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Ramanujan's class invariants, Kronecker's limit formula and modular equations (III)

by

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To the memory of K. G. Ramanathan

1. Introduction. For $|q| < 1$, set

$$(a;q)_\infty := \prod_{k=1}^{\infty}(1 - aq^{k-1}).$$

If n is any positive rational number and $q = \exp(-\pi\sqrt{n})$, *Ramanujan's class invariants* are defined by

$$G_n := 2^{-1/4}q^{-1/24}(-q;q^2)_\infty \tag{1.1}$$

and

$$g_n := 2^{-1/4}q^{-1/24}(q;q^2)_\infty. \tag{1.2}$$

If n is a positive integer, then G_n and g_n can be roughly viewed as generators of the Hilbert class field of the complex quadratic field $K = \mathbb{Q}(\sqrt{-n})$, or more generally, generators of the ring class field of the order of K, that is, $\mathbb{Z}(\sqrt{-n})$, because of their relations with $j(\sqrt{-n})$. For complete accounts, the reader is referred to the important paper of B. Birch [6] and the excellent books of Cox [7] and Lang [8]. In the notation of H. Weber [15], $G_n = 2^{-1/4}f(\sqrt{-n})$ and $g_n = 2^{-1/4}f_1(\sqrt{-n})$ where f, f_1 are called *Weber's functions*, and $f(\sqrt{-n}), f_1(\sqrt{-n})$ are also called *Weber's invariants*. The term "invariant" is due to Weber.

In his monumental book [15], Weber calculated a total of 105 class invariants. Ramanujan [11, 12] calculated a total of 110 class invariants among which 49 are not found in Weber's book. However, Ramanujan's approach is still a mystery today. Using Kronecker's limit formula and modular equa-

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tions of degree 3, 5 and 7, we [4, 5] have proved 18 class invariants of Ramanujan which had not been proved in literature.

Watson [14] employed an "empirical process" to establish 16 class invariants: g_n for $n = 66, 114, 126, 138, 154, 238, 310, 522, 630$ and G_n for $n = 333, 465, 765, 777, 897, 1645, 1677$, which were first stated by Ramanujan [11]. In fact, Watson's "empirical process" is not rigorous. Therefore, it is highly desirable to find rigorous proofs of these class invariants of Ramanujan and Watson. In [16], we have rigorously proved 6 class invariants, namely g_n for $n = 66, 114, 138, 154, 238$ and 310. Note that in all of these 6 cases, the class number h_K is 8 and the genus number is 4. In [16], we have also pointed out that using Theorems 1 and 2 in [4] one can rigorously prove 5 class invariants of Ramanujan and Watson, namely g_n for $n = 126, 522, 630$ and G_n for $n = 333, 765$. Note that in all of the 5 cases, n is a multiple of 9.

The aim of this paper is to provide rigorous proofs for the remaining 5 class invariants of Ramanujan–Watson, that is, G_n for $n = 465, 777, 897$, 1645 and 1677. Note that in all of these 5 cases, the class number h_K is 16 and the genus number is 8. Therefore, the proofs here are more complicated than those in [5] and [16]. We also point out that this paper is a continuation of our previous work, and the reader is referred to [5] and [16] for more details of the background.

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2. Kronecker's limit formula and modular equations. Let $Q(u,v) := y^{-1}(u+vz)(u+v\bar{z})$, where $z = x+iy$, with $y > 0$, and $\bar{z}$ is the complex conjugate of z. The *Epstein zeta-function* $\zeta_Q(s)$ is defined for $\sigma = \operatorname{Re} s > 1$ by

$$\zeta_Q(s) := \sum_{u,v}{}' Q(u,v)^{-s}, \tag{2.1}$$

where $\sum'$ means that the pair $(u,v) = (0,0)$ is excluded from the summation. It is well known that $\zeta_Q(s)$ can be analytically continued to the entire complex plane with a simple pole at $s = 1$.

The celebrated Kronecker first limit formula can be stated as follows:

$$\lim_{s\to 1}\left(\zeta_Q(s) - \frac{\pi}{s-1}\right) = 2\pi(\gamma - \log 2 - \log(\sqrt{y}\,|\eta(z)|^2)),$$

or equivalently,

$$\zeta_Q(s) = \frac{\pi}{s-1} + 2\pi(\gamma - \log 2 - \log(\sqrt{y}\,|\eta(z)|^2)) + O(s-1), \tag{2.2}$$

where γ denotes Euler's constant and $\eta(z)$ is the Dedekind eta-function defined, for $\operatorname{Im} z > 0$, by

$$\eta(z) = q^{1/12}(q^2;q^2)_\infty \quad \text{with } q = e^{\pi i z}. \tag{2.3}$$

Let $K = \mathbb{Q}(\sqrt{-n})$ be a complex quadratic field with a squarefree positive integer n, and let C_K denote the ideal class group of K. Then the discriminant of K is given by

$$d := d_K = \begin{cases} -4n & \text{if } -n \equiv 2,3 \pmod 4, \\ -n & \text{if } -n \equiv 1 \pmod 4. \end{cases}$$

Set

$$\omega := \omega_K = \begin{cases} \sqrt{-n} & \text{if } -n \equiv 2,3 \pmod 4, \\ \dfrac{1+\sqrt{-n}}{2} & \text{if } -n \equiv 1 \pmod 4. \end{cases}$$

Let $A \in C_K$ and $\mathfrak{b} = [a, b+\omega]$ be any nonzero integral ideal $\in A^{-1}$. Set $z = x + iy = (b+\omega)/a$, and

$$F(A) = |\eta(z)|^2/\sqrt{a}. \tag{2.4}$$

Note that $F(A)$ depends only on A. Genus characters are a special class of characters on the ideal class group C_K. A genus character χ has only values of ± 1 and is associated with a decomposition of $d = d_1 d_2$, where $d_1 > 0$, d, d_1 and d_2 are discriminants of $K = \mathbb{Q}(\sqrt{d})$, $K_1 = \mathbb{Q}(\sqrt{d_1})$ and $K_2 = \mathbb{Q}(\sqrt{d_2})$, respectively. Then applying Kronecker's limit formula, we have the following [13, p. 72]:

Theorem 2.1. *For a nonprincipal genus character χ,*

$$\frac{\nu h_1 h_2 \log \varepsilon_1}{\nu_2} = -\sum_{A \in C_K} \chi(A) \log F(A),$$

or

$$\varepsilon_1^{\nu h_1 h_2/\nu_2} = \prod_{A \in C_K} F(A)^{-\chi(A)}, \tag{2.5}$$

where h_1 and h_2 are the class numbers of K_1 and K_2, respectively, ε_1 is the fundamental unit of K_1, and ν, ν_1 and ν_2 are the numbers of roots of unity in K, K_1 and K_2, respectively.

We emphasize that (2.5) is the major ingredient in our proofs. For complete accounts, the reader is referred to the great book of Siegel [13].

We also need modular equations of degrees 3, 5, 7, 13 and 21. For brevity, we state them in terms of class invariants. The reader is referred to Berndt's books [1–3] for details about modular equations.

Theorem 2.2 (modular equation of degree 3; [1, p. 231]).

$$\left(\frac{G_n}{G_{n/9}}\right)^6+\left(\frac{G_{n/9}}{G_n}\right)^6+2\sqrt{2}\left(\frac{1}{(G_nG_{n/9})^3}-(G_nG_{n/25})^3\right)=0. \tag{2.6}$$

Theorem 2.3 (modular equation of degree 5; [1, p. 282]).

$$\left(\frac{G_n}{G_{n/25}}\right)^3+\left(\frac{G_{n/25}}{G_n}\right)^3+2\left(\frac{1}{(G_nG_{n/25})^2}-(G_nG_{n/25})^2\right)=0. \tag{2.7}$$

Theorem 2.4 (modular equation of degree 7; [1, p. 315]).

$$\left(\frac{G_n}{G_{n/49}}\right)^4+\left(\frac{G_{n/49}}{G_n}\right)^4+7-2\sqrt{2}\left(\frac{1}{(G_nG_{n/49})^3}+(G_nG_{n/49})^3\right)=0. \tag{2.8}$$

Theorem 2.5 (modular equation of degree 13; [15, p. 315]). *Let*

$$A=\frac{G_{n/169}}{G_n}+\frac{G_n}{G_{n/169}}\quad \text{and}\quad B=8((G_nG_{n/169})^6-(G_nG_{n/169})^{-6}).$$

Then

$$A(A^6+6A^4+A^2-20)=B. \tag{2.9}$$

Theorem 2.6 (modular equation of degree 21; [3, Entry 36, Chapter 36]. *Set*

$$P=(G_nG_{n/9}G_{n/49}G_{n/441})^3\quad \text{and}\quad Q=\frac{G_nG_{n/9}}{G_{n/49}G_{n/441}}.$$

Let $X=Q+Q^{-1}$. *Then*

$$X^4+7X^3+10X^2=8(P+P^{-1}-2). \tag{2.10}$$

3. Four class invariants of Ramanujan. In this section, we provide rigorous proofs for four class invariants of Ramanujan, namely G_n for $n=465,1645,897$ and 1677. For all of these four cases, as mentioned earlier, the class number h_K is 16, the genus number is 8 and each genus of K contains two ideal classes. In the case where each genus of K contains only one ideal class, G_n and g_n are much simpler. They can be found mainly by making use of (2.5). This idea was first developed by Siegel [13] and was utilized by K. Ramachandra [9] and K. G. Ramanathan [10].

Let $\tau=\sqrt{-n}$. Then, by (1.1) and (2.3), it is easily seen that

$$\left|\frac{\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)}\right|=2^{1/4}G_n. \tag{3.1}$$

By a slight abuse of notation, in what follows we use a representative ideal $\mathfrak{a}$ to denote the ideal class A which contains $\mathfrak{a}$. If $\mathfrak{b}=[a,b+\omega]=\overline{\mathfrak{b}}=[a,b+\overline{\omega}]$, then $\mathfrak{b}$ is called *ambiguous*. It is obvious that if an ambiguous ideal $\mathfrak{b}$ is in A, then $A=A^{-1}$.

Set $K=\mathbb{Q}(\sqrt{-n})$, where n is a squarefree positive integer divisible by a prime p and $-n\equiv 3 \pmod 4$. Then $\omega=\sqrt{-n}$, and $[1,\omega],[2,1+\omega],[p,\omega]$ and $[2p,p+\omega]$ are ambiguous primitive ideals in K. We proved [4] the following theorem.

THEOREM 3.1. *Let $K=\mathbb{Q}(\sqrt{-n})$, where n is a squarefree positive integer with $-n\equiv 3 \pmod 4$ and is divisible by a prime p. Assume that each genus in K contains two ideal classes and $[2,1+\omega]$ is not in the principal genus.*

(i) *If $[p,\omega]$ is in the principal genus, then*

$$G_nG_{n/p^2}=\prod_{\chi(2)=-1}\varepsilon_1^{4h_1h_2/(\nu_2h)}, \tag{3.2}$$

where χ, associated with the decomposition $d=d_1d_2$, runs through all genus characters with $\chi([2,1+\omega])=\chi(2)=-1$, h,h_1 and h_2 are the class numbers of $K=\mathbb{Q}(\sqrt{-n})=\mathbb{Q}(\sqrt{d})$, $K_1=\mathbb{Q}(\sqrt{d_1})$ and $K_2=\mathbb{Q}(\sqrt{d_2})$, respectively, ε_1 is the fundamental unit in K_1, and ν_2 is the number of roots of unity in K_2.

(ii) *If $[2p,p+\omega]$ is in the principal genus, then*

$$\frac{G_n}{G_{n/p^2}}=\prod_{\chi(2)=-1}\varepsilon_1^{4h_1h_2/(\nu_2h)}. \tag{3.3}$$

In order to make our proofs of these four class invariants of Ramanujan–Watson simpler, we state the following elementary lemmas which are easily verified. So we omit the proofs.

LEMMA 3.1. *Let a,b be positive numbers greater than $1/2$. If $x>1$ and*

$$\frac{1}{2}(x^4+x^{-4})=(8a^2-1)(8b^2-1)+16ab\sqrt{(4a^2-1)(4b^2-1)},$$

then

$$x=\left(\sqrt{a+\frac{1}{2}}+\sqrt{a-\frac{1}{2}}\right)\left(\sqrt{b+\frac{1}{2}}+\sqrt{b-\frac{1}{2}}\right).$$

LEMMA 3.2. *Let a,b be positive numbers greater than $1/2$. If $x>1$ and*

$$x^3+x^{-3}$$
$$=(4a-1)(4b-1)\sqrt{(2a+1)(2b+1)}+(4a+1)(4b+1)\sqrt{(2a-1)(2b-1)},$$

then

$$x=\left(\sqrt{a+\frac{1}{2}}+\sqrt{a-\frac{1}{2}}\right)\left(\sqrt{b+\frac{1}{2}}+\sqrt{b-\frac{1}{2}}\right).$$

LEMMA 3.3. *For $B>0$, the equation $Z(Z^6+6Z^4+Z^2-20)=B$ has exactly one root Z with $Z>1$.*

Now we are ready to prove the four class invariants rigorously.

THEOREM 3.2.

$$G_{465} = \left(\frac{1+\sqrt{5}}{2}\right)^{1/4}(2+\sqrt{3})^{1/4}(5\sqrt{5}+2\sqrt{31})^{1/12}\left(\frac{3\sqrt{3}+\sqrt{31}}{2}\right)^{1/4} \times \left(\sqrt{\frac{2+\sqrt{31}}{4}}+\sqrt{\frac{6+\sqrt{31}}{4}}\right)^{1/2} \times \left(\sqrt{\frac{11+2\sqrt{31}}{2}}+\sqrt{\frac{13+2\sqrt{31}}{2}}\right)^{1/2}.$$

Proof. We list all information needed in order to apply Theorem 3.1.

1) Set $K = \mathbb{Q}(\sqrt{-465})$. Then $\omega = \sqrt{-465}$, $d = -1860$, $h = 16$, and each genus of K contains two classes. The principal genus consists of $A_0 = [1, \omega]$ and $A_0' = [10, 5+\omega]$ while $A_1 = [2, 1+\omega]$ and $A_1' = [5, \omega]$ form another genus.

2) There are four genus characters χ with $\chi(2) = -1$, denoted by χ_1, χ_2, χ_3 and χ_4.

(i) For χ_1 associated with the decomposition $-1860 = 5(-372)$, $h_1 = 1$, $\varepsilon_1 = (1+\sqrt{5})/2$ and $h_2 = 4$, $\nu_2 = 2$.

(ii) For χ_2 associated with the decomposition $-1860 = 12(-155)$, $h_1 = 1$, $\varepsilon_1 = 2+\sqrt{3}$ and $h_2 = 4$, $\nu_2 = 2$.

(iii) For χ_3 associated with the decomposition $-1860 = 93(-20)$, $h_1 = 1$, $\varepsilon_1 = (29+3\sqrt{93})/2$ and $h_2 = 2$, $\nu_2 = 2$.

(iv) For χ_4 associated with the decomposition $-1860 = 620(-3)$, $h_1 = 2$, $\varepsilon_1 = 249+20\sqrt{155}$ and $h_2 = 1$, $\nu_2 = 6$.

Applying (3.3) with $p = 5$, we find that

$$\frac{G_{465}}{G_{93/5}} = \left(\frac{1+\sqrt{5}}{2}\right)^{1/2}(2+\sqrt{3})^{1/2}(249+20\sqrt{155})^{1/12}\left(\frac{29+3\sqrt{93}}{2}\right)^{1/4}. \tag{3.4}$$

It follows that

$$Q = \left(\frac{G_{465}}{G_{93/5}}\right)^6 = (2+\sqrt{5})(26+15\sqrt{3})(5\sqrt{5}+2\sqrt{31})(45\sqrt{3}+14\sqrt{31}) \tag{3.5}$$

and

$$Q^{-1} = \left(\frac{G_{93/5}}{G_{465}}\right)^6 = (-2+\sqrt{5})(26-15\sqrt{3})(5\sqrt{5}-2\sqrt{31})(-45\sqrt{3}+14\sqrt{31}). \tag{3.6}$$

By (2.7) with $n = 465$ and simple algebra, one can see that

$$(G_{465}G_{93/5})^4 + (G_{465}G_{93/5})^{-4} = \frac{1}{4}(Q + Q^{-1} + 10). \tag{3.7}$$

Set $X = G_{465}G_{93/5}$, by (3.5), (3.6) and (3.7), we find that

$$\frac{X^4 + X^{-4}}{2} = \frac{1}{2}(47883 + 12360\sqrt{15} + 8600\sqrt{31} + 2220\sqrt{465}). \tag{3.8}$$

Set $A = (4 + \sqrt{31})/4$ and $B = 6 + \sqrt{31}$. It is elementary to see that

$$\begin{aligned}(8A^2 - 1)(8B^2 - 1) &+ 16AB\sqrt{(4A^2 - 1)(4B^2 - 1)} \\ &= \frac{1}{2}(47883 + 12360\sqrt{15} + 8600\sqrt{31} + 2220\sqrt{465}).\end{aligned} \tag{3.9}$$

It is obvious that $X = G_{465}G_{93/5} > 1$. By Lemma 3.1, we find that

$$\begin{aligned}G_{465}G_{93/5} = {} & \left(\sqrt{\frac{2 + \sqrt{31}}{4}} + \sqrt{\frac{6 + \sqrt{31}}{4}}\right) \\ & \times \left(\sqrt{\frac{11 + 2\sqrt{31}}{2}} + \sqrt{\frac{13 + 2\sqrt{31}}{2}}\right).\end{aligned} \tag{3.10}$$

Therefore, by (3.4) and (3.10), we complete the proof.

THEOREM 3.3.

$$\begin{aligned}G_{1645} = {} & (2 + \sqrt{5})^{1/2}(3 + \sqrt{7})^{1/4}\left(\frac{7 + \sqrt{47}}{2}\right)^{1/4}\left(\frac{73\sqrt{5} + 9\sqrt{329}}{2}\right)^{1/8} \\ & \times \left(\sqrt{\frac{119 + 7\sqrt{329}}{8}} + \sqrt{\frac{127 + 7\sqrt{329}}{8}}\right)^{1/2} \\ & \times \left(\sqrt{\frac{743 + 41\sqrt{329}}{8}} + \sqrt{\frac{751 + 41\sqrt{329}}{8}}\right)^{1/2}.\end{aligned}$$

Proof. We record all information needed in order to apply Theorem 3.1.

1) Set $K = \mathbb{Q}(\sqrt{-1645})$. Then $\omega = \sqrt{-465}$, $d = -6580$, $h = 16$, and each genus of K contains two classes. The principal genus consists of $A_0 = [1, \omega]$ and $A_0' = [14, 7 + \omega]$ while $A_1 = [2, 1 + \omega]$ and $A_1' = [7, \omega]$ form another genus.

2) There are four genus characters χ with $\chi(2) = -1$, denoted by χ_1, χ_2, χ_3 and χ_4.

(i) For χ_1 associated with the decomposition $-6580 = 5(-1316)$, $h_1 = 1$, $\varepsilon_1 = (1 + \sqrt{5})/2$ and $h_2 = 4$, $\nu_2 = 2$.

(ii) For χ_2 associated with the decomposition $-6580 = 28(-235)$, $h_1 = 1$, $\varepsilon_1 = 8 + 3\sqrt{7}$ and $h_2 = 2$, $\nu_2 = 2$.

(iii) For χ_3 associated with the decomposition $-6580 = 188(-35)$, $h_1 = 1, \varepsilon_1 = 48 + 7\sqrt{47}$ and $h_2 = 2$, $\nu_2 = 2$.

(iv) For χ_4 associated with the decomposition $-6580 = 1645(-4)$, $h_1 = 2$, $\varepsilon_1 = (26647 + 657\sqrt{1645})/2$ and $h_2 = 1$, $\nu_2 = 4$.

Applying (3.3) with $p = 7$, we find that

$$\frac{G_{1645}}{G_{235/7}} = \left(\frac{1+\sqrt{5}}{2}\right)^3 (8+3\sqrt{7})^{1/4}(48+7\sqrt{47})^{1/4}\left(\frac{26647+657\sqrt{1645}}{2}\right)^{1/8}.$$

It follows that

$$\frac{G_{1645}}{G_{235/7}} = (2+\sqrt{5})(3+\sqrt{7})^{1/2}\left(\frac{7+\sqrt{47}}{2}\right)^{1/12}\left(\frac{73\sqrt{5}+9\sqrt{329}}{2}\right)^{1/4}, \tag{3.11}$$

$$\begin{aligned} Q &:= \left(\frac{G_{1645}}{G_{235/7}}\right)^4 \\ &= (161+72\sqrt{5})(8+3\sqrt{7})(48+7\sqrt{47})\left(\frac{73\sqrt{5}+9\sqrt{329}}{2}\right) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} Q^{-1} &= \left(\frac{G_{235/7}}{G_{1645}}\right)^4 \\ &= (161-72\sqrt{5})(8-3\sqrt{7})(48-7\sqrt{47})\left(\frac{-73\sqrt{5}+9\sqrt{329}}{2}\right). \end{aligned} \tag{3.13}$$

Set $X = G_{1645}G_{235/7}$. By (2.8) with $n = 1645$ and simple algebra, one can see that

$$\begin{aligned} X^3 + X^{-3} &= \frac{1}{2\sqrt{2}}(Q + Q^{-1} + 7) \\ &= 5025667\sqrt{2} + 849492\sqrt{70} \\ &\quad + 327838\sqrt{470} + 277074\sqrt{658}. \end{aligned} \tag{3.14}$$

Next we apply Lemma 3.2. Set $A = (123 + 7\sqrt{329})/8$ and $B = 747 + 41\sqrt{329}$. It is elementary to see that

$$\sqrt{(2A+1)(2B+1)} = \frac{1}{2}\sqrt{47450+2616\sqrt{329}} = \frac{1}{2}(109\sqrt{2}+6\sqrt{658}),$$

$$(4A-1)(4B-1)\sqrt{(2A+1)(2B+1)} = 5025667 + 277074\sqrt{658}.$$

It is also elementary to show that

$$(4A+1)(4B+1)\sqrt{(2A-1)(2B-1)} = 849492\sqrt{70} + 327838\sqrt{470}.$$

Therefore, we have

$$\begin{aligned} X^3 + X^{-3} &= (4A-1)(4B-1)\sqrt{(2A+1)(2B+1)} \\ &\quad + (4A+1)(4B+1)\sqrt{(2A-1)(2B-1)}. \end{aligned}$$

By Lemma 3.2, we find that

$$X = G_{1645}G_{235/7} \tag{3.15}$$
$$= \left(\sqrt{\frac{119+7\sqrt{329}}{8}}+\sqrt{\frac{127+7\sqrt{329}}{8}}\right)$$
$$\times \left(\sqrt{\frac{743+41\sqrt{329}}{8}}+\sqrt{\frac{751+41\sqrt{329}}{8}}\right).$$

Therefore, by (3.11) and (3.15), we have proved the theorem.

THEOREM 3.4.

$$G_{897} = (2+\sqrt{3})^{1/2}\left(\frac{3+\sqrt{13}}{2}\right)^{1/2}\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^{1/4}(4\sqrt{13}+3\sqrt{23})^{1/12}$$
$$\times \left(\sqrt{\frac{60+9\sqrt{39}}{4}}+\sqrt{\frac{56+9\sqrt{39}}{4}}\right)^{1/2}$$
$$\times \left(\sqrt{\frac{8+\sqrt{39}}{4}}+\sqrt{\frac{4+\sqrt{39}}{4}}\right)^{1/2}.$$

Proof. We record all information needed in order to apply Theorem 3.1.

1) Set $K = \mathbb{Q}(\sqrt{-897})$. Then $\omega = \sqrt{-897}$, $d = -3588$, $h = 16$, and each genus of K contains two classes. The principal genus consists of $A_0 = [1,\omega]$ and $A_0' = [13,\omega]$ while $A_1 = [2, 1+\omega]$ and $A_1' = [26, 13+\omega]$ form another genus.

2) There are four genus characters χ with $\chi(2) = -1$, denoted by χ_1, χ_2, χ_3 and χ_4.

(i) For χ_1 associated with the decomposition $-3588 = 13(-276)$, $h_1 = 1$, $\varepsilon_1 = (3+\sqrt{13})/2$ and $h_2 = 4$, $\nu_2 = 2$.

(ii) For χ_2 associated with the decomposition $-3588 = 69(-52)$, $h_1 = 1$, $\varepsilon_1 = (25+3\sqrt{69})/2$ and $h_2 = 2$, $\nu_2 = 2$.

(iii) For χ_3 associated with the decomposition $-3588 = 12(-299)$, $h_1 = 1$, $\varepsilon_1 = 2+\sqrt{3}$ and $h_2 = 8$, $\nu_2 = 2$.

(iv) For χ_4 associated with the decomposition $-3588 = 1196(-3)$, $h_1 = 2$, $\varepsilon_1 = 415+24\sqrt{299}$ and $h_2 = 1$, $\nu_2 = 6$.

Applying (3.2) with $p = 13$, we find that

$$G_{897}G_{69/13} \tag{3.16}$$
$$= \left(\frac{3+\sqrt{13}}{2}\right)\left(\frac{25+3\sqrt{69}}{2}\right)^{1/4}(2+\sqrt{3})(415+24\sqrt{299})^{1/12}$$
$$= \left(\frac{3+\sqrt{13}}{2}\right)\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^{1/2}(2+\sqrt{3})(4\sqrt{13}+3\sqrt{23})^{1/6}.$$

It follows that

$$\begin{aligned} P &:= (G_{897}G_{69/13})^6 \\ &= (649+180\sqrt{13})(36\sqrt{3}+13\sqrt{23}) \\ &\quad \times (1351+780\sqrt{3})(4\sqrt{13}+3\sqrt{23}), \end{aligned} \tag{3.17}$$

$$\begin{aligned} P^{-1} &= (649-180\sqrt{13})(36\sqrt{3}-13\sqrt{23}) \\ &\quad \times (1351-780\sqrt{3})(4\sqrt{13}-3\sqrt{23}). \end{aligned} \tag{3.18}$$

Therefore, we find that

$$\begin{aligned} &B := 8(P-P^{-1}) \\ &= 64(227328075\sqrt{3}+109204875\sqrt{13}+47401173\sqrt{69}+22770787\sqrt{299}). \end{aligned} \tag{3.19}$$

Set

$$A = \frac{11}{2}\sqrt{3}+3\sqrt{13}+\sqrt{69}+\frac{1}{2}\sqrt{299}.$$

Then, by elementary algebra, we find that

$$A(A^6+6A^4+A^2-20)=B.$$

From Theorem 2.5 and Lemma 3.3, we find that

$$A = \frac{G_{897}}{G_{69/13}}+\frac{G_{69/13}}{G_{897}} = \frac{11}{2}\sqrt{3}+3\sqrt{13}+\sqrt{69}+\frac{1}{2}\sqrt{299}$$

and

$$\begin{aligned} \frac{G_{897}}{G_{69/13}} &= \left(\sqrt{\frac{60+9\sqrt{39}}{4}}+\sqrt{\frac{56+9\sqrt{39}}{4}}\right) \\ &\quad \times \left(\sqrt{\frac{8+\sqrt{39}}{4}}+\sqrt{\frac{4+\sqrt{39}}{4}}\right). \end{aligned} \tag{3.20}$$

Now, the theorem follows from (3.16) and (3.20).

THEOREM 3.5.

$$\begin{aligned} G_{1677} &= (4414\sqrt{13}+2427\sqrt{43})^{1/12}\left(\frac{3+\sqrt{13}}{2}\right)^{3/4} \\ &\quad \times (\sqrt{13}+2\sqrt{3})^{1/4}\left(\frac{\sqrt{43}+\sqrt{39}}{2}\right)^{1/4} \\ &\quad \times \left(\sqrt{\frac{355+54\sqrt{43}}{4}}+\sqrt{\frac{351+54\sqrt{43}}{4}}\right)^{1/2} \\ &\quad \times \left(\sqrt{\frac{17+2\sqrt{43}}{4}}+\sqrt{\frac{13+2\sqrt{43}}{4}}\right)^{1/2}. \end{aligned}$$

Proof. We list all information needed in order to apply Theorem 3.1.

1) Set $K = \mathbb{Q}(\sqrt{-1677})$. Then $\omega = \sqrt{-1677}$, $d = -6708$, $h = 16$, and each genus of K contains two classes. The principal genus consists of $A_0 = [1, \omega]$ and $A_0' = [13, \omega]$ while $A_1 = [2, 1+\omega]$ and $A_1' = [26, 13+\omega]$ form another genus.

2) There are four genus characters χ with $\chi(2) = -1$, denoted by χ_1, χ_2, χ_3 and χ_4.

(i) For χ_1 associated with the decomposition $-6708 = 2236(-3)$, $h_1 = 2$, $\varepsilon_1 = 506568295 + 21425556\sqrt{559}$ and $h_2 = 1$, $\nu_2 = 6$.

(ii) For χ_2 associated with the decomposition $-6708 = 13(-516)$, $h_1 = 1$, $\varepsilon_1 = (3 + \sqrt{13})/2$ and $h_2 = 12$, $\nu_2 = 2$.

(iii) For χ_3 associated with the decomposition $-6708 = 156(-43)$, $h_1 = 2$, $\varepsilon_1 = 25 + 4\sqrt{39}$ and $h_2 = 1$, $\nu_2 = 2$.

(iv) For χ_4 associated with the decomposition $-6708 = 1677(-4)$, $h_1 = 4$, $\varepsilon_1 = (41 + \sqrt{1677})/2$ and $h_2 = 1$, $\nu_2 = 4$.

Applying (3.2) with $p = 13$, we find that

$$G_{1677}G_{129/13} = (506568295 + 21425556\sqrt{559})^{1/12} \times \left(\frac{3+\sqrt{13}}{2}\right)^{3/2} (25 + 4\sqrt{39})^{1/4}(41 + \sqrt{1677})^{1/4}$$

or

$$G_{1677}G_{129/13} = (4414\sqrt{13} + 2427\sqrt{43})^{1/6}\left(\frac{3+\sqrt{13}}{2}\right)^{3/2} \times (\sqrt{13} + 2\sqrt{3})^{1/2}\left(\frac{\sqrt{43}+\sqrt{39}}{2}\right)^{1/2}. \tag{3.21}$$

It follows that

$$\begin{aligned} P &:= (G_{1677}G_{129/13})^6 \\ &= (4414\sqrt{13} + 2427\sqrt{43})(23382 + 6485\sqrt{13}) \\ &\quad \times (102\sqrt{3} + 49\sqrt{13})(20\sqrt{43} + 21\sqrt{39}), \end{aligned} \tag{3.22}$$

$$\begin{aligned} P^{-1} &= (4414\sqrt{13} - 2427\sqrt{43})(-23382 + 6485\sqrt{13}) \\ &\quad \times (-102\sqrt{3} + 49\sqrt{13})(20\sqrt{43} - 21\sqrt{39}). \end{aligned} \tag{3.23}$$

Therefore, we find that

$$\begin{aligned} B &:= 8(P - P^{-1}) \\ &= 32(8621996645262 + 4977912082935\sqrt{3} \\ &\quad + 1314842161815\sqrt{43} + 759124475889\sqrt{129}). \end{aligned} \tag{3.24}$$

Set

$$A = 37 + \frac{39}{2}\sqrt{3} + \frac{11}{2}\sqrt{43} + 3\sqrt{129}.$$

Then, by an elementary algebra, we find that

$$A(A^6+6A^4+A^2-20)=B.$$

From Theorem 2.5 and Lemma 3.3, we find that

$$A=\frac{G_{1677}}{G_{129/13}}+\frac{G_{129/13}}{G_{1677}}=37+\frac{39}{2}\sqrt{3}+\frac{11}{2}\sqrt{43}+3\sqrt{129}$$

and

$$\frac{G_{1677}}{G_{129/13}}=\left(\sqrt{\frac{355+54\sqrt{43}}{4}}+\sqrt{\frac{351+54\sqrt{43}}{4}}\right)\times\left(\sqrt{\frac{17+2\sqrt{43}}{4}}+\sqrt{\frac{13+2\sqrt{43}}{4}}\right). \tag{3.25}$$

Now, the theorem follows from (3.21) and (3.25).

4. Class invariant G_{777}. In this section, we shall give a rigorous proof of G_{777}. As we will see, the proof is different from the proofs of the other four invariants and does not use Theorem 3.1.

THEOREM 4.1.

$$G_{777}=(2+\sqrt{3})^{1/4}(6+\sqrt{37})^{1/4}\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)^{1/4}(246\sqrt{7}+107\sqrt{37})^{1/12}$$
$$\times\left(\sqrt{\frac{6+3\sqrt{7}}{4}}+\sqrt{\frac{10+3\sqrt{7}}{4}}\right)^{1/2}$$
$$\times\left(\sqrt{\frac{15+6\sqrt{7}}{2}}+\sqrt{\frac{17+6\sqrt{7}}{2}}\right)^{1/2}.$$

Proof. We list some information needed.

I) Set $K=\mathbb{Q}(\sqrt{-777})$. Then $\omega=\sqrt{-777}$, $d=-3108$, $h=16$, and each genus of K contains two classes. The genus structure and class group C_K are as follows:

1) The principal genus, G_0 consists of $A_0=[1,\omega]$ and $A_0'=[21,\omega]$.
2) The genus G_1 consists of $A_1=[2,1+\omega]$ and $A_1'=[42,21+\omega]$.
3) The genus G_2 consists of $A_2=[3,\omega]$ and $A_2'=[7,\omega]$.
4) The genus G_3 consists of $A_3=[6,3+\omega]$ and $A_3'=[14,7+\omega]$.
5) The genus G_4 consists of $A_4=[11,2+\omega]$ and $A_4'=[11,-2+\omega]$.
6) The genus G_5 consists of $A_5=[13,4+\omega]$ and $A_5'=[13,-4+\omega]$.
7) The genus G_6 consists of $A_6=[22,9+\omega]$ and $A_6'=[22,-9+\omega]$.
8) The genus G_7 consists of $A_7=[26,9+\omega]$ and $A_7'=[26,-9+\omega]$.

II) There are two genus characters χ that we need here. We denote them by χ_1,χ_2.

(i) For χ_1 associated with the decomposition $-3108 = 1036(-3)$, $h_1 = 2$, $\varepsilon_1 = 847225 + 52644\sqrt{259}$ and $h_2 = 1$, $\nu_2 = 6$. It is evident that $\chi_1(A_j) = 1$ for $j = 0, 2, 5, 6$ and $\chi_1(A_j) = -1$ for $j = 1, 3, 4, 7$.

(ii) For χ_2 associated with the decomposition $-3108 = 37(-84)$, $h_1 = 1$, $\varepsilon_1 = 6 + \sqrt{37}$ and $h_2 = 4, \nu_2 = 2$. It is also clear that $\chi_2(A_j) = 1$ for $j = 0, 2, 4, 7$ and $\chi_2(A_j) = -1$ for $j = 1, 3, 5, 6$.

It follows from Theorem 2.1 that

$$\prod_{\chi=\chi_1,\chi_2} \prod_{A\in C_K} F(A)^{-\chi(A)} = \prod_{\chi=\chi_1,\chi_2} \varepsilon_1^{\nu h_1 h_2/\nu_2}.$$

Therefore, we have

$$\left(\frac{F(A_1)F(A_1')F(A_3)F(A_3')}{F(A_0)F(A_0')F(A_2)F(A_2')}\right)^2 = (847225 + 52644\sqrt{259})^{2/3}(6 + \sqrt{37})^4.$$

By (2.4) and (3.1), we find that

$$(G_{777}G_{37/21}G_{259/3}G_{111/7})^4 = (847225 + 52644\sqrt{259})^{2/3}(6 + \sqrt{37})^4,$$

and

$$P := G_{777}G_{37/21}G_{259/3}G_{111/7} = (246\sqrt{7} + 107\sqrt{37})^{1/3}(6 + \sqrt{37}). \tag{4.1}$$

Employing Theorem 2.6 with $n = 777$ and by a laborious calculation, one can see that

$$Q := \frac{G_{777}G_{259/3}}{G_{111/7}G_{37/21}} = \left(\sqrt{\frac{15 + 6\sqrt{7}}{2}} + \sqrt{\frac{17 + 6\sqrt{7}}{2}}\right)^2. \tag{4.2}$$

By (4.1) and (4.2), we find that

$$\begin{aligned} G_{777}G_{259/3} &= (246\sqrt{7} + 107\sqrt{37})^{1/6}(6 + \sqrt{37})^{1/2} \\ &\quad\times \left(\sqrt{\frac{15 + 6\sqrt{7}}{2}} + \sqrt{\frac{17 + 6\sqrt{7}}{2}}\right). \end{aligned} \tag{4.3}$$

Making use of a modular equation of degree 3 (Theorem 2.2) with $n = 777$ and again by a lengthy calculation, we find that

$$\frac{G_{777}}{G_{259/3}} = (2 + \sqrt{3})^{1/2}\left(\frac{\sqrt{7} + \sqrt{3}}{2}\right)^{1/2}\left(\sqrt{\frac{6 + 3\sqrt{7}}{4}} + \sqrt{\frac{10 + 3\sqrt{7}}{4}}\right). \tag{4.4}$$

Hence, the theorem follows from (4.3) and (4.4).

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