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On perfect powers in products with terms from arithmetic progressions

by

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1. Introduction. The purpose of this paper is to obtain certain extensions of a remarkable theorem of Erdős and Selfridge [3, Theorem 1] that a product of two or more consecutive positive integers is never a power. If $n(n+1)\ldots(n+k-1) = y^l$ for positive integers k, l, n, y with $k \ge 2$ and $l \ge 2$, then $\mathrm{ord}_p(n(n+1)\ldots(n+k-1))$ is congruent to $0 \pmod{l}$ for every prime p. Erdős and Selfridge derived their result from the following statement.

Theorem A (Erdős and Selfridge [3, Theorem 2]). *Let $k \ge 3$, $l \ge 2$ and $n \ge 1$ be integers such that $n+k-1 \ge p^{(k)}$ where $p^{(k)}$ is the least prime satisfying the inequality $p^{(k)} \ge k$. Then there is a prime $p \ge k$ for which $\mathrm{ord}_p(n(n+1)\ldots(n+k-1))$ is not congruent to $0 \pmod{l}$.*

In an earlier paper ([2]), Erdős had shown that the equation

$$n(n+1)\ldots(n+k-1) = k!y^l$$

has no solution under necessary conditions (see Section 2).

Theorem B (Erdős [2]). *Let $k \ge 4$, $l \ge 2$, $n \ge k+1$ and $y \ge 1$ be integers. Then*

$$\binom{n+k-1}{k} = y^l$$

does not hold.

We observe that Theorem B is not a consequence of Theorem A whenever k is a prime. The goal of the present paper is to extend Theorems A and B. This extension has the following form. Let $n > 0$, $l \ge 2$, $k \ge k_0$ and $t \ge t_0 = t_0(k)$ be integers where k_0 and t_0 are explicitly given numbers. Let $d_1, \ldots, d_t$ be distinct integers in the interval $[0, k-1]$. Let $d \in \Lambda$ where Λ is an explicitly given finite set of positive integers depending only on k and l.

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Suppose that $(n+d_1d)\dots(n+d_td)$ is divisible by a prime exceeding k. Then there exists a prime $p>k$ for which $\mathrm{ord}_p((n+d_1d)\dots(n+d_td))\not\equiv 0 \pmod{l}$. The precise statements will be given in the next section. As an application of our result we derive the following generalisations of the theorem of Erdős and Selfridge [3, Theorem 1] mentioned in the beginning and of Theorem B. For an integer $\nu>1$, we define $P(\nu)$ to be the greatest prime factor of ν and write $P(1)=1$.

Corollary 1. *The equation*

$$n(n+d)\dots(n+(k-1)d)=y^l \quad \textit{in integers } 1\le d\le 6,\ k\ge 3,\ l\ge 2,\ n\ge 1,\ y\ge 1 \textit{ with } \gcd(n,d)=1$$

has no solution.

Corollary 2. *The equation*

$$n(n+d)\dots(n+(k-1)d)=by^l \quad \textit{in integers } 1\le d\le 6,\ k\ge 4,\ P(b)\le k,\ l\ge 2,\ n\ge 1,\ y\ge 1 \textit{ with } \gcd(n,d)=1$$

has no solution provided that the left hand side of the equation is divisible by a prime exceeding k whenever $d=1$.

2. Results. For an integer $\nu>1$, we define $p(\nu)$ and $\omega(\nu)$ to be the smallest prime factor of ν and the number of distinct prime factors of ν, respectively, and we write $p(1)=1$ and $\omega(1)=0$. Let $b,d,k\ge 2$, $l\ge 2$, $n,t\ge 2$ and y denote positive integers such that $P(b)\le k$ and $\gcd(n,d)=1$. Further, we write $d_1,\dots,d_t$ for distinct integers in the interval $[0,k-1]$. We set

$$k_0=\begin{cases}4 & \text{if } d=1,\\ 3 & \text{if } d>1,\end{cases} \tag{1}$$

$$\alpha(k)=\left[\frac{(.0156)k}{\log k}\right],\qquad \beta(k)=\left[\frac{(.0017)k}{\log k}\right] \tag{2}$$

and

$$t_0\ge\begin{cases}k & \text{for } k\le 8,\ l\ge 3 \text{ and for } k\le 24,\ l=2,\\ k-1 & \text{for } 9\le k\le 11380,\ l\ge 3 \text{ and}\\ & \text{for } 25\le k<870,\ l=2,\\ k-\alpha(k) & \text{for } k\ge 870,\ l=2,\\ k-\beta(k) & \text{for } k>11380,\ l\ge 3.\end{cases} \tag{3}$$

We assume that

$$k\ge k_0,\qquad t\ge t_0.$$

We shall follow the above notation throughout the paper. We prove

THEOREM 1. (a) *Let $k \geq k_0,\ t \geq t_0$ and*

$$\begin{cases} d \in \{1,2,3,4,6\} \text{ and } l \geq 2, \text{ or} \\ d \leq 120,\ d \text{ even and } l \geq 5, \text{ or} \\ d \leq 36,\ d \text{ odd},\ 3 \mid d \text{ and } l \geq 5. \end{cases} \tag{4}$$

Assume that $(n+d_1d)\dots(n+d_td)$ is divisible by a prime exceeding k. Then there exists a prime $p > k$ for which

$$\mathrm{ord}_p((n+d_1d)\dots(n+d_td)) \not\equiv 0 \pmod{l}. \tag{5}$$

(b) *Let $d = 5, k \geq 4$ and*

$$t \geq \begin{cases} k & \text{for } l = 3,\ k \leq 25, \\ t_0 & \text{otherwise.} \end{cases}$$

Suppose that $(n+d_1d)\dots(n+d_td)$ is divisible by a prime exceeding k. Then there exists a prime $p > k$ satisfying (5).

Theorem 1 is equivalent to saying that under the assumptions of Theorem 1, the equation

$$(n+d_1d)\dots(n+d_td) = by^l$$

does not hold. Theorem 1 with $d = 1$ answers a question of Shorey and Tijdeman ([8, §1]). Furthermore, it answers some of the problems raised by Erdős and Selfridge at the end of their paper [3]. We observe that the hypothesis that $(n+d_1d)\dots(n+d_td)$ is divisible by a prime exceeding k is necessary in Theorem 1. Shorey and Tijdeman [9] showed that this hypothesis is satisfied whenever $t = k$, $d > 1$ and $(n,d,k) \neq (2,7,3)$. It is known that $n(n+1) = 2y^2$ has infinitely many solutions. Further, we have $n(n+1)(n+2) = 6y^2$ if $n = 48, y = 140$. The equation $n(n+d) = y^l$ can always be solved with $n = n_1^l,\ d = (n_1+1)^l - n_1^l$ for any positive integer n_1. Thus we see that the assumption $k \geq k_0$ with k_0 as in (1) is necessary in Theorem 1(a). Theorem 1(b) with $k = 3$ remains unproved. We shall derive Theorems A and B from Theorem 1 in Section 7. In view of the examples given above, the assumption $k \geq 4$ of Theorem B is necessary. Now we consider Theorem B with $k \geq 2,\ l \geq 2,\ n \leq k$ and $y \geq 1$. It is clear from the examples given above that the equation in Theorem B has solutions if $n \leq 4$. Further, by the relation $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$, we derive from Theorem B that the equation in Theorem B does not hold if $n \geq 5$.

When k is large, better bounds than (4) can be obtained for d so that the assertion of Theorem 1 is valid. We have

THEOREM 2. *Let $k \geq 11380$, $t \geq t_0$ and*

$$d \leq \begin{cases} (.3)k^{1/3} & \text{if } l = 2, \\ (1.75)k^{1/3} & \text{if } l = 3, \\ 295k^{l-3} & \text{if } l \geq 5. \end{cases} \tag{6}$$

Suppose $(n+d_1d)\dots(n+d_td)$ *is divisible by a prime* $> k$. *Then there exists a prime* $p > k$ *satisfying* (5).

If k exceeds a large effectively computable absolute constant (unspecified), we refer to Shorey and Tijdeman [10] and Shorey and Nesterenko [7] for better bounds for d and t, respectively.

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3. Basic lemmas. In this section, we prove lemmas for the proofs of Theorems 1 and 2. We first observe that there is no loss of generality in assuming that l is a prime number, which we suppose throughout the paper. Also we assume that

$$P((n+d_1d)\dots(n+d_td)) > k \tag{7}$$

and

$$\mathrm{ord}_p((n+d_1d)\dots(n+d_td)) \equiv 0 \pmod{l} \quad \text{for every prime } p > k. \tag{8}$$

We shall use the above assumptions (7) and (8) without any further reference in this section. By (8), we write

$$n + d_id = a_ix_i^l, \quad P(a_i) \le k, \quad a_i \text{ is } l\text{th power free for } 1 \le i \le t \tag{9}$$

and

$$n + d_id = A_iX_i^l, \quad P(A_i) \le k, \quad \gcd\Big(\prod_{p\le k} p, X_i\Big) = 1 \quad \text{for } 1 \le i \le t. \tag{10}$$

Let $S = \{a_i \mid 1 \le i \le t\}$ and $S' = \{A_i \mid 1 \le i \le t\}$. Let t' be the number of distinct elements of S. We order the distinct elements of S as $a'_1 < a'_2 < \dots < a'_{t'}$. Using an argument of Erdős ([3, Lemma 2]), we find that there exist sets $S_1 \subset S$ and $S'_1 \subset S'$ with $|S_1|$ and $|S'_1|$ greater than or equal to $t - \pi(k)$ such that

$$\prod_{a_i\in S_1} a_i \le (k-1)! \quad \text{and} \quad \prod_{A_i\in S'_1} A_i \le (k-1)!. \tag{11}$$

From (7) and (9) we have $n + (k-1)d \ge (k+1)^l$, which implies that

$$n > k^l \quad \text{if } d \le lk^{l-2}. \tag{12}$$

We begin with a lemma on Stirling's formula, upper bounds for $\pi(x)$ and $\vartheta(x) = \sum_{p\le x} \log p$ and a lower bound for the nth prime p_n, the proofs of which can be found in [5, p. 447] and [6, pp. 69, 71].

LEMMA 1. *For any integer $M > 1$, we have*

(i) $\log M! < \log\sqrt{2\pi} + \left(M + \frac{1}{2}\right)\log M - M + \dfrac{1}{12M}$,

(ii) $\log M! > \log\sqrt{2\pi} + \left(M + \frac{1}{2}\right)\log M - M$,

(iii) $\pi(M) < \dfrac{M}{\log M}\left(1 + \dfrac{3}{2\log M}\right)$,

(iv) $\vartheta(M) < (1.01624)M$,

(v) $p_M > M\log M$.

The next lemma deals with the distinctness property of a_i's and A_i's.

LEMMA 2. (a) *Let $l \geq 2$ and $d \leq lk^{l-2}$. Then the a_i for $1 \leq i \leq t$ and A_i for $1 \leq i \leq t$ are distinct.*

(b) *Let $l = 2$.*

(i) *If $k \geq 11380$ and $2^{\omega(d)}d^2 \leq (.00039)k(\log k)^2$, then the a_i for $1 \leq i \leq t$ are distinct.*

(ii) *If $d = 3$, then the number of distinct a_i's is at least $t, t-1, t-2$ according as $k = 3, 4 \leq k \leq 22$, $k \geq 23$.*

(iii) *If $d = 4$, then the a_i for $1 \leq i \leq t$ are distinct.*

(iv) *If $d = 5$, then the number of distinct a_i's is at least $t-2, t-3$ according as $4 \leq k \leq 38$, $k \geq 39$. Further, if $n > \frac{25}{4}k^2 - 15k + 9$, then the a_i for $1 \leq i \leq t$ are distinct.*

(v) *If $d = 6$, then the number of distinct a_i's is at least $t - 1$.*

Proof. (a) Let $a_i = a_j$ for $1 \leq i, j \leq t$ and $i \neq j$. We may assume without loss of generality that $n + d_i d > n + d_j d$. Then $x_i > x_j$ and

$$dk > d(d_i - d_j) = (n + d_i d) - (n + d_j d) = a_j(x_i^l - x_j^l) > l a_j x_j^{l-1}.$$

Thus we derive from (12) that

$$dk > l(a_j x_j^l)^{(l-1)/l} \geq l n^{(l-1)/l} > lk^{l-1},$$

which is a contradiction. The proof for the distinctness of the A_i's is similar.

(b) (i) Let $k \geq 11380$ and $2^{\omega(d)}d^2 \leq (.00039)k(\log k)^2$. By Lemma 2(a), we may assume that $d \geq 3$. By an argument of Shorey and Tijdeman [10, p. 315], we show that

$$n + (k-1)d > \frac{(.0001)k^3(\log k)^2}{2^{\omega(d)}}. \tag{13}$$

From $n + (k-1)d \geq (k+1)^2$ it follows that

$$n + d_\mu d \geq (k+1)^2/35 \quad \text{for } k/35 \leq d_\mu < k.$$

Let $T_1 = \{\mu \mid k/35 \leq d_\mu < k,\ X_\mu = 1\}$ and $T_2 = \{\mu \mid k/35 \leq d_\mu < k,$

$X_\mu \neq 1\}$. By an argument of Erdős [3, Lemma 2], we have

$$|T_1| \le \frac{k\log k}{\log\frac{(k+1)^2}{35}} + \pi(k),$$

which, by (3), (2) and Lemma 1(iii), implies that $|T_2| > (.2278)k$. For $\mu \in T_2$, we have $X_\mu > k$ and X_μ's are pairwise distinct. Further, we may assume that X_μ is prime for $\mu \in T_2$, otherwise, (13) follows. Then we can find a subset T_3 of T_2 such that

$$|T_3| \ge \frac{1}{35}(.2278)k$$

and by Lemma 1(v), we get for $\mu \in T_3$,

$$X_\mu \ge \frac{34}{35}(.2278)k\log\left(\frac{34}{35}(.2278)k\right),$$

i.e.,

$$X_\mu \ge (.1854)k\log k.$$

We argue as in [10, pp. 315–316] to conclude that for every A_μ with $\mu \in T_3$, there exist at most $2^{\omega(d)+1}$ i's belonging to T_3 with $A_i = A_\mu$. Thus there are at least $(.0032)k/2^{\omega(d)}$ distinct A_i's. Hence

$$n + (k-1)d \ge \frac{(.0032)(.1854)^2k^3(\log k)^2}{2^{\omega(d)}},$$

which implies (13).

Now we proceed to show that the a_i's, for $1 \le i \le t$ are distinct. Let $a_i = a_j$ for $1 \le i, j \le t$ with $i \neq j$. We assume without loss of generality that $x_i > x_j$. By (13), we have

$$\begin{aligned} kd &> a_ix_i^2 - a_jx_j^2 \ge a_j((x_j+1)^2 - x_j^2) > 2a_jx_j \ge 2(a_jx_j^2)^{1/2} \\ &> 2\left(\frac{(.0001)k^3(\log k)^2}{2^{\omega(d)}} - kd\right)^{1/2} \end{aligned}$$

which implies that

$$2^{\omega(d)}d^2\left(1 + \frac{4}{kd}\right) > (.0004)k(\log k)^2.$$

Since $k \ge 11380$ and $d \ge 3$, it follows that $2^{\omega(d)}d^2 > (.00039)k(\log k)^2$. This contradiction proves the distinctness of a_i.

For the proofs of (ii) to (v) we suppose $a_i = a_j$ for $1 \le i, j \le t$ and $i \neq j$. We assume without loss of generality that $n + d_id > n + d_jd$ and hence $x_i > x_j$. Let $x_i = x_j + h$ for some positive integer h. Then

$$\begin{aligned} (k-1)d &\ge (d_i - d_j)d = (n + d_id) - (n + d_jd) \\ &= a_j(x_i^2 - x_j^2) = a_j((x_j+h)^2 - x_j^2) \end{aligned} \tag{14}$$

$$= 2ha_jx_j + a_jh^2 = 2ha_j^{1/2}(a_jx_j^2)^{1/2} + a_jh^2$$
$$\geq 2ha_j^{1/2}n^{1/2} + a_jh^2.$$

(ii) Let $d = 3$. From $n+(k-1)3 \geq (k+1)^2$, it follows that $n \geq k^2-k+4$. We use this in (14) to get $h = 1$, $a_j \leq 2$. Since $h = 1$ the number of i with $a_i = a_j$ and $i \neq j$ is at most one. If $a_i = a_j = 2$, it follows from (14) that $k^2 - 22k - 7 \geq 0$, which implies that $k \geq 23$. Similarly, if $a_i = a_j = 1$, we get $k \geq 4$. The result follows.

(iii) Let $d = 4$. Since a_i for $1 \leq i \leq t$ are odd, it follows from $(d_i - d_j)4 = a_jh(2x_j + h)$ that h is even. We have $n \geq k^2 - 2k + 5$, which is used in (14) to give $h \leq 1$, a contradiction.

(iv) Let $d = 5$. We have $n \geq k^2 - 3k + 6$. We use this in (14) to get $h \leq 2$. We observe from (14) that for $h = 2$, $a_j = 1$ and for $h = 1$, $a_j \in \{1, 2, 3, 4, 6\}$, $a_i = a_j = 6$ holds only for $k \geq 39$. Further, it follows from (14) that when $h = 1$, we have $2x_j + 1 \equiv 0 \pmod 5$. Thus $x_j \equiv 2 \pmod 5$ implying that $n \equiv n + d_j5 = a_jx_j^2 \equiv -a_j \pmod 5$. Thus a_j belongs to $\{1, 6\}$ or $\{2\}$ or $\{3\}$ or $\{4\}$. Now, the first part of the assertion follows easily. The second part is an easy consequence of (14).

(v) Let $d = 6$. Here a_i for $1 \leq i \leq t$ are odd and h is even. Further, $n \geq k^2 - 4k + 7$ and it follows from (14) that $h = 2$, $a_j = 1$, which proves the result. ∎

As an immediate consequence of (i) of Lemma 2(b), we get

COROLLARY 3. *Let* $l = 2$, $k \geq 11380$ *and* $d \leq (.3)k^{1/3}$. *Then the* a_i *for* $1 \leq i \leq t$ *are distinct.*

In the next lemma, we improve (12) for $l \geq 3$ and $k \geq 9$.

LEMMA 3. *Let* $l \geq 3$, $k \geq 9$ *and* $d \leq lk^{l-2}$. *Then*

$$n > \begin{cases} \gamma(k,l)k^l & \text{if } d \text{ is odd,} \\ (2\gamma(k,l) - 1)k^l & \text{if } d \text{ is even,} \end{cases}$$

where $\gamma(k,l) = t - \pi(k) - k/l$.

Proof. By Lemma 2(a), we see that the A_i for $1 \leq i \leq t$ are distinct. Further, from (11) and (12) we observe that

$$|\{A_i \mid X_i = 1,\ 1 \leq i \leq t\}| \leq \frac{k \log k}{\log n} + \pi(k) \leq \frac{k}{l} + \pi(k).$$

Thus the set $\{A_i \mid X_i \neq 1,\ 1 \leq i \leq t\}$ has cardinality $\geq \gamma(k,l)$. Also, for every A_i in this set, $X_i \geq k + 1$. We note that A_i's are odd if d is even. Hence from the distinctness of A_i's it follows that

$$n + (k-1)d \geq \begin{cases} \gamma(k,l)(k+1)^l & \text{if } d \text{ is odd,} \\ (2\gamma(k,l) - 1)(k+1)^l & \text{if } d \text{ is even.} \end{cases}$$

Using (3) and Lemma 1(iii), we check that $\gamma(k,l) \geq 1$ for $k \geq 9$. The result now follows since $d \leq lk^{l-2}$. ■

LEMMA 4. *Let $l \geq 3$, and $k \geq 9$ whenever $l = 3$, $d > 1$. Suppose l' is a positive integer satisfying*

$$l' \leq \begin{cases} l-1 & \text{if } d = 1 \text{ or } l = 3, \\ l-2 & \text{if } d > 1 \text{ and } l \geq 5, \end{cases}$$

and

$$d \leq \begin{cases} \frac{3}{2}(\gamma(k,3))^{1/3} - \frac{1}{2k} & \text{if } l = 3,\ d \text{ odd}, \\ \frac{3}{2}(2\gamma(k,3)-1)^{1/3} - \frac{1}{2k} & \text{if } l = 3,\ d \text{ even}, \\ k^{l-l'-1} & \text{if } l \geq 5. \end{cases}$$

Then the ratio of any two products $a_{i_1} \ldots a_{i_{l'}}$ and $a_{j_1} \ldots a_{j_{l'}}$ corresponding to distinct l'-tuples $(i_1, \ldots, i_{l'})$ and $(j_1, \ldots, j_{l'})$ with $1 \leq i_1 \leq \ldots \leq i_{l'} \leq t$ and $1 \leq j_1 \leq \ldots \leq j_{l'} \leq t$ is not an lth power of a rational number.

Proof. The assumption on d implies that $d \leq lk^{l-2}$. Thus (12) and Lemma 3 are valid. Let $1 \leq i_1 \leq \ldots \leq i_{l'} \leq t$ and $1 \leq j_1 \leq \ldots \leq j_{l'} \leq t$ with $(i_1, \ldots, i_{l'}) \neq (j_1, \ldots, j_{l'})$ and

$$a_{i_1} \ldots a_{i_{l'}} = a_{j_1} \ldots a_{j_{l'}} (t_1/t_2)^l$$

where t_1 and t_2 are positive integers with $\gcd(t_1, t_2) = 1$. We put

$$A = \frac{a_{i_1} \ldots a_{i_{l'}}}{t_1^l} = \frac{a_{j_1} \ldots a_{j_{l'}}}{t_2^l}. \tag{15}$$

We note that A is a positive integer. First, we show that

$$(n + d_{i_1} d) \ldots (n + d_{i_{l'}} d) \neq (n + d_{j_1} d) \ldots (n + d_{j_{l'}} d). \tag{16}$$

Suppose (16) does not hold. Then we cancel any term on the left hand side which equals some term on the right hand side. There remains at least one term on the left hand side, say, $n + d_{i_1} d$. We note that for $1 \leq r \leq l'$, $\gcd(n + d_{i_1} d, n + d_{j_r} d) \leq k$ since $\gcd(n, d) = 1$. Thus

$$n + d_{i_1} d \leq \gcd(n + d_{i_1} d, n + d_{j_1} d) \ldots \gcd(n + d_{i_1} d, n + d_{j_{l'}} d) \leq k^{l'}$$

which, by (12), gives a contradiction. Thus (16) holds.

We may assume without loss of generality that

$$(n + d_{i_1} d) \ldots (n + d_{i_{l'}} d) > (n + d_{j_1} d) \ldots (n + d_{j_{l'}} d),$$

i.e.,

$$a_{i_1} \ldots a_{i_{l'}} (x_{i_1} \ldots x_{i_{l'}})^l > a_{j_1} \ldots a_{j_{l'}} (x_{j_1} \ldots x_{j_{l'}})^l.$$

Hence by (15), we get $Ax^l > Ay^l$ where $x = t_1 x_{i_1} \ldots x_{i_{l'}}$ and $y = t_2 x_{j_1} \ldots \ldots x_{j_{l'}}$. So $x > y$. Thus

$$\begin{aligned}(n + d_{i_1} d) \ldots (n + d_{i_{l'}} d) &- (n + d_{j_1} d) \ldots (n + d_{j_{l'}} d) \\ &\geq A((y+1)^l - y^l) > lAy^{l-1} > l(Ay^l)^{(l-1)/l} > ln^{(l-1)l'/l}.\end{aligned}$$

On the other hand, using (12), $d \le k^{l-l'-1}$ if $l \ge 5$ and $d \le \frac{3}{2}k^{1/3}$ if $l = 3$, we get

$$\begin{aligned}(n+d_{i_1}d)\dots(n+d_{i_{l'}}d) &- (n+d_{j_1}d)\dots(n+d_{j_{l'}}d)\\ &< (n+kd)^{l'} - n^{l'} = l'n^{l'-1}kd + \binom{l'}{2}n^{l'-2}(kd)^2 + \dots\\ &\le ln^{l'-1}kd - n^{l'-1}kd + \binom{l'}{2}n^{l'-2}(kd)^2\Big\{1 + \frac{l'kd}{3n} + \dots\Big\}\\ &< ln^{l'-1}kd - n^{l'-1}kd + l'(l'-1)n^{l'-2}(kd)^2 < ln^{l'-1}kd,\end{aligned}$$

which, together with the lower bound given above, implies that $n^{(l-l')/l} < kd$. When $l = 3$ from the upper bound and the lower bound inequalities we in fact get $k^2d^2 + 2nkd > 3n^{4/3}$ if $l' = 2$ and $kd > 3n^{2/3}$ if $l' = 1$. Now we use (12) if either $d = 1$ or $l \ge 5$, and Lemma 3 if $d > 1$, $l = 3$, to get a contradiction. ∎

From Lemma 4 it is clear that the a_ia_j for $1 \le i \le j \le t$ are all distinct if either $l \ge 3, d = 1$ or

$$d \le \begin{cases} (1.4)(\gamma(k,3))^{1/3} & \text{if } l = 3 \text{ and } k \ge 9,\\ k^{l-3} & \text{if } l \ge 5.\end{cases}$$

This restriction on d is relaxed in the following lemma.

LEMMA 5. *Let $l \ge 3$ and $k \ge 9$ whenever $d > 1$. Assume that*

$$d \le \frac{7}{5}\cdot 4^{1/l}(\gamma(k,l))^{1-2/l}k^{l-3}.$$

Then the a_ia_j for $1 \le i, j \le t$ are distinct.

Proof. We observe that d, as given in the lemma, implies that $d \le lk^{l-2}$. Hence by Lemma 2(a), the a_i for $1 \le i \le t$ are distinct. Suppose $a_ia_j = a_ra_s$ for $(i,j) \ne (r,s)$ with $1 \le i,j \le t, 1 \le r,s \le t$ and $a_i \le a_j, a_r \le a_s$. Then we observe that $a_ia_j = a_ra_s \ge 4$. As shown in Lemma 4, we have

$$(n+d_id)(n+d_jd) \ne (n+d_rd)(n+d_sd).$$

We may suppose that $(n+d_id)(n+d_jd) > (n+d_rd)(n+d_sd)$. Thus $x_ix_j > x_rx_s$. Hence

$$\begin{aligned}2knd + k^2d^2 &> (n+d_id)(n+d_jd) - (n+d_rd)(n+d_sd)\\ &> la_ra_s(x_rx_s)^{l-1} = l(a_ra_s)^{1/l}(a_rx_r^la_sx_s^l)^{(l-1)/l}\\ &> l(a_ra_s)^{1/l}n^{2(l-1)/l} > l4^{1/l}n^{2(l-1)/l}.\end{aligned}$$

Thus we have

$$k^2d^2 > l4^{1/l}n^{2(l-1)/l}\left(1 - \frac{2kd}{l4^{1/l}n^{1-2/l}}\right)$$
$$> l4^{1/l}n^{2(l-1)/l}\left(1 - \frac{2kd}{3 \cdot 4^{1/l}n^{1-2/l}}\right).$$

For $d = 1$, we use (12) to get a contradiction. Thus we may assume that $d > 1$. Using Lemma 3 and our assumption on d we get

$$k^2d^2 > \frac{l}{15}4^{1/l}(\gamma(k,l))^{2-2/l}k^{2l-2}$$

in which we apply the bound for d and $l \geq 3$ to obtain

$$\frac{735}{25} > \frac{l}{4^{1/l}}(\gamma(k,l))^{2/l}k^2 > 1.8898k^{2+2/l}\left(\frac{t}{k} - \frac{\pi(k)}{k} - \frac{1}{3}\right)^{2/3}.$$

We use $t \geq t_0$, (3), the exact value of $\pi(k)$ for $k \leq 20$ and the upper bound for $\pi(k)$ from Lemma 1(iii) for $k > 20$ to check that

$$\left(\frac{t}{k} - \frac{\pi(k)}{k} - \frac{1}{3}\right)^{2/3} > .2311.$$

Thus we have $k^{2+2/l} \leq 68$. This is a contradiction since $k \geq 9$. This proves the lemma. ■

We need the following graph theoretic lemma from [3].

LEMMA 6. *Suppose G is a bipartite graph of s white vertices and r black vertices which contains no rectangles. Then the number of edges is at most* $s + \binom{r}{2}$.

We use the above lemma as follows. Let $x \geq 1$ be an arbitrary real number. We construct two sets U and V of positive integers $\leq x$ such that all positive integers $\leq x$ can be written as uv with $u \in U$ and $v \in V$. We take (U, V) to be the bipartite graph G with black vertices as elements of U and white vertices as elements of V. Let $\{c_1, \ldots, c_h\}$ be a set of positive integers $\leq x$ with the property that the c_ic_j for $1 \leq i, j \leq h$ are distinct. We say that there is an edge between an element $u \in U$ and $v \in V$ if $uv = c_i$ for $1 \leq i \leq h$. By the distinctness of c_ic_j's it follows that G has no rectangle. Thus it follows from Lemma 6 that $h \leq |V| + \binom{|U|}{2}$.

Now we explain the construction of the sets U and V. Let $2 = p_1 < p_2 < \ldots$ be the sequence of all primes. More generally, let $p'_1 < p'_2 < \ldots$ be the sequence of all primes coprime to d. Since $\gcd(n, d) = 1$, we observe that $a_1, \ldots, a_t$ given by (9) are composed of primes $p'_1, p'_2, \ldots$ For positive integers m and T, we denote by $U = U(m, T)$ the set of integers $\leq T$ which are composed of $p_1, \ldots, p_m$. We observe that $1 \in U$. Further, we understand that an empty product equals 1. We construct a set V as follows. With every prime $p_i, 1 \le i \le m$, we associate an integer $r_i(T)$ such that $p_i r_i(T)$ is the smallest integer $> T$ with $P(p_i r_i(T)) = p_i$. We put

$$r_{m+1}(T) = 1/p_{m+1}, \qquad V_i = \{p_i w \mid w \le x/r_i(T),\ p(p_i w) = p_i\}$$

for $1 \le i \le m$,

$$V_{m+1} = \{w \mid w \le x,\ p(w) = 1 \text{ or } p(w) \ge p_{m+1}\} \quad \text{and} \quad V = \bigcup_{i=1}^{m+1} V_i.$$

Then we see that for $1 \le i \le m+1$,

$$(17) \qquad \begin{aligned} |V_i| &= \left|\left\{w \;\middle|\; w \le \frac{x}{p_i r_i(T)},\ \gcd(w, p_1 \ldots p_{i-1}) = 1\right\}\right| \\ &= \frac{\varphi(p_1 \ldots p_{i-1})}{p_1 \ldots p_{i-1}} \left[\frac{x}{p_i r_i(T)}\right] + E_i \end{aligned}$$

where E_i's are error terms and φ is the Euler totient function. Since $V_1, \ldots, V_{m+1}$ are pairwise disjoint, we have

$$(18) \qquad |V| = \sum_{i=1}^{m+1} \left(\frac{\varphi(p_1 \ldots p_{i-1})}{p_1 \ldots p_{i-1}} \left[\frac{x}{p_i r_i(T)}\right] + E_i\right).$$

We observe that if $X = p_1 \ldots p_{i-1} X' + z$ where $X = [x/(p_i r_i(T))]$ and $0 \le z < p_1 \ldots p_{i-1}$ then $|V_i| = \varphi(p_1 \ldots p_{i-1}) X' + \varrho(z)$ where $\varrho(z)$ is the number of integers $\le z$ and coprime to $p_1 \ldots p_{i-1}$. Hence

$$|V_i| = \frac{\varphi(p_1 \ldots p_{i-1})}{p_1 \ldots p_{i-1}} X + \varrho(z) - \frac{\varphi(p_1 \ldots p_{i-1})}{p_1 \ldots p_{i-1}} z.$$

Thus we see from (17) that for $1 \le i \le m+1$,

$$E_i \le \frac{1}{p_1 \ldots p_{i-1}} \max\{p_1 \ldots p_{i-1} \varrho(z) - \varphi(p_1 \ldots p_{i-1}) z\}$$

where the maximum is taken over all $0 \le z < p_1 \ldots p_{i-1}$ with $\gcd(z, p_1 \ldots p_{i-1}) = 1$. To find this maximum, we first enumerate all the integers $< p_1 \ldots p_{i-1}$ which are coprime to $p_1 \ldots p_{i-1}$. This is done by the method of sieving. Given an integer $z < p_1 \ldots p_{i-1}$, we test if z is divisible by p_j for $1 \le j \le i-1$. If at any stage, the test is positive, then z is deleted. If the test fails for all $j, 1 \le j \le i-1$, then z is retained. Thus we obtain integers $z_1 < z_2 < \ldots < z_{\delta_i}$ where $\delta_i = \varphi(p_1 \ldots p_{i-1})$ which are coprime to $p_1 \ldots p_{i-1}$. Then we compute $p_1 \ldots p_{i-1} \mu - \varphi(p_1 \ldots p_{i-1}) z_\mu$ for $1 \le \mu \le \delta_i$ and take the maximum which depends only on i. Bounds for $E_1, \ldots, E_6$ already appear in [3]. Bounds for $E_7, \ldots, E_{11}$ have been calculated using DEC AXP 3000 / 800 OSF / 1V3.0 at the Tata Institute of Fundamental Research. The times taken for the calculation of E_{10} and E_{11} are about 4 minutes and about 2 hours 8 minutes respectively, while other calculations, put together, took less than a minute. We record in the following lemma the bounds for E_i's which may be of independent interest.

LEMMA 7.

$$E_1 \le 0, \quad E_2 \le \frac{1}{2}, \quad E_3 \le \frac{2}{3}, \quad E_4 \le \frac{14}{15}, \quad E_5 \le \frac{53}{35}, \quad E_6 \le \frac{194}{77},$$
$$E_7 \le \frac{3551}{1001}, \quad E_8 \le \frac{92552}{17017},$$
$$E_9 \le \frac{2799708}{323323}, \quad E_{10} \le \frac{9747144}{676039}, \quad E_{11} \le \frac{58571113}{2800733}.$$

In the next lemma, we construct several sets U and V as described above by choosing m and T suitably which enable us to obtain good lower bounds for a'_h which sharpen considerably the ones given in Erdős and Selfridge [3, (15), (16)].

LEMMA 8. *Let $l \ge 3$ and $k \ge 9$ whenever $d > 1$. Assume that $d \le \frac{7}{5} \cdot 4^{1/l}(\gamma(k,l))^{1-2/l}k^{l-3}$. Then $a'_h \ge \mu(h-\nu)$ where (μ,ν) equals*

(i) $(1, 0)$ *for* $h \le 16$,
(ii) $(1.7777, 7)$ *for* $17 \le h \le 57$,
(iii) $(2.2153, 17)$ *for* $58 \le h \le 177$,
(iv) $(2.5484, 38)$ *for* $178 \le h \le 281$,
(v) $(2.9205, 69)$ *for* $282 \le h \le 800$,
(vi) $(3.32, 157)$ *for* $801 \le h \le 1335$,
(vii) $(3.565, 238)$ *for* $1336 \le h \le 1790$,
(viii) $(4.1135, 445)$ *for* $1791 \le h \le 2617$,
(ix) $(4.2444, 512)$ *for* $2618 \le h \le 3786$,
(x) $(4.3878, 619)$ *for* $3787 \le h \le 5711$,
(xi) $(4.4964, 742)$ *for* $5712 \le h \le 7491$,
(xii) $(4.6189, 921)$ *for* $7492 \le h \le 9183$,
(xiii) $(4.6425, 963)$ *for* $h \ge 9184$.

Proof. By Lemma 2(a), elements of S are distinct. Hence $t' = t$ and $a'_h \ge h$ is valid for $1 \le h \le t$. (See the first line in Table 1.) By Lemma 5, we find that $a'_i a'_j$ for $1 \le i, j \le t$ are distinct. Let $x \ge 1$ be an arbitrary real number. As explained earlier, we can use Lemma 6 to get an upper bound for the number of a'_h which are $\le x$.

We illustrate below the construction of the sets U and V which yields (iii). We take U to be the set of all integers ≤ 8 and composed of only 2 and 3. Thus $m = 2$, $T = 8$, $U = U(2, 8)$ and $|U| = 6$. Next, $r_1(t) = 8$, $r_2(t) = 3$ and $r_3(t) = 1/5$. Further, we have $V = V_1 \cup V_2 \cup V_3$ with $V_1 = \{2w \mid 2w \le x/8\}$, $V_2 = \{3w \mid 3w \le x/3,\ p(3w) = 3\}$ and $V_3 = \{w \mid w \le x,\ p(w) = 1$ or

$p(w) \geq 5\}$. From (18) and Lemma 7, we get

$$|V| \leq \left\{\frac{1}{16} + \frac{1}{18} + \frac{1}{3}\right\}x + \frac{7}{6} < (.4514)x + 2.$$

Now, we show that every integer $\leq x$ is representable as uv with $u \in U$ and $v \in V$. Let $x' = 2^a 3^b x'' \leq x$ with $(x'', 6) = 1$. We give below the value of u in all possible cases. The value of v is given by x'/u. We have for $a \geq 3$, $u = 8$; $a = 2$, $u = 4$; $a = 1$, $u = 6$ if $b \geq 1$; $a = 1$, $u = 2$ if $b = 0$; $a = 0$, $u = 3$ if $b \geq 1$; $a = 0$, $u = 1$ if $b = 0$.

Now, we use Lemma 6 to derive that the number of a'_h which are less than or equal to x is bounded by $(.4514)x + 17$. Taking $x = a'_h$, we get $a'_h \geq 2.2153(h - 17)$. The proof of other values of (μ, ν) are similar. We give below in Table 1 the values of m and T which are used to obtain the values of (μ, ν) listed in (i) to (xiii) of the lemma. Also, we give the cardinalities of the respective sets U and V. ∎

Table 1

Assertion No.	m	T	$r = \|U\|$	$s = \|V\|$	μ	ν	Least value of h
(i)	–	–	–	–	1	0	1
(ii)	1	8	4	$.5625x{+}1$	1.7777	7	17
(iii)	2	8	6	$.4514x{+}2$	2.2153	17	58
(iv)	2	16	9	$.3924x{+}2$	2.5484	38	178
(v)	3	16	12	$.3424x{+}3$	2.9205	69	282
(vi)	4	24	18	$.3012x{+}4$	3.32	157	801
(vii)	5	27	22	$.2805x{+}7$	3.565	238	1336
(viii)	6	36	30	$.2431x{+}10$	4.1135	445	1791
(ix)	7	36	32	$.2356x{+}16$	4.2444	512	2618
(x)	8	39	35	$.2279x{+}24$	4.3878	619	3787
(xi)	9	42	38	$.2224x{+}39$	4.4964	742	5712
(xii)	10	46	42	$.2165x{+}60$	4.6189	921	7492
(xiii)	10	48	43	$.2154x{+}60$	4.6425	963	9184

Let $m \geq 1$ be an integer. For $d = 1$, we define $A_m = \{a'_h \mid P(a'_h) \leq p_m\}$ and $f(k, m) = |A_m|$. Since $t' = t$, we have

$$f(k, m) \geq t - \sum_{h \geq m+1} \left(\left[\frac{k}{p_h}\right] + \varepsilon_h\right) := f_0(k, m) \tag{19}$$

where $\varepsilon_h = 0$ if $p_h > k$ and for $p_h \leq k$, $\varepsilon_h = 0$ or 1 according as $p_h \mid k$ or not for $h \geq m + 1$. Further, we define $B_m = \{a'_h \mid P(a'_h) \leq p'_m\}$ and $g(k, m) = |B_m|$. Then

$$g(k, m) \geq t' - \sum_{h \geq m+1} \left(\left[\frac{k}{p'_h}\right] + \varepsilon'_h\right) := g_0(k, m) \tag{20}$$

where $\varepsilon'_h = 0$ if $p'_h > k$ and for $p'_h \le k$, $\varepsilon'_h = 0$ or 1 according as $p'_h \mid k$ or not for $h \ge m+1$. It is easily seen that $g_0(k,m) \ge f_0(k,m)$ whenever $t' = t$. Suppose d is divisible by either 2 or 3. Then $p'_i \ge p_{i+1}$ for $i \ge 2$. Thus for $m \ge 2$ and $t' = t$ we get

$$g_0(k, m-1) \ge f_0(k,m) \quad \text{if } 2 \mid d \text{ or } 3 \mid d. \tag{21}$$

As k increases, $f_0(k,m)$ and $g_0(k,m)$ become ≤ 0 and hence useless. For these values of k, we proceed as follows. Let $p_1 < \ldots < p_{m_1} \le k^{3/10} < p_{m_1+1} < \ldots < p_{m_1+m_2} \le \sqrt{k}$. For $d = 1$, we define $A = \{a'_i \mid P(a'_i) \le \sqrt{k}$ and a'_i is divisible by at most one of the primes p_{m_1+j} for $1 \le j \le m_2$ which divides a'_i only to the first power$\}$ and $F(k) = |A|$. Then we note that (see [3, p. 298])

$$\begin{aligned} F(k) \ge t - \sum_{\sqrt{k}<p\le k} \left(\left[\frac{k}{p}\right] + 1\right) - \left\{\frac{k}{2}\left(\sum_{i=1}^{m_2} \frac{1}{p^2_{m_1+i}} + \left(\sum_{i=1}^{m_2} \frac{1}{p_{m_1+i}}\right)^2\right)\right\} \\ - \binom{m_2+1}{2} \\ := F_0(k, m_1, m_2). \end{aligned} \tag{22}$$

For $d > 1$, we let $p'_1 < \ldots < p'_{m'_1} \le k^{3/10} < p'_{m'_1+1} < \ldots < p'_{m'_1+m'_2} \le \sqrt{k}$ be all the primes $\le \sqrt{k}$ and coprime to d. We observe that $m'_1 \le m_1$ and $m'_2 \le m_2$. Further, for $m_1 \ge 2, m'_1 \le m_1 - 1$ if $2 \mid d$ or $3 \mid d$. We define $B = \{a'_i \mid P(a'_i) \le \sqrt{k}$ and a'_i is divisible by at most one of the primes $p'_{m'_1+j}$ for $1 \le j \le m'_2$ which divides a'_i only to the first power$\}$ and $G(k) = |B|$. Then as before, we have

$$G(k) \ge G_0(k, m'_1, m'_2)$$

where $G_0(k, m'_1, m'_2)$ is got from the expression for $F_0(k, m_1, m_2)$ by replacing t, m_1, m_2, p_{m_1+i} by $t', m'_1, m'_2, p'_{m'_1+i}$, respectively. When $t' = t$, we have

$$G_0(k, m'_1, m'_2) \ge F_0(k, m_1, m_2). \tag{23}$$

Following the argument of [3], we have

Lemma 9. *Suppose the hypothesis of Lemma* 4 *holds. Then*

(i) *For* $d = 1$, $l \ge 3$, $m \ge 1$, $f(k,m) \ge 1$ *and* $F(k) \ge 1$, *we have*

$$\binom{f(k,m) + l - 2}{l-1} \le l^m \tag{24}$$

and

$$\binom{F(k) + l - 2}{l-1} \le l^{m_1} \binom{l + m_2 - 1}{l-1}. \tag{25}$$

(ii) *For* $d > 1$, $l \geq 3$, $m' \geq 1$, $g(k, m') \geq 1$ *and* $G(k) \geq 1$, *we have*

$$\binom{g(k,m') + l' - 1}{l'} \leq l^{m'} \tag{26}$$

and

$$\binom{G(k) + l' - 1}{l'} \leq l^{m'_1} \binom{l' + m'_2}{l'}. \tag{27}$$

The next result was quoted by Erdős in [2]. This result was proved by A. Meyl in 1878. We refer to [1, p. 25] for further details. This result is independent of the assumptions (7) and (8).

LEMMA 10. *The only solutions of the equation*

$$n(n+1)(n+2) = 6y^2$$

in integers $n > 1$, $y > 1$ *are* $n = 2$, $y = 2$; $n = 48$, $y = 140$.

4. An algorithm. In this section we provide an algorithm to test that (7) does not hold whenever (8) holds.

ALGORITHM. *Let* c, d, k, l *be given with* $c < k^l$ *and* $d < (k+1)^l/(k-1)$.

Step 1. *Find all primes* $q_1, \ldots, q_\theta, q_{\theta+1}, \ldots, q_{\theta+\eta}$ *which are coprime to* d *and such that* $q_1 < \ldots < q_\theta \leq k < q_{\theta+1} < \ldots < q_{\theta+\eta}$ *and* $q_{\theta+i}^l < ck^l$ *for* $1 \leq i \leq \eta$.

Step 2. *For* $1 \leq h \leq \eta$, *form the sets*

$D_h = \{q_1^{\beta_1} \ldots q_\theta^{\beta_\theta} q_{\theta+h}^l \mid q_1^{\beta_1} \ldots q_\theta^{\beta_\theta} q_{\theta+h}^l \leq ck^l$ *for integers* $\beta_i \geq 0,\ 1 \leq i \leq \theta\}$ *and let* $D = \bigcup_{h=1}^{\eta} D_h$.

Step 3. *For every* $q \in D$, *we find some* $j = j(q)$ *with* $1 \leq j \leq k-1$ *such that* $P(q + jd)$ *and* $P(q - (k-j)d)$ *are* $> q_{\theta+\eta}$.

In Step 3 we observe that $q - (k-j)d$ is positive since $q \geq (k+1)^l$ and $d < (k+1)^l/(k-1)$. The above Algorithm yields the following result.

LEMMA 11. *Let* c, d, k, l, n *and* t *be given such that* $t = k, n + (k-1)d \leq ck^l, c < k^l$ *and* $d < (k+1)^l/(k-1)$. *If* (8) *and Step* 3 *hold*, *then* (7) *does not hold.*

Proof. For any $p > k$, we observe from (8) that

$$\mathrm{ord}_p(n(n+d)\ldots(n+(k-1)d)) = 0 \text{ or } l$$

since $c < k^l$. Further, we note that if $q_{\theta+h}$ with $1 \leq h \leq \eta$ divides a term in the product $n(n+d)\ldots(n+(k-1)d)$, then no other $q_{\theta+h'}$ for $h' \neq h$, $1 \leq h' \leq \eta$ divides the same term. Thus every term $n + id$ is of the form $q' q_{\theta+h}^l$ or q' where $P(q') \leq q_\theta$. Thus

$$P(n(n+d)\ldots(n+(k-1)d)) \leq q_{\theta+\eta}.$$

Suppose $n + id = q$ for some i with $0 \le i \le k-1$ and $q \in D$. Then $n+(i+j)d = q+jd$ is a term in the product $n(n+d)\dots(n+(k-1)d)$ if $i+j \le k-1$. Therefore $P(n(n+d)\dots(n+(k-1)d)) > q_{\theta+\eta}$ if $i+j \le k-1$. This is a contradiction. Let $i+j > k-1$. Then $n+(i+j-k)d = q-(k-j)d$. Since $0 \le i+j-k \le k-2$, we see that $n+(i+j-k)d$ is a term in the product $n(n+d)\dots(n+(k-1)d)$. Therefore $P(n(n+d)\dots(n+(k-1)d)) > q_{\theta+\eta}$, which is a contradiction. Hence $n+id \notin D$ for $0 \le i \le k-1$. This implies that $P(n(n+d)\dots(n+(k-1)d)) \le q_\theta \le k$, which contradicts (7). ∎

5. Proof of Theorems 1 and 2 for $l = 2$. We assume that (7) and (8) hold and we arrive at a contradiction if either the assumptions of Theorem 1 or of Theorem 2 hold. Thus Lemma 2 and Corollary 3 are valid and we conclude that the a'_i for $1 \le i \le t$ are distinct whenever $d \in \{1,2,4\}$ or $d \le (.3)k^{1/3}$ with $k \ge 11380$. Further, they are square free. We observe that out of 36 consecutive integers there are at most 24 square free integers. Writing the hth square free integer, say s_h, as $s_h = 36f_1 + f_2$ with $0 \le f_2 < 36$, we find that

$$h \le 24f_1 + \min(f_2, 24) \le \frac{2}{3}(s_h - f_2) + \min(f_2, 24).$$

Thus $s_h \ge \frac{3}{2}(h-8)$. Hence for $t \ge 9$,

$$\prod_{i=1}^{t} a'_i \ge (1\cdot 2\cdot 3\cdot 5\cdot 6\cdot 7\cdot 10\cdot 11)\left(\frac{3}{2}\right)^{t-8}(t-8)! \tag{28}$$

$$= 138600\left(\frac{3}{2}\right)^{t-8}(t-8)!.$$

By following the argument of [10, p. 323], we also have

$$\prod_{i=1}^{t} a'_i \le 2^{\frac{8}{3}-\frac{2k}{3}+\frac{2\log k}{\log 2}}\, 3^{\frac{9}{4}-\frac{k}{4}+\frac{2\log k}{\log 3}}\,(k-1)!\prod_{p\le k} p.$$

From Lemma 1(iv), we have $\prod_{p\le k} p \le (2.78)^k$, which implies that

$$\prod_{i=1}^{t} a'_i \le (75.23)k^4(k-1)!(1.34)^k. \tag{29}$$

Let $k \ge 870$. Since $t \ge k - \alpha(k)$, we deduce from (28) and (29) that

$$(71.88)(1.119)^k \le k^{\alpha(k)+11}(1.5)^{\alpha(k)}. \tag{30}$$

Since $\alpha(k) \le (.0156)k/\log k$, by taking the kth root on both sides of (30), we find that (30) is not satisfied. Let $680 \le k \le 869$. Then $t \ge k-1$ and we see from (28) and (29) that $(47.92)(1.119)^k \le k^{12}$, which is not possible. Thus we may assume that $k < 680$.

First, we consider the case $d = 1$ and $k < 680$. Since the a'_i are distinct and square free, we have $f(k, m) \leq 2^m$ for all m. Thus if $f_0(k, m) \geq 2^m + 1$ for some m, we get a contradiction by (19). We check using (19) with $t = k$ for $k \leq 24, t = k - 1$ for $25 \leq k < 680$ that

$$(31) \quad \begin{cases} f_0(k, 2) \geq 5 \text{ for } 9 \leq k \leq 22, \ f_0(k, 3) \geq 9 \text{ for } 23 \leq k \leq 78, \\ f_0(k, 4) \geq 17 \text{ for } 79 \leq k \leq 276, \ f_0(k, 5) \geq 33 \text{ for } 277 \leq k \leq 493, \\ f_0(k, 6) \geq 65 \text{ for } 494 \leq k < 680. \end{cases}$$

Here and at many other places checkings were done using PARI-GP. We are left with $4 \leq k \leq 8$. Then $t = k$. We use repeatedly the following two facts without mention to deal with these values of k. The product of four consecutive integers is never a square (see [3, p. 300]). There are at most four terms from $\{n + d_i \mid 1 \leq i \leq t\}$ with a'_i composed of only 2 and 3, and they must belong to $\{y_1^2, 2y_2^2, 3y_3^2, 6y_4^2\}$ for some positive integers y_1, y_2, y_3 and y_4, since a'_i are distinct and square free. Thus the product of the four terms is a square. We observe that $k \neq 4$. Let $k = 5$. Then $P(a'_i) \leq 5$. Here we may assume that $5 \nmid n$ and $5 \nmid (n+4)$. Suppose $5 \mid (n+2)$. Then $n(n+1)(n+3)(n+4) = X_1^2$ for some positive integer X_1. Thus $\left(n^2 + 4n + \frac{3}{2}\right)^2 - \frac{9}{4} = X_1^2$. This is impossible. Let $5 \mid (n + 1)$. Then $n \equiv 4 \pmod 5$. Hence $n = y_1^2$ or $6y_4^2$. Let $n = y_1^2$. Then $n + 2 = 6y_4^2, n + 3 = 2y_2^2$ or $3y_3^2$, which is impossible since $n+2$ and $n+3$ are coprime. Let $n = 6y_4^2$. Then $n+2 = y_1^2, n+3 = 3y_3^2$ and $n + 4 = 2y_2^2$. This means $(n + 2)(n + 3)(n + 4) = 6X_2^2$ for some positive integer X_2, which is not possible by Lemma 10. Let $5 \mid (n + 3)$. Arguing as before, we have $n = 2y_2^2, n + 1 = 3y_3^2, n + 2 = y_1^2, n + 4 = 6y_4^2$ implying $n(n + 1)(n + 2) = 6X_3^2$ for some positive integer X_3, which by Lemma 10 implies that $n = 2$. In this case $P(n(n + 1)(n + 2)(n + 3)(n + 4)) = 5$, contradicting our assumption (7).

Thus $k \neq 5$. For $k = 6$, we observe that 5 divides n and $n + 5$. But this means $(n + 1)(n + 2)(n + 3)(n + 4)$ is a square, which is impossible. Let $k = 7$. Then we observe that there exist distinct i_1, i_2, and i_3 between 0 and 6 such that $7 \mid (n + i_1),\ 5 \mid (n + i_2)$ and $5 \mid (n + i_3)$. We consider the possibility $7 \mid (n + 1),\ 5 \mid n,\ 5 \mid (n + 5)$. Then $n \equiv 6 \pmod 7$. Therefore $\{n + 4, n + 6\} = \{3y_3^2, 6y_4^2\}$, which is impossible. The other possibilities can be excluded similarly. Let $k = 8$. Then we derive that $7 \mid n,\ 7 \mid (n + 7)$ and $5 \mid (n+1),\ 5 \mid (n+6)$. Consequently, $(n+2)(n+3)(n+4)(n+5)$ is a square, which is not possible.

Let $d \in \{2, 4\}$ and $k < 680$. Then the a'_i are odd and square free integers. Consequently, we derive that $k \geq 9$. We observe from (21) and (20) that for $m \geq 2,\ f_0(k, m) \leq g_0(k, m - 1) \leq g(k, m - 1) \leq 2^{m-1}$, which is not possible by (31).

Let $d \in \{3, 6\}$. By (ii) and (v) of Lemma 2(b), there are at least $t - 2$ distinct a'_i. Further, they are square free integers. We proceed as at the beginning of this section with t replaced by $t-2$ to obtain $k < 900$. Since $t' \geq t-2$ we deduce from (20) and (19) for $m \geq 2$ that $f_0(k,m) - 2 \leq g_0(k,m-1) \leq g(k,m-1) \leq 2^{m-1}$, which, together with (31), implies that $3 \leq k \leq 8$ and $680 \leq k < 900$. We consider $680 \leq k < 900$. We check that $f_0(k,6) \geq 35$, which is sufficient to get a contradiction. Let $4 \leq k \leq 8$. By (ii) and (v) of Lemma 2(b), there are at least $t-1$ distinct a_i'. Hence the number of a_i' composed only of p_1' is ≥ 3 while at most two such a_i' are possible. If $k = 3$, $d = 3$, all the three a_i' are distinct and composed of only the prime 2, which is not possible. If $k = 3$, $d = 6$, by (v) of Lemma 2(b), at least two a_i' are distinct. This is not possible since $P(a_i') \leq 3$ and $\gcd(a_i', 6) = 1$.

Let $d = 5$. By (iv) of Lemma 2(b), there are at least $t-3$ distinct, square free a_i'. The argument at the beginning of this section with t replaced by $t-3$ yields $k < 1000$. From (20) and (19) we observe that for $m \geq 3$, $f_0(k,m) - 3 \leq g_0(k,m-1) \leq g(k,m-1) \leq 2^{m-1}$ and hence by (31), we have $4 \leq k \leq 22$ and $680 \leq k < 1000$. We check that $f_0(k,6) \geq 36$ for $680 \leq k < 1000$, which is sufficient to get a contradiction. Let $4 \leq k \leq 22$. The number of distinct a_i' is at least $t-2$. We observe that the number of a_i' composed of p_1' and p_2' is at least 5 for $9 \leq k \leq 22$ while this number cannot exceed 4. Thus we may assume that $4 \leq k \leq 8$. Suppose $n > \frac{25}{4}k^2 - 15k + 9$. Then by (iv) of Lemma 2(b), all a_i' are distinct and hence the number of a_i' composed of p_1' and p_2' is at least 5 for $5 \leq k \leq 8$, which is a contradiction. For $k = 4$, we note that for $0 \leq i \leq 3$, $n + i5 \in \{y_1^2, 2y_2^2, 3y_3^2, 6y_4^2\}$ where y_1, y_2, y_3, y_4 are some positive integers. Hence $n(n+5)(n+10)(n+15)$ is a perfect square, say X^2. We put $Y = n^2 + 15n + 25$ to observe that $Y^2 - X^2 = 625$. Since $\gcd(X,Y) = 1$, we have $Y - X = 1, Y + X = 625$, which implies that $Y = 313$, but $n^2 + 15n + 25 = 313$ has no solution in integers. Thus we may assume that $4 \leq k \leq 8$ and $n \leq \frac{25}{4}k^2 - 15k + 9$. Let $k = 8$. Then $n + (k-1)d \leq 324$. We apply the Algorithm of Section 3 to get $c = 5.07 < 8^2$, $\theta = 3$, $\eta = 3$, $q_1 = 2$, $q_2 = 3$, $q_3 = 7$, $q_4 = 11$, $q_5 = 13$, $q_6 = 17$ and $D = \{11^2, 2 \cdot 11^2, 13^2, 17^2\}$. We take $j = 4$ for $q \in \{11^2, 17^2\}$ and $j = 1$ for $q \in \{2 \cdot 11^2, 13^2\}$ to check Step 3. Hence by Lemma 11, assumption (7) does not hold, which is a contradiction. Thus $k \neq 8$. Here and in the sequel, checkings involving the Algorithm were done using Mathematica. We apply the above argument for $4 \leq k \leq 7$ to complete the proof for $d = 5$. This concludes the proof for $l = 2$. ∎

6. Proof of Theorems 1 and 2 for $l \geq 3$. We assume that (7) and (8) hold and we arrive at a contradiction if either the assumptions of Theorem 1 or of Theorem 2 hold.

First we consider the case where $k \geq 11380$. Then

$$d \leq \frac{7}{5} \cdot 4^{1/l} (\gamma(k,l))^{1-2/l} k^{l-3}.$$

Hence Lemma 8 is valid. We set

$$Q(k) = \prod_{h=1}^{\delta(k)} a'_h \quad \text{where } \delta(k) = k - \beta(k) - \pi(k).$$

We use Lemmas 8, 1(i) and 1(ii) to get

$$\begin{aligned}\log Q(k) &\ge \log\left\{16!(1.7777)^{41}\frac{50!}{9!}\dots(4.6425)^{\delta(k)-9183}\frac{(\delta(k)-963)!}{8220!}\right\}\\ &\ge 6227.23 + (\log 4.6425)\delta(k) + \log(\delta(k)-963)!\\ &\ge 6227.23 + (1.5352)\delta(k) + \log(\delta(k)-963)!.\end{aligned}$$

Thus

$$Q(k) > k! \tag{32}$$

is valid if

$$6227.23 + (1.5352)\delta(k) + \log(\delta(k)-963)! > \log k!,$$

which, again by Lemma 1(i) and (ii), is valid if

$$\begin{aligned}&6227.23 + (1.5352)\delta(k)\\ &> \left(k+\frac{1}{2}\right)\log k - k - (\delta(k)-962.5)\log(\delta(k)-963) + \delta(k) - 963 + \frac{1}{12k},\end{aligned}$$

i.e.,

$$\begin{aligned}&6227 + (1.5352)\delta(k)\\ &\quad > (\beta(k)+\pi(k)+963)(\log k - 1) + (\delta(k)-962.5)\log\frac{k}{\delta(k)-963}.\end{aligned} \tag{33}$$

Let $k \ge 14250$. Then

$$\log\frac{k}{\delta(k)-963} < .2092 \quad \text{and} \quad \pi(k) < \frac{1.157k}{\log k}$$

by Lemma 1(iii). Using these estimates we check that (33) and hence (32) are valid for $k \ge 14250$. Next, we use the exact value of $\pi(k)$ from [4] to see that (33) and therefore (32) is valid for $k = 11380$. Thus we need to check (32) for $k \in [11381, 14249] =: I$. We note that for $k \in I$, $\beta(k) = 2$ and

$$Q(k+1) = \begin{cases} Q(k) & \text{if } k+1 \text{ is a prime,}\\ Q(k)a'_{k-\pi(k)-1} & \text{if } k+1 \text{ is not a prime.}\end{cases} \tag{34}$$

Suppose (32) is valid for some $k \in I$. Then from (34) and Table 1, we note that $Q(k+1) > (k+1)!$ whenever $k+1$ is not a prime. Thus (32) is valid for all $k \in I$ if it is valid for all the primes in I. There are 301 primes in I

and (33) is checked to be valid for all these primes. Thus (32) is valid for $k \geq 11380$. On the other hand, we see from $|S_1| \geq t - \pi(k)$, $t \geq t_0$, (3) and (11) that

$$Q(k) \leq \prod_{a_i \in S_1} a_i \leq (k-1)!.$$

This is a contradiction. Thus $k < 11380$. Using the lower bounds for a_i given by [3, (15), (16)], Erdős and Selfridge obtained $k \leq 30000$. In fact, these lower bounds yield $k \leq 30600$ and an application of the preceding argument sharpens to $k \leq 30000$.

It remains to prove Theorem 1 for $k < 11380$. First, let $d = 1$. Using (19) we check that

$$(35) \quad \begin{cases} f_0(k,2) \geq 4,\ 4 \leq k \leq 22;\ f_0(k,3) \geq 8,\ 23 \leq k \leq 102; \\ f_0(k,4) \geq 16,\ 103 \leq k \leq 282;\ f_0(k,5) \geq 22,\ 283 \leq k \leq 612; \\ f_0(k,6) \geq 38,\ 613 \leq k \leq 1102;\ f_0(k,7) \geq 66,\ 1103 \leq k \leq 1636; \\ f_0(k,8) \geq 115, 1637 \leq k \leq 2238. \end{cases}$$

Hence

$$(36) \quad \binom{f_0(k,m)+l-2}{l-1} > l^m$$

for $l = 3, k, m$ chosen as in (35). We note by induction on l that (36) is valid for all $l > 3, k, m$ as in (35) since

$$f_0(k,m) > m + 1 + \frac{3m(m-1)}{2(9-m)}$$

and hence

$$f_0(k,m) + l - 1 > l\left(1 + \frac{m}{l} + \binom{m}{2}\frac{1}{l^2} + \dots\right) = l\left(1 + \frac{1}{l}\right)^m,$$

thereby showing that

$$\binom{f_0(k,m)+l-1}{l} > (l+1)^m$$

by (36). But this contradicts (24) by (19). Thus we may assume that $k \geq 2239$.

In Table 2, we give the values of m_1, m_2, the range of k and using the definition of $F_0(k, m_1, m_2)$ from (22) a lower bound for $F_0(k, m_1, m_2)$, say $F_0^*(m_1, m_2)$, for that range of k.

Table 2

m_1	m_2	k	$F_0^*(m_1, m_2)$
4	11	2239–2808	112
4	12	2809–2960	121
5	11	2961–3480	195
5	12	3481–3720	210
5	13	3721–4488	226
5	14	4489–5040	241
5	15	5041–5165	257
6	14	5166–5328	418
6	15	5329–6240	445
6	16	6241–6888	472
6	17	6889–7920	499
6	18	7921–9408	526
6	19	9409–10200	553
6	20	10201–10608	580
6	21	10609–11379	607

We check that

$$\binom{F_0^*(m_1, m_2) + l - 2}{l - 1} > l^{m_1} \binom{l + m_2 - 1}{l - 1}$$

for $l = 3,\ m_1,\ m_2,\ F_0^*(m_1, m_2)$ as in Table 2. Since

$$F_0^*(m_1, m_2) > 1 + m_1 + m_2 + \frac{m_1 m_2}{3} + \frac{m_1(m_1 - 1)(m_2 + 3)}{2(9 - m_1)},$$

we have

$$F_0^*(m_1, m_2) + l - 1 > (l + m_2)\left(1 + \frac{1}{l}\right)^{m_1}$$

and hence the inequality

$$\binom{F_0^*(m_1, m_2) + l - 2}{l - 1} > l^{m_1} \binom{l + m_2 - 1}{l - 1}$$

is valid for all $l > 3$, m_1, m_2, $F_0^*(m_1, m_2)$ as in Table 2. This contradicts (25) in view of (22). Thus Theorem 1 is valid for $d = 1$.

Let $d > 1$ and $k < 11380$. We first prove Theorem 1(a). Let d be as in (4). By Lemma 2(a), $t' = t$ and hence a'_i are distinct for $1 \le i \le t$. Let $l \ge 5$. Then we observe that the hypothesis of Lemma 4 is valid with

$$l' = \begin{cases} l - 3 & \text{for } k \ge 11,\ d \text{ even and } k \ge 6,\ d \text{ odd}, \\ l - 2 & \text{for } k \ge 121,\ d \text{ even and } k \ge 37,\ d \text{ odd}. \end{cases}$$

We use (20), (21) and (35) to obtain

$$\binom{g(k, m - 1) + l' - 1}{l'} > l^{m-1}$$

with m chosen as in (35) for $k \le 2238$. This contradicts (26) with $m' = m-1$.

Let $k > 2238$. We use $G(k) \ge G_0(k, m_1', m_2')$, (23) and Table 2 to obtain

$$\binom{G(k)+l'-1}{l'} > l^{m_1-1}\binom{l'+m_2}{l'}$$

with m_1 and m_2 chosen as in Table 2. This contradicts (27) since $m_1' \le m_1 - 1$ and $m_2' \le m_2$. Thus we may assume that $3 \le k \le 10$, d even or $3 \le k \le 5$, d odd, $3 \mid d$. Suppose that $3 \le k \le 10$, d even. The number of a_i' divisible by p_1' (≥ 3) is at most 4 if $k = 10$; 3 if $k = 7, 8, 9$; 2 if $k = 4, 5, 6$ and 1 if $k = 3$. From (20) we find that the number of a_i' divisible by p_1' is at least 6 if $k = 10$; 4 if $4 \le k \le 9$; and 3 if $k = 3$. This is a contradiction. Suppose that $3 \le k \le 5$, d odd, $3 \mid d$. The number of a_i' divisible by p_1' (≥ 2) is at most 3 if $k = 5$; 2 if $k \in \{3, 4\}$; while by (20), this number is at least 4 if $k = 5; 3$ if $k \in \{3, 4\}$ since $3 \mid d$. This contradiction proves Theorem 1(a) for $l \ge 5$.

Let $l = 3$. We take $l' = 2$ and $d \in \{2, 3, 4, 6\}$. The hypothesis of Lemma 4 is valid for $k \ge 40$ if $d \in \{2, 3, 4\}$ and for $k \ge 100$ if $d = 6$. For $k \le 2238$, we use (20), (21) and (35) to obtain

$$\binom{g(k, m-1)+1}{2} > 3^{m-1}$$

with m chosen as in (35) and this contradicts (26) with $m' = m - 1$. For $k > 2238$, we use (23) and Table 2 to obtain

$$\binom{G(k)+1}{2} > l^{m_1-1}\binom{m_2+2}{2}$$

with m_1 and m_2 chosen as in Table 2. This contradicts (27) since $m_1' \le m_1 - 1, m_2' \le m_2$.

Thus we may suppose that $k \le 39$ if $d \in \{2, 3, 4\}$ and $k < 100$ if $d = 6$. We know that a_i' for $1 \le i \le t$ are cube free. Hence $g(k, 1) \le 3$ and $g(k, 2) \le 9$. We check using (21) and (19) that $g_0(k, 1) \ge 4$ for $4 \le k \le 40$, $g_0(k, 2) \ge 10$ for $41 \le k < 100$ if $d = 6$ since in this case $g_0(k, 2) \ge f_0(k, 4)$. Thus we may assume that $k = 3$. If $d = 2, 4$, then either $a_i' = 1$ or $3 \mid a_i'$ for $1 \le i \le 3$. This is a contradiction since at most one a_i' is divisible by 3 and a_i' are distinct. If $d = 3$, then either $a_i' = 1$ or $2 \mid a_i'$ for $1 \le i \le 3$. Hence $n = 2y_1^3,\ n + 3 = y_2^3,\ n + 6 = 4y_3^3$ or $n = 4y_1^3,\ n + 3 = y_2^3,\ n + 6 = 2y_3^3$ for some positive integers y_1, y_2, y_3. This means $y_2^6 = (n + 3)^2 = (2y_1y_3)^3 + 9$, which is not possible since two cubes > 1 cannot differ by 9. If $d = 6$, then $a_1' = a_2' = a_3' = 1$, which is a contradiction. This completes the proof of Theorem 1(a) for $l = 3$.

Now we prove Theorem 1(b) for $k < 11380$. Let $d = 5$ and $k \ge 4$. First, we consider the case $l \ge 5$. We observe that the hypothesis of Lemma 4 is valid with $l' = l - 2$ for $k \geq 5$. We note from (20) and (19) that

$$(37) \qquad \begin{cases} g(k, m-1) \geq g_0(k, m-1) \geq f_0(k, m) & \text{for } m \geq 3, \\ g(k, 2) \geq 5 & \text{for } 5 \leq k \leq 22. \end{cases}$$

We use (37) and (35) to obtain

$$\binom{g(k,2) + l' - 1}{l'} \geq l^2 \qquad \text{for } 5 \leq k \leq 22$$

and

$$\binom{g(k,m-1) + l' - 1}{l'} > l^{m-1} \qquad \text{for } 23 \leq k \leq 2238$$

with m chosen as in (35). This contradicts (26).

Let $k > 2238$. We use (23) and Table 2 to obtain

$$\binom{G(k) + l' - 1}{l'} > l^{m_1 - 1} \binom{l' + m_2}{l'},$$

which contradicts (27) since $m_1' \leq m_1 - 1$ and $m_2' \leq m_2$ for $m_1 \geq 3$ as $5 \nmid a_i'$ for $1 \leq i \leq t$. Thus we may assume that $k = 4$. It is not possible to apply Lemma 4 with $l' = l - 2$ since the assumption $d \leq k^{l - l' - 1}$ with $l' = l - 2$ of Lemma 4 is not valid for $d = 5$ and $k = 4$. But we observe that $g(k, 2) = 4$ and by (7), $n \geq 7^l - 15 > 6^l$. By following the proof of Lemma 4, we find that Lemma 4 holds with $l' = l - 2$. We check that

$$\binom{g(k,2) + l - 3}{l - 2} > l^2 \qquad \text{for } l \geq 7.$$

This contradicts (26). Thus $l = 5$. Let $n \geq (12.5)^5$. Then we use the upper bound $60n^3 + 6 \cdot 15^2 n^2 + 4 \cdot 15^3 n + 15^4$ for $(n + d_{i_1} d) \ldots (n + d_{i_{l'}} d) - (n + d_{j_1} d) \ldots (n + d_{j_{l'}} d)$ to see that the assertion of Lemma 4 holds with $l' = l - 1$. Hence

$$\binom{g(k,2) + l - 2}{l - 1} > l^2,$$

which contradicts (26). Thus we may assume that $n < (12.5)^5$. We apply the Algorithm to get $c = 298.04 < 4^5, \theta = 2, \eta = 2, q_1 = 2, q_2 = 3, q_3 = 7, q_4 = 11$ and $D = \{7^5, 2 \cdot 7^5, 3 \cdot 7^5, 4 \cdot 7^5, 6 \cdot 7^5, 8 \cdot 7^5, 9 \cdot 7^5, 12 \cdot 7^5, 16 \cdot 7^5, 18 \cdot 7^5, 11^5\}$. We take $j = 1$ for every $q \in D$ to check Step 3. Hence by Lemma 11, assumption (7) does not hold, which is a contradiction. This proves Theorem 1(b) for $l \geq 5$.

Let $l = 3$ and $l' = 2$. The hypothesis of Lemma 4 is valid for $k > 100$ and we argue as in the case $l = 3$ of Theorem 1(a) to exclude the cases $100 < k < 11380$. Thus $k \leq 100$. Now we use the estimate $n > \gamma(k, 3)k^3$ of

Lemma 3 and Lemma 1(iii) in the proof of Lemma 3 to obtain for $k \geq 79$,

$$|\{A_i \mid 1 \leq i \leq t,\ X_i \neq 1\}| \geq k - 1 - \pi(k) - \frac{k \log k}{3 \log k + \log \gamma(k,3)} \geq (.5681)k.$$

Hence $n > (.5681)k^4$. We use this estimate in the inequality $k^2d^2 + 2nkd > 3n^{4/3}$ of Lemma 4 to observe that the assertion of Lemma 4 is valid whenever

$$d \leq \frac{3}{2}(.5681k)^{1/3} - \frac{1}{2k}.$$

We use (37) and (35) with $m = 3$ to check that

$$\binom{g(k,2)+1}{2} > 3^2 \quad \text{for } 79 \leq k \leq 100,$$

which contradicts (26) with $l' = m' = 2$. Thus $k < 79$. Now we check using (20) that $g_0(k,2) \geq 10$ for $25 \leq k < 79, k \in \{21, 22\}$ and this is not possible since a_i' are cube free.

Thus we are left with $4 \leq k \leq 20$ and $k \in \{23, 24\}$. We see that if $n > 40k^3$, then the hypothesis of Lemma 4 is satisfied. Further, $g(k,2) \geq 4$ and hence

$$\binom{g(k,2)+1}{2} > 3^2,$$

which contradicts (26) with $l' = m' = 2$. Thus we may assume $n \leq 40k^3$. As earlier, we apply the Algorithm to eliminate the cases $4 \leq k \leq 20$ and $k \in \{23, 24\}$. We illustrate the case $k = 7$. Then $n + (k-1)d \leq 13750,\ c = 40.1 < 7^3,\ \theta = 3,\ \eta = 5,\ q_1 = 2,\ q_2 = 3,\ q_3 = 7,\ q_4 = 11,\ q_5 = 13,\ q_6 = 17,\ q_7 = 19,\ q_8 = 23,$ $D = \{11^3, 2 \cdot 11^3, 3 \cdot 11^3, 4 \cdot 11^3, 6 \cdot 11^3, 7 \cdot 11^3, 8 \cdot 11^3, 9 \cdot 11^3, 13^3, 2 \cdot 13^3, 3 \cdot 13^3, 4 \cdot 13^3, 6 \cdot 13^3, 17^3, 2 \cdot 17^3, 19^3, 2 \cdot 19^3, 23^3\}$. We check that Step 3 is valid with $j = 1$ whenever $q \in D$ but $q \notin \{6 \cdot 11^3, 3 \cdot 13^3, 19^3\}$ and with $j = 2$ otherwise. Hence by Lemma 11, assumption (7) does not hold, which is a contradiction. This completes the proof of Theorem 1(b).

Proof of Corollary 1. We observe from the equation of Corollary 1 that

$$\text{ord}_p(n(n+d)\ldots(n+(k-1)d)) \equiv 0 \pmod{l} \tag{38}$$

for every prime p. We apply the result of Shorey and Tijdeman [9] to deduce that $n(n+d)\ldots(n+(k-1)d)$ is divisible by a prime exceeding k for $1 < d \leq 6$. When $d = 1$ and $n \leq k$, by Bertrand's postulate, there exists a prime p with $n \leq (n+k)/2 \leq p < n+k$. Then p divides $n(n+1)\ldots(n+k-1)$ only to the first power, which contradicts (38). Thus we may suppose that $n > k$ whenever $d = 1$. Then by a theorem of Sylvester, there exists a prime exceeding k dividing $n(n+1)\ldots(n+k-1)$. Now we apply Theorem 1 to get a contradiction to (38) except in the cases $k = 3, d \in \{1, 5\}$. To deal with these cases, we write as usual $n = a_1x_1^l,\ n+d = a_2x_2^l,\ n+2d = a_3x_3^l$ where a_1, a_2, a_3 are lth power free integers. It follows from the equation of Corollary 1 that $P(a_i) \le 2$. It is easy to check that a_i' are distinct. Hence $(a_1, a_2, a_3) \in \{(2, 1, 2^{l-1}), (2^{l-1}, 1, 2)\}$ and $l \ge 3$. Then $(2x_1x_3)^l = n(n+2d) = (n+d)^2 - d^2 = x_2^{2l} - d^2$, implying $x_2^2 > 2x_1x_3$. Hence $(2x_1x_3+1)^l - (2x_1x_3)^l \le d^2$, showing that $d = 5,\ l = 3,\ x_1x_3 = 1$, which is impossible. ∎

Proof of Corollary 2. By our assumption when $d = 1$ and by the result of Shorey and Tijdeman [9], we see that $P(n(n+d)\dots(n+(k-1)d)) > k$ for $1 \le d \le 6$. Hence by Theorem 1, there exists a prime $p > k$ such that

$$\operatorname{ord}_p(n(n+d)\dots(n+(k-1)d)) \not\equiv 0 \pmod{l}.$$

By the equation in Corollary 2, p divides $n(n+d)\dots(n+(k-1)d)$ to an order which is $\equiv 0 \pmod{l}$ since $P(b) \le k$. This is a contradiction. ∎

7. Proofs of Theorems A and B

Proof of Theorem A. Suppose $n \le k$. Then there exists a prime $p = p^{(k)}$ with $n \le (n+k)/2 \le k \le p < n+k$. Therefore p divides $n(n+1)\dots(n+k-1)$ only to the first power. Hence the theorem follows. We may therefore, assume that $n > k$. Then, by a theorem of Sylvester, there exists a prime $p > k$ dividing $n(n+1)\dots(n+k-1)$. Now, the theorem follows from Theorem 1(a) with $d = 1,\ t = k$ whenever $k \ge 4$. Thus we need to consider $k = 3$. We assume that $\operatorname{ord}_p(n(n+1)(n+2)) \equiv 0 \pmod{l}$ for every prime $p \ge 3$. We write $n + i = b_ix_i^l$ where b_i is lth power free, $P(b_i) \le 2$ for $0 \le i \le 2$. We see as in Lemma 2(a) that b_1, b_2, b_3 are distinct. Hence $l \ge 3$. Then it follows as in Lemma 4 that the products $b_{i_1}\dots b_{i_{l-1}}$ are all distinct. We note that n is even and thus $b_1 = 2,\ b_2 = 1,\ b_3 = 2^{\alpha-1}$ or $b_1 = 2^{\alpha-1},\ b_2 = 1,\ b_3 = 2$ for some integer α with $2 \le \alpha \le l$. Then we have $(b_1)^{\alpha-1}(b_2)^{l-\alpha} = b_3(b_2)^{l-2}$ or $(b_3)^{\alpha-1}(b_2)^{l-\alpha} = b_1(b_2)^{l-2}$, respectively. This contradicts the fact that $b_{i_1}\dots b_{i_{l-1}}$ are distinct. ∎

Proof of Theorem B. Since $n \ge k+1$, the left hand side of the equation in Theorem B is divisible by a prime exceeding k by a theorem of Sylvester. Hence the hypothesis of Theorem 1 with $d = 1$ is satisfied and the assertion follows. ∎

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