously

$$\Sigma_3 = \sum_{d_1,d_2\le H} d_1d_2 \sum_{\substack{k_1,k_2\le H\\ (q,k_1)=d_1\\ (q,k_2)=d_2}} \sum_{q>Q} \frac{\log q}{k_1k_2q^2} = \Sigma_4 + \Sigma_5, \tag{34}$$

where

$$\Sigma_4 = \sum_{\substack{d_1,d_2\le H\\ [d_1,d_2]>Q}} d_1d_2 \sum_{\substack{k_1,k_2\le H\\ (q,k_1)=d_1\\ (q,k_2)=d_2}} \sum_{q>Q} \frac{\log q}{k_1k_2q^2},$$

$$\Sigma_5 = \sum_{[d_1,d_2]\le Q} d_1d_2 \sum_{\substack{k_1,k_2\le H\\ (q,k_1)=d_1\\ (q,k_2)=d_2}} \sum_{q>Q} \frac{\log q}{k_1k_2q^2}.$$

We have

$$\Sigma_4 \ll \sum_{\substack{d_1,d_2\le H\\ [d_1,d_2]>Q}} d_1d_2 \sum_{\substack{k_1,k_2\le H\\ k_1\equiv 0\,(d_1)\\ k_2\equiv 0\,(d_2)}} \frac{1}{k_1k_2} \sum_{q>Q/[d_1,d_2]} \frac{\log(q[d_1,d_2])}{q^2[d_1,d_2]^2} \tag{35}$$

$$\ll (\log x) \sum_{\substack{d_1,d_2\le H\\ [d_1,d_2]>Q}} \frac{1}{[d_1,d_2]^2} \sum_{k_1\le H/d_1} \frac{1}{k_1} \sum_{k_2\le H/d_2} \frac{1}{k_2} \sum_{q=1}^{\infty} \frac{(1+\log q)}{q^2}$$

$$\ll (\log^3 x) \sum_{h>Q} \frac{1}{h^2} \sum_{[d_1,d_2]=h} 1 \ll (\log^3 x) \sum_{h>Q} \frac{\tau_3(h)}{h^2} \ll \frac{\log^4 x}{Q}.$$

For the sum Σ_5 we find

$$\Sigma_5 \ll (\log^3 x) \sum_{[d_1,d_2]\le Q} \frac{1}{[d_1,d_2]^2} \sum_{q>Q/[d_1,d_2]} \frac{\log q}{q^2} \ll \frac{\log^4 x}{Q}\cdot \Sigma^*$$

where Σ^* is defined by (32). Consequently,

$$\Sigma_5 \ll \frac{\log^7 x}{Q}. \tag{36}$$

Finally, combining (30), (31), (33)–(36) and using the definitions of Q and τ we get

$$\mathcal{E}_1 \ll \frac{x^2}{(\log x)^A}.$$

4.2. *The estimate of* $\mathcal{E}_2$. It is clear that

$$\mathcal{E}_2 \le \sum_{k_1,k_2\le H} \Big| \int_{E_2} S_{k_1}(\alpha)S_{k_2}(\alpha)S(-2\alpha)\,d\alpha \Big|.$$

Using the definition of $S_{k_2}(\alpha)$ we get

$$\begin{aligned}
\mathcal{E}_2 &\le \sum_{k_1,k_2\le H} \sum_{\substack{x<p\le 3x\\ p\equiv -2\,(k_2)}} \Big| \int_{E_2} S_{k_1}(\alpha)S(-2\alpha)e(\alpha p)\,d\alpha \Big| \\
&\le \sum_{k_1,k_2\le H} \sum_{(x+2)/k_2<r\le (3x+2)/k_2} \Big| \int_{E_2} S_{k_1}(\alpha)S(-2\alpha)e(-2\alpha)e(\alpha r k_2)\,d\alpha \Big| \\
&\le \sum_{k_1\le H} \sum_{n\le 3x+2} \Big(\sum_{k_2\le H} \sum_{\substack{(x+2)/k_2<r\le (3x+2)/k_2\\ rk_2=n}} 1 \Big) \\
&\quad\times \Big| \int_{E_2} S_{k_1}(\alpha)S(-2\alpha)e(-2\alpha)e(\alpha n)\,d\alpha \Big| \\
&\le \sum_{k\le H} \sum_{n\le 3x+2} \tau(n) \Big| \int_{E_2} S_{k}(\alpha)S(-2\alpha)e(-2\alpha)e(\alpha n)\,d\alpha \Big| .
\end{aligned}$$

By Cauchy's inequality we get

$$\mathcal{E}_2 \le \Bigg(\sum_{k\le H} \sum_{n\le 3x+2} \frac{\tau^2(n)}{k} \Bigg)^{1/2} \Bigg(\sum_{k\le H} k \sum_{n\le 3x+2} \Big| \int_{-1/\tau}^{1-1/\tau} f(\alpha)e(\alpha n)\,d\alpha \Big|^2 \Bigg)^{1/2},$$

where

$$f(\alpha) = \begin{cases} S_k(\alpha)S(-2\alpha)e(-2\alpha) & \text{if } \alpha\in E_2,\\ 0 & \text{if } \alpha\in E_1.\end{cases}$$

We now apply Bessel's inequality to obtain

$$\begin{aligned}
(37)\qquad \mathcal{E}_2 &\ll x^{1/2}(\log^2 x) \Big(\sum_{k\le H} k \int_{E_2} |S_k(\alpha)S(2\alpha)|^2\,d\alpha \Big)^{1/2} \\
&\le x^{1/2}(\log^2 x) \Big(\sum_{k\le H} k \sum_{\substack{x<p_1,p_2\le 3x\\ p_1\equiv p_2\equiv -2\,(k)}} \Big| \int_{E_2} |S(2\alpha)|^2 e((p_1-p_2)\alpha)\,d\alpha \Big| \Big)^{1/2} \\
&= x^{1/2}(\log^2 x)\Sigma^{1/2}, \qquad \text{say.}
\end{aligned}$$

We have

$$\begin{aligned}
(38)\qquad \Sigma &= \sum_{k\le H} k \sum_{\substack{|r|\le 2x\\ r\equiv 0\,(k)}} \Big(\sum\sum_{\substack{x<p_1,p_2\le 3x\\ p_1\equiv p_2\equiv -2\,(k)\\ p_1-p_2=r}} 1 \Big) \Big| \int_{E_2} |S(2\alpha)|^2 e(\alpha r)\,d\alpha \Big| \\
&\ll x \sum_{k\le H} \sum_{\substack{|r|\le 2x\\ r\equiv 0\,(k)}} \Big| \int_{E_2} |S(2\alpha)|^2 e(\alpha r)\,d\alpha \Big| \\
&= x(\Sigma' + \Sigma''),
\end{aligned}$$

where

$$\Sigma' = \sum_{k\le H}\Big(\int_{E_2}|S(2\alpha)|^2\,d\alpha\Big),$$

$$\Sigma'' = \sum_{k\le H}\sum_{\substack{1\le|r|\le 2x\\ r\equiv 0\,(k)}}\Big|\int_{E_2}|S(2\alpha)|^2 e(\alpha r)\,d\alpha\Big|.$$

Obviously

$$\Sigma' \le H\int_0^1|S(2\alpha)|^2\,d\alpha \ll \frac{Hx}{\log x}. \tag{39}$$

By the Cauchy inequality we find

$$\begin{aligned}
\Sigma'' &\ll \sum_{k\le H}\sum_{\substack{1\le r\le 2x\\ r\equiv 0\,(k)}}\Big|\int_{E_2}|S(2\alpha)|^2 e(r\alpha)\,d\alpha\Big| \\
&= \sum_{1\le r\le 2x}\Big(\sum_{\substack{k\le H\\ k|r}}1\Big)\Big|\int_{E_2}|S(2\alpha)|^2 e(r\alpha)\,d\alpha\Big| \\
&\ll \sum_{1\le r\le 2x}\tau(r)\Big|\int_{E_2}|S(2\alpha)|^2 e(r\alpha)\,d\alpha\Big| \\
&\ll \Big(\sum_{1\le r\le 2x}\tau^2(r)\Big)^{1/2}\Big(\sum_{1\le r\le 2x}\Big|\int_{-1/\tau}^{1-1/\tau} g(\alpha)e(r\alpha)\,d\alpha\Big|^2\Big)^{1/2},
\end{aligned}$$

where

$$g(\alpha) = \begin{cases} |S(2\alpha)|^2 & \text{if } \alpha\in E_2,\\ 0 & \text{if } \alpha\in E_1.\end{cases}$$

We again apply the Bessel inequality to obtain

$$\begin{aligned}
\Sigma'' &\ll x^{1/2}(\log x)^{3/2}\Big(\int_{E_2}|S(2\alpha)|^4\,d\alpha\Big)^{1/2} \\
&\ll x^{1/2}(\log x)^{3/2}\sup_{\alpha\in E_2}|S(2\alpha)|\Big(\int_0^1|S(2\alpha)|^2\,d\alpha\Big)^{1/2} \\
&\ll x(\log x)\sup_{\alpha\in E_2}|S(2\alpha)|.
\end{aligned} \tag{40}$$

Using the definitions of Q, τ and E_2 we can prove in the same way as in [6, Theorem 3, X] that

$$\sup_{\alpha\in E_2}|S(2\alpha)| \ll \frac{x}{(\log x)^{2A+7}}. \tag{41}$$

From (37)–(41) we obtain

$$\mathcal{E}_2 \ll \frac{x^2}{(\log x)^A}.$$

The Theorem is proved.

Finally, the authors would like to thank the Ministry of Science and Education of Bulgaria for financial support under grant MM–430.

References

[1] J. Brüdern and E. Fouvry, *Lagrange's Four Squares Theorem with almost prime variables*, J. Reine Angew. Math. 454 (1994), 59–96.

[2] H. Davenport, *Multiplicative Number Theory* (revised by H. Montgomery), 2nd ed., Springer, 1980.

[3] H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, London, 1974.

[4] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., Oxford Univ. Press, 1979.

[5] D. R. Heath-Brown, *Three primes and an almost-prime in arithmetic progression*, J. London Math. Soc. (2) 23 (1981), 396–414.

[6] A. A. Karatsuba, *Principles of Analytic Number Theory*, Nauka, Moscow, 1983 (in Russian).

[7] H. Maier and C. Pomerance, *Unusually large gaps between consecutive primes*, Trans. Amer. Math. Soc. 322 (1990), 201–237.

[8] D. I. Tolev, *On the number of representations of an odd integer as a sum of three primes, one of which belongs to an arithmetic progression*, Proc. Steklov Math. Inst., to appear.

[9] J. G. van der Corput, *Über Summen von Primzahlen und Primzahlquadraten*, Math. Ann. 116 (1939), 1–50.

[10] I. M. Vinogradov, *Representation of an odd number as a sum of three primes*, Dokl. Akad. Nauk SSSR 15 (1937), 169–172 (in Russian).

Department of Mathematics
Plovdiv University "P. Hilendarski"
"Tsar Asen" 24
Plovdiv 4000, Bulgaria
E-mail: tpeneva@ulcc.uni-plovdiv.bg
dtolev@ulcc.uni-plovdiv.bg

Received on 24.3.1997 (3150)