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Exact m-covers and the linear form $\sum_{s=1}^{k} x_s/n_s$

by

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1. Introduction. For $a, n \in \mathbb{Z}$ with $n > 0$, we let

$$a + n\mathbb{Z} = \{\ldots, a - 2n, a - n, a, a + n, a + 2n, \ldots\}$$

and call it an *arithmetic sequence*. Given a finite system

$$A = \{a_s + n_s\mathbb{Z}\}_{s=1}^{k} \tag{1}$$

of arithmetic sequences, we assign to each $x \in \mathbb{Z}$ the corresponding covering multiplicity $\sigma(x) = |\{1 \le s \le k : x \in a_s + n_s\mathbb{Z}\}|$ ($|S|$ means the cardinality of a set S), and call $m(A) = \inf_{x\in\mathbb{Z}} \sigma(x)$ the *covering multiplicity* of A. Apparently

$$\sum_{s=1}^{k} \frac{1}{n_s} = \frac{1}{N} \sum_{x=0}^{N-1} \sigma(x) \ge m(A) \tag{2}$$

where N is the least common multiple of those *common differences* (or *moduli*) $n_1, \ldots, n_k$. For a positive integer m, (1) is said to be an *m-cover* of $\mathbb{Z}$ if its covering multiplicity is not less than m, and an *exact m-cover* of $\mathbb{Z}$ if $\sigma(x) = m$ for all $x \in \mathbb{Z}$. Note that $k \ge m$ if (1) forms an m-cover of $\mathbb{Z}$. Clearly the *covering function* $\sigma : \mathbb{Z} \to \mathbb{Z}$ is constant if and only if (1) forms an exact m-cover of $\mathbb{Z}$ for some $m = 1, 2, \ldots$ An exact 1-cover of $\mathbb{Z}$ is a partition of $\mathbb{Z}$ into residue classes.

P. Erdős ([E]) proposed the concept of *cover* (i.e., 1-cover) of $\mathbb{Z}$ in the 1930's, Š. Porubský ([P]) introduced the notion of exact m-cover of $\mathbb{Z}$ in the 1970's, and the author ([Su3]) studied m-covers of $\mathbb{Z}$ for the first time. The most challenging problem in this field is to describe those $n_1, \ldots, n_k$ in an m-cover (or exact m-cover) (1) of $\mathbb{Z}$ (cf. [Gu]). In [Su2, Su3, Su4] the author revealed some connections between (exact) m-covers of $\mathbb{Z}$ and

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Egyptian fractions. Here we concentrate on exact m-covers of $\mathbb{Z}$. In [Su3, Su4] results for exact m-covers of $\mathbb{Z}$ were obtained by studying general m-covers of $\mathbb{Z}$ and noting that an exact m-cover (1) of $\mathbb{Z}$ is an m-cover of $\mathbb{Z}$ with $\sum_{s=1}^{k} 1/n_s = m$. In Section 4 of the present paper we shall directly characterize exact m-covers of $\mathbb{Z}$ in various ways. (Note that in the famous book [Gu] R. K. Guy wrote that characterizing exact 1-covers of $\mathbb{Z}$ is a main outstanding unsolved problem in the area.) This enables us to make further progress. With the help of the linear form $\sum_{s=1}^{k} x_s/n_s$ (studied in the next section), we will provide some new properties of exact m-covers of $\mathbb{Z}$ (see Section 3). The fifth section is devoted to proofs of the main theorems stated in Section 3.

For a complex number x and nonnegative integer n, as usual,

$$\binom{x}{n} := \frac{1}{n!}\prod_{j=0}^{n-1}(x-j)$$

$\left(\binom{x}{0} \text{ is } 1\right)$. For real x we use $[x]$ and $\{x\}$ to represent the integral part and the fractional part of x respectively. For two integers a, b not both zero, (a, b) denotes the greatest common divisor of a and b.

Now we state our central results for an exact m-cover (1) of $\mathbb{Z}$:

(I) For $a = 0, 1, 2, \ldots$ and $t = 1, \ldots, k$ there are at least $\binom{m-1}{[a/n_t]}$ subsets I of $\{1, \ldots, k\}$ for which $t \notin I$ and $\sum_{s\in I} 1/n_s = a/n_t$, where the lower bounds are best possible.

(II) If $\emptyset \neq I \subseteq \{1, \ldots, k\}$ and $(n_s, n_t) \mid a_s - a_t$ for all $s, t \in I$, then

$$\left\{\left\{\sum_{s\in J}\frac{1}{n_s}\right\} : J \subseteq \{1, \ldots, k\} \setminus I\right\} \supseteq \left\{\frac{r}{[n_s]_{s\in I}} : r = 0, 1, \ldots, [n_s]_{s\in I} - 1\right\}$$

where $[n_s]_{s\in I}$ is the least common multiple of those n_s with $s \in I$.

(III) For any rational c, the number of solutions of the equation $\sum_{s=1}^{k} x_s/n_s = c$ with $x_s \in \{0, 1, \ldots, n_s - 1\}$ for $s = 1, \ldots, k$, is the sum of finitely many (not necessarily distinct) prime factors of $n_1, \ldots, n_k$ if $c \neq 0, 1, 2, \ldots$, and at least $\binom{k-m}{n}$ if c equals a nonnegative integer n.

2. On the linear form $\sum_{s=1}^{k} x_s/n_s$. In this section we shall say something general about the linear form $\sum_{s=1}^{k} x_s/n_s$ where $n_1, \ldots, n_k$ are positive integers.

Let us first introduce more notations. For x, y in the rational field $\mathbb{Q}$, if $x - y \in \mathbb{Z}$ then we write $x \equiv y \pmod 1$. For $n = 1, 2, \ldots$ we set $R(n) = \{0, \ldots, n-1\}$. When we deal with a finite collection $\{n_s\}_{s\in I}$ of positive integers, the least common multiple $[n_s]_{s\in I}$ and the product $\prod_{s\in I} n_s$ will be regarded as 1 if I is empty.

DEFINITION. Two (finite) sequences $\{n_s\}_{s=1}^k$ and $\{m_t\}_{t=1}^l$ of positive integers are said to be *equivalent* if $k = l$ and $(n_s, n_t) = (m_s, m_t)$ for all $s, t = 1, \ldots, k$ with $s \neq t$. We call $\{n_s\}_{s=1}^k$ a *normal* sequence if n_t divides $[n_s]_{s=1,\, s\neq t}^k$ for every $t = 1, \ldots, k$.

PROPOSITION 2.1. *Let $n_1, \ldots, n_k$ be arbitrary positive integers. Then $\{(n_t, [n_s]_{s=1,\, s\neq t}^k)\}_{t=1}^k$ is the only normal sequence equivalent to $\{n_s\}_{s=1}^k$.*

Proof. For each $t = 1, \ldots, k$ we let

$$n'_t = (n_t, [n_s]_{s=1,\, s\neq t}^k) = [(n_s, n_t)]_{s=1,\, s\neq t}^k.$$

Clearly n'_t divides $[n'_s]_{s=1,\, s\neq t}^k$ because $(n_s, n_t) \mid n'_s$ for all $s = 1, \ldots, k$ with $s \neq t$. For $i, j = 1, \ldots, k$ with $i \neq j$, $(n'_i, n'_j) = (n_i, n_j)$ since $n_i \mid [n_s]_{s=1,\, s\neq j}^k$ and $n_j \mid [n_s]_{s=1,\, s\neq i}^k$. Hence $\{n'_s\}_{s=1}^k$ is normal and equivalent to $\{n_s\}_{s=1}^k$. If so is $\{m_s\}_{s=1}^k$ where $m_1, \ldots, m_k$ are positive integers, then

$$m_t = (m_t, [m_s]_{s=1,\, s\neq t}^k) = [(m_s, m_t)]_{s=1,\, s\neq t}^k = [(n_s, n_t)]_{s=1,\, s\neq t}^k = n'_t$$

for every $t = 1, \ldots, k$. We are done.

PROPOSITION 2.2. *Let $n_1, \ldots, n_k$ be positive integers. For $\theta \in \mathbb{Q}$ the equation*

$$\sum_{s=1}^k \frac{x_s}{n_s} \equiv \theta \pmod 1 \quad \text{with } x_s \in R(n_s) \text{ for } s = 1, \ldots, k \tag{3}$$

is solvable if and only if $[n_1, \ldots, n_k]\theta \in \mathbb{Z}$, and in the solvable case the number of solutions is $n_1 \ldots n_k/[n_1, \ldots, n_k]$, which does not change if we replace $\{n_s\}_{s=1}^k$ by an equivalent sequence.

Proof. We argue by induction. The case $k = 1$ is trivial. Let $k > 1$ and assume Proposition 2.2 for smaller values of k. Observe that

$$\frac{1}{[n_1, \ldots, n_k]}\mathbb{Z} = \frac{([n_1, \ldots, n_{k-1}], n_k)}{[n_1, \ldots, n_{k-1}]n_k}\mathbb{Z} = \frac{1}{n_k}\mathbb{Z} + \frac{1}{[n_1, \ldots, n_{k-1}]}\mathbb{Z}.$$

So $[n_1, \ldots, n_k]\theta \in \mathbb{Z}$ if and only if $[n_1, \ldots, n_{k-1}](\theta - x/n_k) \in \mathbb{Z}$ for some $x \in \mathbb{Z}$. For any $a \in \mathbb{Z}$ with $0 \leq a < n_k$, the congruence

$$\sum_{s=1}^{k-1} \frac{x_s}{n_s} \equiv \theta - \frac{a}{n_k} \pmod 1$$

is solvable if and only if

$$[n_1, \ldots, n_{k-1}]\left(\theta - \frac{a}{n_k}\right) \in \mathbb{Z},$$

i.e.

$$[n_1, \ldots, n_{k-1}]a \equiv [n_1, \ldots, n_{k-1}]n_k\theta \pmod{n_k}.$$

Hence (3) is solvable if and only if $[n_1,\ldots,n_k]\theta\in\mathbb{Z}$. In the solvable case there are exactly $([n_1,\ldots,n_{k-1}],n_k)=[(n_1,n_k),\ldots,(n_{k-1},n_k)]$ numbers $a\in R(n_k)$ satisfying the last congruence, thus by the induction hypothesis (3) has exactly

$$\frac{n_1\ldots n_{k-1}}{[n_1,\ldots,n_{k-1}]}([n_1,\ldots,n_{k-1}],n_k)=\frac{n_1\ldots n_k}{[n_1,\ldots,n_k]}$$

solutions. As $n_1\ldots n_{k-1}/[n_1,\ldots,n_{k-1}]$ depends only on those (n_i,n_j) with $1\le i<j<k$, the number $n_1\ldots n_k/[n_1,\ldots,n_k]$ depends only on the $(n_s,n_t),\ 1\le s<t\le k$. This ends the proof.

Corollary 2.1. *Let a be an integer and $n_1,\ldots,n_k$ positive integers. Then $a/[n_1,\ldots,n_k]$ can be written uniquely in the form $q+\sum_{s=1}^k x_s/n_s$ with $q\in\mathbb{Z}$ and $x_s\in R(n_s)$ for $s=1,\ldots,k$ if and only if $(n_s,n_t)=1$ for all $s,t=1,\ldots,k$ with $s\ne t$.*

Proof. By Proposition 2.2, equation (3) with $\theta=a/[n_1,\ldots,n_k]$ has a unique solution if and only if $n_1\ldots n_k=[n_1,\ldots,n_k]$. So the desired result follows.

Corollary 2.2. *Let $n_1,\ldots,n_k$ be positive integers. Then the number of solutions of the equation*

$$(4)\qquad \sum_{s=1}^k\frac{x_s}{n_s}\equiv 0 \pmod 1\qquad \text{with } x_s\in\mathbb{Z} \text{ and } 0<x_s<n_s \text{ for } s=1,\ldots,k$$

equals

$$(-1)^k+\sum_{t=1}^k(-1)^{k-t}\sum_{1\le i_1<\ldots<i_t\le k}\frac{n_{i_1}\ldots n_{i_t}}{[n_{i_1},\ldots,n_{i_t}]}$$

which depends only on those (n_s,n_t) with $1\le s<t\le k$.

Proof. For $I\subseteq\{1,\ldots,k\}$ let $\#I$ denote the number of solutions of the diophantine equation $\sum_{s\in I}x_s/n_s\equiv 0\pmod 1$ with $x_s\in\{1,\ldots,n_s-1\}$ for $s\in I$, and consider $\#\emptyset$ to be 1. By Proposition 2.2, $\sum_{J\subseteq I}\#J=\prod_{s\in I}n_s/[n_s]_{s\in I}$ for all $I\subseteq\{1,\ldots,k\}$, therefore $\#\{1,\ldots,k\}$ coincides with

$$\sum_{J\subseteq\{1,\ldots,k\}}\sum_{s=0}^{k-|J|}(-1)^{k-|J|-s}\binom{k-|J|}{s}\#J$$
$$=\sum_{J\subseteq\{1,\ldots,k\}}\sum_{J\subseteq I\subseteq\{1,\ldots,k\}}(-1)^{k-|I|}\#J$$

$$= \sum_{I\subseteq\{1,\ldots,k\}} (-1)^{k-|I|} \sum_{J\subseteq I} \#J = \sum_{I\subseteq\{1,\ldots,k\}} (-1)^{k-|I|} \frac{\prod_{s\in I} n_s}{[n_s]_{s\in I}}$$

$$= (-1)^k + \sum_{t=1}^{k} (-1)^{k-t} \sum_{1\le i_1<\ldots<i_t\le k} \frac{n_{i_1}\ldots n_{i_t}}{[n_{i_1},\ldots,n_{i_t}]}.$$

In view of Proposition 2.2, the number $\#\{1,\ldots,k\}$ remains the same if an equivalent sequence is substituted for $\{n_s\}_{s=1}^k$. The proof is now complete.

Remark 1. Equation (4) is closely related to diagonal hypersurfaces over a finite field. The formula for the number of solutions of (4) was obtained by R. Lidl and H. Niederreiter [LN], R. Stanly (cf. C. Small [Sm]), Q. Sun, D.-Q. Wan and D.-G. Ma [SWM] with much more complicated methods. The fact that the number does not vary if we replace $\{n_s\}_{s=1}^k$ by the corresponding normal sequence, was recently noted by A. Granville, S.-G. Li and Q. Sun [GLS]. For necessary and sufficient conditions for the solvability of (4), the reader is referred to [SW] where the authors determined when (4) has a unique solution.

COROLLARY 2.3. *Let* (1) *be a system of arithmetic sequences with* $(n_s,n_t)\,|\,a_s-a_t$ *for all* $s,t=1,\ldots,k$. *Then for any* $\theta\in\mathbb{Q}$ *with* $0\le\theta<1$ *we have*

$$(5)\qquad \Bigl|\sum_{\substack{x_s\in R(n_s)\ for\ s=1,\ldots,k\\ \{\sum_{s=1}^k x_s/n_s\}=\theta}} e^{2\pi i\sum_{s=1}^k a_s x_s/n_s}\Bigr| = \begin{cases} \dfrac{n_1\ldots n_k}{[n_1,\ldots,n_k]} & \text{if } [n_1,\ldots,n_k]\theta\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases}$$

Proof. By the Chinese Remainder Theorem in general form, the intersection $\bigcap_{s=1}^k a_s+n_s\mathbb{Z}$ is nonempty if and only if $a_s+n_s\mathbb{Z}\cap a_t+n_t\mathbb{Z}\ne\emptyset$ for all $s,t=1,\ldots,k$. (For a proof see, e.g., [Su1].) Since $(n_s,n_t)\,|\,a_s-a_t$ for $s,t=1,\ldots,k$, $\bigcap_{s=1}^k a_s+n_s\mathbb{Z}$ must contain an integer x. With the help of Proposition 2.2,

$$\sum_{\substack{x_s\in R(n_s)\text{ for }s=1,\ldots,k\\ \{\sum_{s=1}^k x_s/n_s\}=\theta}} e^{2\pi i\sum_{s=1}^k a_s x_s/n_s} = \sum_{\substack{x_s\in R(n_s)\text{ for }s=1,\ldots,k\\ \{\sum_{s=1}^k x_s/n_s\}=\theta}} e^{2\pi i x\theta}$$

vanishes if $[n_1,\ldots,n_k]\theta\notin\mathbb{Z}$, and otherwise equals $\frac{n_1\ldots n_k}{[n_1,\ldots,n_k]}e^{2\pi i x\theta}$. So (5) holds.

To conclude this section we make a few comments. For system (1), $M(A)=\sup_{x\in\mathbb{Z}}\sigma(x)$ does not change if an equivalent sequence takes the place of $\{n_s\}_{s=1}^k$, because for $\emptyset\ne I\subseteq\{1,\ldots,k\}$ the set $\bigcap_{s\in I} a_s+n_s\mathbb{Z}$ is nonempty if and only if $(n_s,n_t)\,|\,a_s-a_t$ for all $s,t\in I$. Observe that (1) forms an exact m-cover of $\mathbb{Z}$ if and only if $\sum_{s=1}^k 1/n_s = m \ge M(A)$. So whether $n_1,\dots,n_k$ are the moduli of an exact m-cover of $\mathbb{Z}$ only depends on $\sum_{s=1}^k 1/n_s$ and the $k(k-1)/2$ numbers $(n_s,n_t),\ 1\le s<t\le k$. For a given exact m-cover (1) of $\mathbb{Z}$, replacing $\{n_s\}_{s=1}^k$ by the unique normal sequence $\{n'_s\}_{s=1}^k$ equivalent to it we find that

$$\sum_{s=1}^k \frac{1}{n'_s} \le M(A) \le m = \sum_{s=1}^k \frac{1}{n_s}.$$

As $n'_s \le n_s$ for $s=1,\dots,k$, the sequence $\{n_s\}_{s=1}^k$ must be identical with $\{n'_s\}_{s=1}^k$ and hence normal. In the light of the above, the reader should not be surprised by connections between the exact m-cover (1) of $\mathbb{Z}$ and the linear form $\sum_{s=1}^k x_s/n_s$.

3. Main theorems and their consequences. In this section we let (1) be an exact m-cover of $\mathbb{Z}$; we also let $I\subseteq\{1,\dots,k\}$ and $\bar I=\{1,\dots,k\}\setminus I$. For any rational c, we let $I^*(c)$ be the number of solutions $\langle x_s\rangle_{s\in I}$ to the diophantine equation

$$\sum_{s\in I}\frac{x_s}{n_s}=c \quad \text{with } x_s\in R(n_s) \text{ for all } s\in I, \tag{6}$$

and $I_*(c)=|\{J\subseteq I:\sum_{s\in J}1/n_s=c\}|$ be the number of solutions $\langle\delta_s\rangle_{s\in I}$ to the equation

$$\sum_{s\in I}\frac{\delta_s}{n_s}=c \quad \text{with } \delta_s\in R(2)=\{0,1\} \text{ for all } s\in I. \tag{7}$$

(When $I=\emptyset$ and $c=0$ we view each of (6) and (7) as having only the zero solution.) We also set

$$I_*^{(0)}(c)=\left|\left\{J\subseteq I: 2\,|\,|J| \text{ and } \sum_{s\in J}\frac{1}{n_s}=c\right\}\right| \tag{8}$$

and

$$I_*^{(1)}(c)=\left|\left\{J\subseteq I: 2\nmid|J| \text{ and } \sum_{s\in J}\frac{1}{n_s}=c\right\}\right|. \tag{9}$$

Let us present our main theorems whose proofs will be given later, and derive a number of interesting corollaries from them.

THEOREM 3.1. *Let c be a rational number.*

(i) *When $|I|\le m$, if $I^*(c-n)=1$ for a nonnegative integer n then*

$$\bar I_*(c)+\sum_{\substack{l=0\\ l\ne n}}^{m-|I|}\binom{m-|I|}{l}I^*(c-l)\ge\binom{m-|I|}{n}; \tag{10}$$

in particular, if c can be uniquely written in the form $n+\sum_{s\in I} x_s/n_s$ where n and x_s lie in $\{0,1,\dots,m-|I|\}$ and $\{0,1,\dots,n_s-1\}$ respectively, then

$$\bar{I}_*(c) \geq \binom{m-|I|}{n}.$$

(ii) *When $|I|\geq m$, if $\bar{I}_*(c-n)=1$ for a nonnegative integer n then*

$$I^*(c)+\sum_{\substack{l=0\\ l\neq n}}^{|I|-m}\binom{|I|-m}{l}\bar{I}_*(c-l)\geq\binom{|I|-m}{n}; \tag{11}$$

in particular, if c can be uniquely expressed in the form $n+\sum_{s\in J}1/n_s$ where $J\subseteq\bar{I}$ and $n\in\{0,1,\dots,|I|-m\}$, then

$$I^*(c)\geq\binom{|I|-m}{n}.$$

Below there are corollaries involving the cases $|I|\leq m$, $|I|=m$ and $|I|\geq m$.

COROLLARY 3.1. *Assume that those n_s with $s\in I$ are pairwise relatively prime. Then $|I|\leq m$ and*

$$\left|\left\{J\subseteq\bar{I}:\sum_{s\in J}\frac{1}{n_s}=n+\sum_{s\in I}\frac{x_s}{n_s}\right\}\right|\geq\binom{m-|I|}{n} \tag{12}$$

for all $n=0,1,2,\dots$ and $x_s\in R(n_s)$ with $s\in I$; in particular,

$$\left\{\sum_{s\in J}\frac{1}{n_s}:J\subseteq\bar{I}\right\}\supseteq\left\{\frac{a}{[n_s]_{s\in I}}:a\in\mathbb{Z}\ \&\ |I|\leq\frac{a}{[n_s]_{s\in I}}\leq m-|I|\right\} \tag{13}$$

and

$$\left|\left\{J\subseteq\bar{I}:\sum_{s\in J}\frac{1}{n_s}\equiv\frac{a}{\prod_{s\in I}n_s}\pmod 1\right\}\right|\geq 2^{m-|I|}\quad\textit{for every } a\in\mathbb{Z}. \tag{14}$$

Proof. By the Chinese Remainder Theorem, $\bigcap_{s\in I}a_s+n_s\mathbb{Z}\neq\emptyset$ if $I\neq\emptyset$. Since any integer lies in exactly m members of (1), $|I|$ does not exceed m. Let $N=[n_s]_{s\in I}=\prod_{s\in I}n_s$. By Corollary 2.1, for each $a\in\mathbb{Z}$ the number a/N can be expressed uniquely in the form $q+\sum_{s\in I}x_s/n_s$ with $q\in\mathbb{Z}$ and $x_s\in R(n_s)$ for $s\in I$. Whenever $x_s\in R(n_s)$ for all $s\in I$, by Theorem 3.1, (12) holds for every nonnegative integer n. If $|I|N\leq a\leq(m-|I|)N$ then the corresponding integer $q=a/N-\sum_{s\in I}x_s/n_s$ lies in the interval $[0,\ m-|I|]$ and hence

$$\left|\left\{J\subseteq\bar{I}:\sum_{s\in J}\frac{1}{n_s}=\frac{a}{N}=q+\sum_{s\in I}\frac{x_s}{n_s}\right\}\right|\geq\binom{m-|I|}{q}>0.$$

This yields (13). For (14) we observe that

$$\begin{aligned}\Big|\Big\{J\subseteq \bar I:\sum_{s\in J}\frac{1}{n_s}\equiv\frac{a}{N}\pmod 1\Big\}\Big|&\\ \ge \sum_{n=0}^{m-|I|}\Big|\Big\{J\subseteq\bar I:\sum_{s\in J}\frac{1}{n_s}=n+\sum_{s\in I}\frac{x_s}{n_s}\Big\}\Big|&\\ \ge\sum_{n=0}^{m-|I|}\binom{m-|I|}{n}=2^{m-|I|}.&\end{aligned}$$

This concludes the proof.

Applying Corollary 3.1 with $I=\emptyset$ we immediately get the theorem of Sun [Su2].

Putting $I=\{t\}$ $(1\le t\le k)$ in Corollary 3.1 we then obtain result (I) stated in the first section. In the case $m=1$, result (I) was first observed by the author in [Su4]. When $m>1$, we noted in [Su4] that, providing $n_1<\ldots<n_{k-l}<n_{k-l+1}=\ldots=n_k$, for every $r=0,1,\ldots,n_k-1$ there exists a $J\subseteq\{1,\ldots,k-1\}$ with $\sum_{s\in J}1/n_s\equiv r/n_k\pmod 1$. In [Su4] we even conjectured that, if (1) forms an m-cover of $\mathbb{Z}$ with $\sigma(x)=m$ for all $x\equiv a_t\pmod{n_t}$ where $1\le t\le k$, then

$$\Big\{\Big\{\sum_{s\in I}\frac{1}{n_s}\Big\}:I\subseteq\{1,\ldots,k\}\setminus\{t\}\Big\}\cap\frac{1}{n_t}\mathbb{Z} =\Big\{\frac{r}{n_t}:r=0,\ldots,n_t-1\Big\}. \tag{15}$$

Result (I) confirms the conjecture for exact m-covers of $\mathbb{Z}$. The lower bounds are best possible as is shown by the following example.

Example. Let $k>m>0$ be integers. Let $a_s=0$ and $n_s=1$ for $s=1,\ldots,m-1$, $a_s=2^{s-m}$ and $n_s=2^{s-m+1}$ for $s=m,\ldots,k-1$, also $a_k=0$ and $n_k=2^{k-m}$. It is clear that $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ forms an exact m-cover of $\mathbb{Z}$. As each nonnegative integer can be expressed uniquely in the binary form, the reader can easily check that for $a=0,1,2,\ldots$ and $t=1,\ldots,k$ we always have

$$\Big|\Big\{J\subseteq\{1,\ldots,k\}\setminus\{t\}:\sum_{s\in J}\frac{1}{n_s}=\frac{a}{n_t}\Big\}\Big|=\binom{m-1}{[a/n_t]}.$$

Corollary 3.2. *Suppose that $|I|=m$. Then no number occurs exactly once among the $2^{k-m}n_1\ldots n_m$ rationals*

$$\sum_{s\in I}\frac{x_s}{n_s},\quad x_s\in R(n_s)\ \text{for}\ s\in I;\qquad \sum_{s\in J}\frac{1}{n_s},\quad J\subseteq\bar I. \tag{16}$$

Proof. If $I^*(\sum_{s\in I} x_s/n_s) = 1$ where $x_s \in R(n_s)$ for $s \in I$ then $\bar{I}_*(\sum_{s\in I} x_s/n_s) \geq \binom{m-|I|}{0} = 1$ by Theorem 3.1(i). If $J \subseteq \bar{I}$ and $\bar{I}_*(\sum_{s\in J} 1/n_s) = 1$, then $I^*(\sum_{s\in J} 1/n_s) \geq \binom{|I|-m}{0} = 1$ by Theorem 3.1(ii). We are done.

Corollary 3.3. *Assume that $|I| \geq m$. For any $J \subseteq \bar{I}$, if*

$$\left|\sum_{s\in J'} \frac{1}{n_s} - \sum_{s\in J} \frac{1}{n_s}\right| \in \{0, 1, \ldots, |I| - m\} \quad \text{for no } J' \subseteq \bar{I} \text{ with } J' \neq J, \tag{17}$$

then

$$I^*\left(n + \sum_{s\in J} \frac{1}{n_s}\right) \geq \binom{|I| - m}{n} \quad \text{for } n = 0, 1, 2, \ldots \tag{18}$$

and hence

$$\prod_{s\in I} n_s \geq 2^{|I|-m}[n_s]_{s\in I}. \tag{19}$$

Proof. Let J be a subset of $\bar{I}$ which satisfies (17). Note that $\binom{|I|-m}{n} = 0$ for every integer $n > |I| - m$. For $n \in \mathbb{Z}$ with $0 \leq n \leq |I| - m$, if $J' \subseteq \bar{I}$ and $n' \in \{0, 1, \ldots, |I| - m\}$ then by (17),

$$n + \sum_{s\in J} \frac{1}{n_s} = n' + \sum_{s\in J'} \frac{1}{n_s} \Rightarrow J = J' \text{ and } n = n'.$$

So (18) holds in view of the latter part of Theorem 3.1, and thus by Proposition 2.2,

$$\begin{aligned}\frac{\prod_{s\in I} n_s}{[n_s]_{s\in I}} &\geq \left|\left\{\langle x_s\rangle_{s\in I} : x_s \in R(n_s) \text{ for } s \in I \ \&\ \sum_{s\in I} \frac{x_s}{n_s} \equiv \sum_{s\in J} \frac{1}{n_s} \pmod 1\right\}\right| \\ &\geq \sum_{n=0}^{|I|-m} I^*\left(n + \sum_{s\in J} \frac{1}{n_s}\right) \geq \sum_{n=0}^{|I|-m} \binom{|I| - m}{n} = 2^{|I|-m}.\end{aligned}$$

Putting $I = \{1, \ldots, k\}$ and $J = \emptyset$ in Corollary 3.3 we obtain the second half of result (III). When $1 \leq t \leq k$ and $n_t > 1$, Corollary 3.3 in the case $I = \{1, \ldots, k\} \setminus \{t\}$ and $J = \{t\}$ also yields an interesting result.

Let $p(1) = 1$ and $p(n)$ denote the smallest (positive) prime factor of n for $n = 2, 3, \ldots$ For a positive integer n we also put

$$D(n) = \Big\{\sum_{p\mid n} pm_p : \text{all the } m_p \text{ are nonnegative integers}\Big\}. \tag{20}$$

Theorem 3.2. *Let c be a rational number.*

(i) *If* $|I|\le m$, *then either*

$$\bar I_*(c)+\sum_{n=0}^{m-|I|} I^*(c-n)\ge p([n_1,\ldots,n_k]) \tag{21}$$

or

$$\bar I_*^{(0)}(c)-\bar I_*^{(1)}(c)=\sum_{n=0}^{m-|I|}(-1)^n\binom{m-|I|}{n}I^*(c-n); \tag{22}$$

moreover

$$\bar I_*(c)+\sum_{n=0}^{m-|I|}\binom{m-|I|}{n}I^*(c-n)\in D([n_1,\ldots,n_k]) \tag{23}$$

if $|S|,|T|\le 1$ *and* $S\cap T=\emptyset$ *where*

$$S=\{n \bmod 2 : n\in\mathbb{Z},\ 0\le n\le m-|I| \text{ and } I^*(c-n)\ne 0\}$$

and

$$T=\Big\{|J| \bmod 2 : J\subseteq \bar I \text{ and } \sum_{s\in J}\frac{1}{n_s}=c\Big\}.$$

(ii) *If* $|I|\ge m$, *then either*

$$I^*(c)+\sum_{n=0}^{|I|-m}\bar I_*(c-n)\ge p([n_1,\ldots,n_k]) \tag{24}$$

or

$$I^*(c)=\sum_{n=0}^{|I|-m}(-1)^n\binom{|I|-m}{n}(\bar I_*^{(0)}(c-n)-\bar I_*^{(1)}(c-n)); \tag{25}$$

furthermore

$$I^*(c)+\sum_{n=0}^{|I|-m}\binom{|I|-m}{n}\bar I_*(c-n)\in D([n_1,\ldots,n_k]) \tag{26}$$

if $c\ne n+\sum_{s\in J}1/n_s$ *for any* $n=0,1,\ldots,|I|-m$ *and* $J\subseteq\bar I$ *with* $n\equiv|J|$ (mod 2).

Corollary 3.4. *Let* $|I|\le m$ *and* $J\subseteq \bar I$. *Suppose that* $\sum_{s\in J}1/n_s$ *cannot be expressed in the form* $n+\sum_{s\in I}x_s/n_s$ *where* $n\in\{0,1,\ldots,m-|I|\}$ *and* $x_s\in R(n_s)$ *for* $s\in I$. *Put*

$$\mathcal{J}=\Big\{J'\subseteq\bar I : \sum_{s\in J'}\frac{1}{n_s}=\sum_{s\in J}\frac{1}{n_s}\Big\}.$$

Then either $|\mathcal{J}|\ge p([n_1,\ldots,n_k])$ *or* $|\mathcal{J}|\equiv 0$ (mod 2); *either* $|J'|\not\equiv|J|$ (mod 2) *for some* $J'\in\mathcal{J}$, *or* $|\mathcal{J}|$ *can be expressed as the sum of some* (*not necessarily distinct*) *prime divisors of* $[n_1,\ldots,n_k]$.

Proof. Let $c=\sum_{s\in J}1/n_s$. As $\bar I_*(c)=\bar I_*^{(0)}(c)+\bar I_*^{(1)}(c)$, and $I^*(c-n)=0$ for every $n=0,1,\ldots,m-|I|$, the desired results follow from Theorem 3.2(i).

Remark 2. In the case $I=\emptyset$ Corollary 3.4 was obtained by the author in [Su4].

COROLLARY 3.5. *Assume that $|I|=m$. Let l be the total number of ways in which the rational c is expressed in the form $\sum_{s\in I}x_s/n_s$ or $\sum_{s\in\bar I}\delta_s/n_s$ where $x_s\in R(n_s)$ for $s\in I$ and $\delta_s\in\{0,1\}$ for $s\in\bar I$. Then we have*

$$l\ge p([n_1,\ldots,n_k])\quad\text{or}\quad l=2\left|\left\{J\subseteq\bar I:\sum_{s\in J}\frac{1}{n_s}=c\right\}\right|,\tag{27}$$

and l can be written as the sum of finitely many (not necessarily distinct) prime divisors of $n_1,\ldots,n_k$ providing $\sum_{s\in J}1/n_s=c$ for no $J\subseteq\bar I$ with $|J|\equiv 0\pmod 2$.

Proof. Obviously $l=I^*(c)+\bar I_*(c)$, and (22) or (25) says that $\bar I_*^{(0)}(c)-\bar I_*^{(1)}(c)=I^*(c)$, i.e. $l=2\bar I_*^{(0)}(c)$. Therefore Theorem 3.2 yields Corollary 3.5.

COROLLARY 3.6. *Let $|I|\ge m$. Suppose that $\sum_{s\in I}m_s/n_s$ cannot be expressed in the form $n+\sum_{s\in J}1/n_s$ with $n\in\{0,1,\ldots,|I|-m\}$ and $J\subseteq\bar I$, where $m_s\in R(n_s)$ for each $s\in I$. Then*

$$\left|\left\{\langle x_s\rangle_{s\in I}:x_s\in R(n_s)\text{ for }s\in I\text{ and }\sum_{s\in I}\frac{x_s}{n_s}=\sum_{s\in I}\frac{m_s}{n_s}\right\}\right|\tag{28}$$

must be a finite sum of (not necessarily distinct) prime divisors of $[n_1,\ldots,n_k]$.

Proof. Let $c=\sum_{s\in I}m_s/n_s$. Note that $\bar I_*(c-n)=0$ for each $n=0,1,\ldots,|I|-m$. By Theorem 3.2(ii), $I^*(c)$ belongs to $D([n_1,\ldots,n_k])$.

Clearly Corollary 3.6 in the case $I=\{1,\ldots,k\}$ gives the first half of result (III).

THEOREM 3.3. (i) *If $(n_s,n_t)\,|\,a_s-a_t$ for all $s,t\in I$, then*

$$\sum_{n=0}^{m-1}\bar I_*\left(n+\frac{r}{[n_s]_{s\in I}}\right)=\left|\left\{J\subseteq\bar I:\left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\frac{r}{[n_s]_{s\in I}}\right\}\right|\ge\frac{\prod_{s\in I}n_s}{[n_s]_{s\in I}}\tag{29}$$

for each $r=0,1,\ldots,[n_s]_{s\in I}-1$.

(ii) *Assume* $|I| = m$, $0 \le \theta < 1$, *and* $[n_s]_{s\in I}\theta \notin \mathbb{Z}$ *or* $(n_i, n_j) \nmid a_i - a_j$ *for some* $i, j \in I$. *Then either*

$$\sum_{n=0}^{m-1} \bar{I}_*(n+\theta) = \left|\left\{J \subseteq \bar{I} : \left\{\sum_{s\in J} \frac{1}{n_s}\right\} = \theta\right\}\right| \ge p([n_s]_{s\in\bar{I}}) \tag{30}$$

or

$$\left|\left\{J \subseteq \bar{I} : 2 \mid |J| \ \&\ \left\{\sum_{s\in J} \frac{1}{n_s}\right\} = \theta\right\}\right| = \left|\left\{J \subseteq \bar{I} : 2 \nmid |J| \ \&\ \left\{\sum_{s\in J} \frac{1}{n_s}\right\} = \theta\right\}\right|$$

and hence

$$\sum_{n=0}^{m-1} \bar{I}_*(n+\theta) = \left|\left\{J \subseteq \bar{I} : \left\{\sum_{s\in J} \frac{1}{n_s}\right\} = \theta\right\}\right| \equiv 0 \pmod 2; \tag{31}$$

moreover,

$$\sum_{n=0}^{m-1} \bar{I}_*(n+\theta) = \left|\left\{J \subseteq \bar{I} : \left\{\sum_{s\in J} \frac{1}{n_s}\right\} = \theta\right\}\right| \in D([n_s]_{s\in\bar{I}}) \tag{32}$$

if all the $|J| \bmod 2$ *with* $J \subseteq \bar{I}$ *and* $\{\sum_{s\in J} 1/n_s\} = \theta$ *are the same.*

Remark 3. When those n_s with $s \in I$ are pairwise relatively prime, Theorem 3.3(i) yields the lower bound 1 in (29) while (14) gives the bound $2^{m-|I|}$.

COROLLARY 3.7. *If* $I \ne \emptyset$ *and* $(n_s, n_t) \mid a_s - a_t$ *for all* $s, t \in I$, *then*

$$\prod_{s\in I} n_s \le 2^{k-|I|}, \qquad [n_s]_{s\in I} \mid [n_s]_{s\in\bar{I}}, \tag{33}$$

and

$$\left\{\left\{\sum_{s\in J} \frac{1}{n_s}\right\} : J \subseteq \bar{I}\right\} \supseteq \left\{0, \frac{1}{[n_s]_{s\in I}}, \ldots, \frac{[n_s]_{s\in I} - 1}{[n_s]_{s\in I}}\right\}. \tag{34}$$

Proof. (34) follows immediately from Theorem 3.3(i). Since $\sum_{s\in J} 1/n_s \equiv 1/[n_s]_{s\in I} \pmod 1$ for some $J \subseteq \bar{I}$, $[n_s]_{s\in I}$ must divide $[n_s]_{s\in\bar{I}}$. For the inequality in (33) we notice that

$$\begin{aligned}
2^{k-|I|} &\ge \left|\bigcup_{r=0}^{[n_s]_{s\in I}-1} \left\{J \subseteq \bar{I} : \left\{\sum_{s\in I} \frac{1}{n_s}\right\} = \frac{r}{[n_s]_{s\in I}}\right\}\right| \\
&= \sum_{r=0}^{[n_s]_{s\in I}-1} \left|\left\{J \subseteq \bar{I} : \left\{\sum_{s\in I} \frac{1}{n_s}\right\} = \frac{r}{[n_s]_{s\in I}}\right\}\right| \\
&\ge \sum_{r=0}^{[n_s]_{s\in I}-1} \frac{\prod_{s\in I} n_s}{[n_s]_{s\in I}} = \prod_{s\in I} n_s.
\end{aligned}$$

R e m a r k 4. By checking (33) and (34) with I taken to be $K=\{1,\dots,m-1,k\}$ and $K\cup\{k-1\}$ in the previous example, we find that Corollary 3.7 is sharp. When (1) forms an exact 1-cover of $\mathbb{Z}$ and $I\subseteq\{1,\dots,k\}$ contains at least two elements, we cannot have $(n_s,n_t)\mid a_s-a_t$ for all $s,t\in I$ with $s\neq t$, and (34) fails to hold because for all $J\subseteq\bar{I}$ we have

$$\sum_{s\in J}\frac{1}{n_s}\le\sum_{s\in\bar{I}}\frac{1}{n_s}=1-\sum_{s\in I}\frac{1}{n_s}<1-\frac{1}{[n_s]_{s\in I}}=\frac{[n_s]_{s\in I}-1}{[n_s]_{s\in I}}.$$

For any $a,n\in\mathbb{Z}$ with $n>0$, each integer in $a+n\mathbb{Z}$ belongs to exactly m members of (1) and hence

$$A_{a(n)}=\left\{b_s+\frac{n_s}{(n,n_s)}\mathbb{Z}\right\}_{s\in J}$$

also forms an exact m-cover of $\mathbb{Z}$ where $J=\{1\le s\le k:(n,n_s)\mid a-a_s\}$, $b_s\in\mathbb{Z}$ and $a+b_sn\equiv a_s\pmod{n_s}$ for $s\in J$. Instead of $A=A_{0(1)}$ we may apply our results to $A_{a(n)}$ so as to get more general ones. See [Su4] for examples of such transformations.

4. Characterizations of exact m-covers of $\mathbb{Z}$

THEOREM 4.1. *Let* (1) *be a system of arithmetic sequences. Let* $I\subseteq\{1,\dots,k\}$ *and* $\bar{I}=\{1,\dots,k\}\setminus I$. *If* $|I|\le m$ *then* (1) *is an exact m-cover of* $\mathbb{Z}$ *if and only if*

$$\sum_{\substack{J\subseteq\bar{I}\\ \sum_{s\in J}1/n_s=c}}(-1)^{|J|}e^{2\pi i\sum_{s\in J}a_s/n_s}=\sum_{n=0}^{m-|I|}(-1)^n\binom{m-|I|}{n}\sum_{\substack{x_s\in R(n_s)\ for\ s\in I\\ \sum_{s\in I}x_s/n_s=c-n}}e^{2\pi i\sum_{s\in I}a_sx_s/n_s}\tag{35}$$

is valid for all rational $c\ge 0$. *If* $|I|\ge m$, *then* (1) *forms an exact m-cover of* $\mathbb{Z}$ *if and only if*

$$\sum_{\substack{x_s\in R(n_s)\ for\ s\in I\\ \sum_{s\in I}x_s/n_s=c}}e^{2\pi i\sum_{s\in I}a_sx_s/n_s}=\sum_{n=0}^{|I|-m}(-1)^n\binom{|I|-m}{n}\sum_{\substack{J\subseteq\bar{I}\\ \sum_{s\in J}1/n_s=c-n}}(-1)^{|J|}e^{2\pi i\sum_{s\in J}a_s/n_s}\tag{36}$$

holds for all rational $c\ge 0$.

Proof. Put $N=[n_1,\dots,n_k]$. We assert that (1) forms an exact m-cover of $\mathbb{Z}$ if and only if we have the identity

$$\prod_{s=1}^{k}(1-z^{N/n_s}e^{2\pi ia_s/n_s})=(1-z^N)^m. \tag{37}$$

Apparently any zero of the left hand side of (37) is an Nth root of unity. Observe that for every integer x the number $e^{-2\pi ix/N}$ is a zero of the left hand side of (37) with multiplicity m if and only if x lies in $a_s+n_s\mathbb{Z}$ for exact m of $s=1,\dots,k$. So the assertion follows from Viète's theorem.

Now consider the case $|I|\le m$. Clearly the following identities are equivalent:

$$\prod_{s=1}^{k}(1-z^{N/n_s}e^{2\pi ia_s/n_s})=(1-z^N)^{m-|I|}\prod_{s\in I}(1-(z^{N/n_s}e^{2\pi ia_s/n_s})^{n_s}),$$

$$\prod_{s\in\bar I}(1-z^{N/n_s}e^{2\pi ia_s/n_s})=(1-z^N)^{m-|I|}\prod_{s\in I}\sum_{m_s=0}^{n_s-1}z^{m_sN/n_s}e^{2\pi im_sa_s/n_s},$$

$$\sum_{J\subseteq\bar I}(-1)^{|J|}z^{\sum_{s\in J}N/n_s}e^{2\pi i\sum_{s\in J}a_s/n_s}$$
$$=\sum_{n=0}^{m-|I|}(-1)^n\binom{m-|I|}{n}z^{nN}\prod_{s\in I}\sum_{m_s=0}^{n_s-1}z^{m_sN/n_s}e^{2\pi ia_sm_s/n_s}.$$

By the assertion the first one holds if and only if (1) forms an exact m-cover of $\mathbb{Z}$. Since the third one is valid if and only if (35) is true for every rational $c\ge 0$, we get the desired result.

For the case $|I|\ge m$, that (1) forms an exact m-cover of $\mathbb{Z}$ is equivalent to any of the identities given below:

$$\prod_{s\in\bar I}(1-z^{N/n_s}e^{2\pi ia_s/n_s})\cdot\prod_{s\in I}(1-z^{N/n_s}e^{2\pi ia_s/n_s})=(1-z^N)^m,$$

$$\prod_{s\in\bar I}(1-z^{N/n_s}e^{2\pi ia_s/n_s})\cdot\prod_{s\in I}(1-(z^{N/n_s}e^{2\pi ia_s/n_s})^{n_s})$$
$$=(1-z^N)^m\prod_{s\in I}\sum_{m_s=0}^{n_s-1}z^{m_sN/n_s}e^{2\pi ia_sm_s/n_s},$$

$$\sum_{n=0}^{|I|-m}(-1)^n\binom{|I|-m}{n}z^{nN}\sum_{J\subseteq\bar I}(-1)^{|J|}z^{\sum_{s\in J}N/n_s}e^{2\pi i\sum_{s\in J}a_s/n_s}$$
$$=\prod_{s\in I}\sum_{m_s=0}^{n_s-1}z^{m_sN/n_s}e^{2\pi ia_sm_s/n_s}.$$

As the last one holds if and only if (36) does for all rational $c \geq 0$, we are done.

Remark 5. In the case $I = \emptyset$ and $c \in \{1, \ldots, m\}$, that (35) holds for any exact m-cover (1) of $\mathbb{Z}$ was first observed by the author in [Su2] with the help of the Riemann zeta function.

The characterization of exact m-cover (1) of $\mathbb{Z}$ given in Theorem 4.1 involves a fixed subset I of $\{1, \ldots, k\}$. Now we present a new one which depends on all the $I \subseteq \{1, \ldots, k\}$ with $|I| = m$.

THEOREM 4.2. *Let* (1) *be a system of arithmetic sequences. Then* (1) *forms an exact m-cover of $\mathbb{Z}$ if and only if the relation*

$$\sum_{\substack{J \subseteq \{1, \ldots, k\} \setminus I \\ \{\sum_{s \in J} 1/n_s\} = \theta}} (-1)^{|J|} e^{2\pi i \sum_{s \in J} a_s/n_s} = \sum_{\substack{x_s \in R(n_s) \text{ for } s \in I \\ \{\sum_{s \in I} x_s/n_s\} = \theta}} e^{2\pi i \sum_{s \in I} a_s x_s/n_s} \tag{38}$$

holds for all $\theta \in [0, 1)$ and $I \subseteq \{1, \ldots, k\}$ with $|I| = m$.

Proof. Let $N = [n_1, \ldots, n_k]$ and $\bar{I} = \{1, \ldots, k\} \setminus I$ for all $I \subseteq \{1, \ldots, k\}$. First suppose that (1) forms an m-cover of $\mathbb{Z}$. Let x be any integer and I a subset of $\{1, \ldots, k\}$ with $|I| = m$. By taking $z = r^{1/N} e^{2\pi i x/N}$ in (37), we get the equality

$$\prod_{s=1}^{k} (1 - r^{1/n_s} e^{2\pi i (x + a_s)/n_s}) = (1 - r)^m$$

for all $r \geq 0$. If $I = \{1 \leq s \leq k : n_s \mid x + a_s\}$, then

$$\begin{aligned}
&\prod_{s \in \bar{I}} (1 - e^{2\pi i (x + a_s)/n_s}) \Big/ \prod_{s \in I} \sum_{x_s = 0}^{n_s - 1} e^{2\pi i (x + a_s) x_s/n_s} \\
&= \lim_{r \to 1} \prod_{s \in \bar{I}} (1 - r^{1/n_s} e^{2\pi i (x + a_s)/n_s}) \Big/ \prod_{s \in I} \lim_{\bar{r} \to e^{2\pi i (x + a_s)/n_s}} \frac{1 - \bar{r}^{n_s}}{1 - (\bar{r}^{n_s})^{1/n_s}} \\
&= \lim_{r \to 1} \prod_{s \in \bar{I}} (1 - r^{1/n_s} e^{2\pi i (x + a_s)/n_s}) \cdot \prod_{s \in I} \frac{1 - r^{1/n_s}}{1 - r} \\
&= \lim_{r \to 1} (1 - r)^{-|I|} \prod_{s=1}^{k} (1 - r^{1/n_s} e^{2\pi i (x + a_s)/n_s}) \\
&= \lim_{r \to 1} (1 - r)^{-|I|} (1 - r)^m = 1.
\end{aligned}$$

If $I\ne\{1\le s\le k: n_s\,|\,x+a_s\}$, then $n_s\,|\,x+a_s$ for some $s\in\bar I$ and $n_t\nmid x+a_t$ for some $t\in I$, therefore

$$\prod_{s\in\bar I}(1-e^{2\pi i(x+a_s)/n_s})=0=\prod_{t\in I}\sum_{x_t=0}^{n_t-1}e^{2\pi i(x+a_t)x_t/n_t}.$$

So we always have the identity

$$\prod_{s\in\bar I}(1-e^{2\pi i(x+a_s)/n_s})=\prod_{s\in I}\sum_{x_s=0}^{n_s-1}e^{2\pi i(x+a_s)x_s/n_s}. \tag{39}$$

Next assume (39) for all $x\in\mathbb{Z}$ and $I\subseteq\{1,\dots,k\}$ with $|I|=m$. For each integer x, if $|\{1\le s\le k: n_s\,|\,x+a_s\}|>m$, then we can choose a proper subset I of $\{1\le s\le k: n_s\,|\,x+a_s\}$ with cardinality m for which the left hand side of (39) is zero but the right hand side of (39) is nonzero; if $|\{1\le s\le k: n_s\,|\,x+a_s\}|<m$, then we can select an $I\supset\{1\le s\le k: n_s\,|\,x+a_s\}$ with $|I|=m$ for which the left hand side of (39) is nonzero while the right hand side of (39) is zero. So (1) forms an exact m-cover of $\mathbb{Z}$.

Now let I be any subset of $\{1,\dots,k\}$ with $|I|=m$. For every $x\in\mathbb{Z}$,

$$\begin{aligned}\prod_{s\in\bar I}(1-e^{2\pi i(x+a_s)/n_s})&=\sum_{J\subseteq\bar I}(-1)^{|J|}e^{2\pi i(\sum_{s\in J}a_s/n_s+x\sum_{s\in J}1/n_s)}\\&=\sum_{r=0}^{N-1}e^{2\pi irx/N}\sum_{\substack{J\subseteq\bar I\\ \{\sum_{s\in J}1/n_s\}=r/N}}(-1)^{|J|}e^{2\pi i\sum_{s\in J}a_s/n_s}\end{aligned}$$

while $\prod_{s\in I}\sum_{x_s=0}^{n_s-1}e^{2\pi i(x+a_s)x_s/n_s}$ coincides with

$$\begin{aligned}&\sum_{x_s\in R(n_s)\text{ for }s\in I}e^{2\pi i(\sum_{s\in I}a_sx_s/n_s+x\sum_{s\in I}x_s/n_s)}\\&\quad=\sum_{r=0}^{N-1}e^{2\pi irx/N}\sum_{\substack{x_s\in R(n_s)\text{ for }s\in I\\ \{\sum_{s\in I}x_s/n_s\}=r/N}}e^{2\pi i\sum_{s\in I}a_sx_s/n_s}.\end{aligned}$$

If (38) holds for all $\theta\in[0,1)$ then (39) follows from the above for each $x\in\mathbb{Z}$. Conversely, providing (39) for all $x\in\mathbb{Z}$, for any $a=0,1,\dots,N-1$ we have

$$\begin{aligned}&N\sum_{\substack{J\subseteq\bar I\\ \{\sum_{s\in J}1/n_s\}=a/N}}(-1)^{|J|}e^{2\pi i\sum_{s\in J}a_s/n_s}\\&\quad=\sum_{r=0}^{N-1}\sum_{\substack{J\subseteq\bar I\\ \{\sum_{s\in J}1/n_s\}=r/N}}(-1)^{|J|}e^{2\pi i\sum_{s\in J}a_s/n_s}\sum_{x=0}^{N-1}e^{2\pi i(r-a)x/N}\end{aligned}$$

$$= \sum_{x=0}^{N-1} e^{-2\pi iax/N}\Big(\sum_{r=0}^{N-1} e^{2\pi irx/N} \sum_{\substack{J\subseteq \bar I\\ \{\sum_{s\in J} 1/n_s\}=r/N}} (-1)^{|J|}e^{2\pi i\sum_{s\in J} a_s/n_s}\Big)$$

$$= \sum_{x=0}^{N-1} e^{-2\pi iax/N}\Big(\sum_{r=0}^{N-1} e^{2\pi irx/N} \sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \{\sum_{s\in I} x_s/n_s\}=r/N}} e^{2\pi i\sum_{s\in I} a_sx_s/n_s}\Big)$$

$$= \sum_{r=0}^{N-1} \sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \{\sum_{s\in I} x_s/n_s\}=r/N}} e^{2\pi i\sum_{s\in I} a_sx_s/n_s} \sum_{x=0}^{N-1} e^{2\pi i(r-a)x/N}$$

$$= N \sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \{\sum_{s\in I} x_s/n_s\}=a/N}} e^{2\pi i\sum_{s\in I} a_sx_s/n_s},$$

therefore (38) is valid for every $\theta\in[0,1)$.

Combining the above we obtain the desired result.

5. Proofs of Theorems 3.1–3.3

Proof of Theorem 3.1. (i) Assume $|I|\le m$ and $I^*(c-n)=1$ where n is a nonnegative integer. Let $\langle m_s\rangle_{s\in I}$ be the unique tuple for which $\sum_{s\in I} m_s/n_s = c-n$ and $m_s\in R(n_s)$ for all $s\in I$. Since $\binom{m-|I|}{n}=0$ if $n>m-|I|$, by (35) we have

$$\sum_{\substack{J\subseteq \bar I\\ \sum_{s\in J} 1/n_s=c}} (-1)^{|J|}e^{2\pi i\sum_{s\in J} a_s/n_s} - (-1)^n\binom{m-|I|}{n} e^{2\pi i\sum_{s\in I} a_sm_s/n_s}$$

$$= \sum_{\substack{l=0\\ l\ne n}}^{m-|I|} (-1)^l\binom{m-|I|}{l} \sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \sum_{s\in I} x_s/n_s=c-l}} e^{2\pi i\sum_{s\in I} a_sx_s/n_s}.$$

Therefore $\bar I_*(c)+\sum_{l=0,\,l\ne n}^{m-|I|}\binom{m-|I|}{l} I^*(c-l)$ is greater than or equal to

$$\Bigg|\sum_{\substack{J\subseteq \bar I\\ \sum_{s\in J} 1/n_s=c}} (-1)^{|J|}e^{2\pi i\sum_{s\in J} a_s/n_s} - \sum_{\substack{l=0\\ l\ne n}}^{m-|I|} (-1)^l\binom{m-|I|}{l} \sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \sum_{s\in I} x_s/n_s=c-l}} e^{2\pi i\sum_{s\in I} a_sx_s/n_s}\Bigg|$$

$$= \Bigg|(-1)^n\binom{m-|I|}{n} e^{2\pi i\sum_{s\in I} a_sm_s/n_s}\Bigg| = \binom{m-|I|}{n}.$$

(ii) Now we suppose $|I| \ge m$ and $\bar{I}_*(c-n) = 1$ where n is a nonnegative integer. Let I' be the unique subset of $\bar{I}$ such that $\sum_{s\in I'} 1/n_s = c-n$. By (36) we have

$$\sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \sum_{s\in I} x_s/n_s = c}} e^{2\pi i\sum_{s\in I} a_s x_s/n_s} - (-1)^n\binom{|I|-m}{n}(-1)^{|I'|}e^{2\pi i\sum_{s\in I'} a_s/n_s}$$
$$= \sum_{\substack{l=0\\ l\ne n}}^{|I|-m} (-1)^l\binom{|I|-m}{l}\sum_{\substack{J\subseteq \bar{I}\\ \sum_{s\in J} 1/n_s = c-l}} (-1)^{|J|}e^{2\pi i\sum_{s\in J} a_s/n_s}.$$

Thus (11) follows.

Lemma. *Let $c_1,\ldots,c_k$ be nonnegative integers and $d_1,\ldots,d_l$ positive integers. Assume that there exist nonzero numbers $z_1,\ldots,z_k$ for which $\sum_{s=1}^k c_s z_s^t = 0$ for those positive integers t not divisible by any of $d_1,\ldots,d_l$. Then $c_1+\ldots+c_k$ is the sum of some* (*not necessarily distinct*) *numbers among $d_1,\ldots,d_l$.*

This is Lemma 9 of [Su4] and the initial idea is due to Y.-G. Chen.

Proof of Theorem 3.2. Let d be an integer prime to $N = [n_1,\ldots,n_k]$. Since any integer can be written in the form $dx+Ny$ where $x,y\in\mathbb{Z}$, and $dx+Ny\equiv da_s \pmod{n_s}$ if and only if $x\equiv a_s \pmod{n_s}$, it follows that $A_d = \{da_s+n_s\mathbb{Z}\}_{s=1}^k$ also forms an exact m-cover of $\mathbb{Z}$. When $|I|\le m$, by applying Theorem 4.1 to A_d we get

$$\sum_{\substack{J\subseteq \bar{I}\\ \sum_{s\in J} 1/n_s = c}} (-1)^{|J|}e^{2\pi i d\sum_{s\in J} a_s/n_s}$$
$$= \sum_{n=0}^{m-|I|} (-1)^n\binom{m-|I|}{n}\sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \sum_{s\in I} x_s/n_s = c-n}} e^{2\pi i d\sum_{s\in I} a_s x_s/n_s},$$

that is, $\sum_{w\in W_1} B_1(c,w)e^{2\pi i dw}$ is zero, where W_1 is the union of the sets

$$\left\{\left\{\sum_{s\in J}\frac{a_s}{n_s}\right\} : J\subseteq \bar{I}\ \&\ \sum_{s\in J}\frac{1}{n_s} = c\right\}$$

and

$$\left\{\left\{\sum_{s\in I}\frac{a_s x_s}{n_s}\right\} : x_s\in R(n_s)\text{ for } s\in I,\ c-\sum_{s\in I}\frac{x_s}{n_s}\in\{0,1,\ldots,m-|I|\}\right\},$$

and

$$B_1(c,w) = \sum_{\substack{J\subseteq \bar I\\ \sum_{s\in J}1/n_s=c\\ \{\sum_{s\in J}a_s/n_s\}=w}} (-1)^{|J|}$$
$$- \sum_{n=0}^{m-|I|}(-1)^n\binom{m-|I|}{n}\Bigg|\Bigg\{\langle x_s\rangle_{s\in I}: x_s\in R(n_s)\text{ for } s\in I,$$
$$\sum_{s\in I}\frac{x_s}{n_s}=c-n\ \&\ \Bigg\{\sum_{s\in I}\frac{a_sx_s}{n_s}\Bigg\}=w\Bigg\}\Bigg|$$

for $w\in W_1$. If $|I|\ge m$, then by applying Theorem 4.1 to A_d we obtain the equality

$$\sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \sum_{s\in I}x_s/n_s=c}} e^{2\pi id\sum_{s\in I}a_sx_s/n_s}$$
$$=\sum_{n=0}^{|I|-m}(-1)^n\binom{|I|-m}{n}\sum_{\substack{J\subseteq\bar I\\ \sum_{s\in J}1/n_s=c-n}}(-1)^{|J|}e^{2\pi id\sum_{s\in J}a_s/n_s},$$

i.e., $\sum_{w\in W_2}B_2(c,w)e^{2\pi idw}=0$, where W_2 is the union of

$$\Bigg\{\Bigg\{\sum_{s\in J}\frac{a_s}{n_s}\Bigg\}: J\subseteq\bar I \text{ and } \sum_{s\in J}\frac{1}{n_s}=c-n\text{ for some } n=0,1,\ldots,|I|-m\Bigg\}$$

and

$$\Bigg\{\Bigg\{\sum_{s\in I}\frac{a_sx_s}{n_s}\Bigg\}: x_s\in R(n_s)\text{ for } s\in I\text{ and }\sum_{s\in I}\frac{x_s}{n_s}=c\Bigg\},$$

and

$$B_2(c,w)$$
$$=\Bigg|\Bigg\{\langle x_s\rangle_{s\in I}: x_s\in R(n_s)\text{ for } s\in I,\ \sum_{s\in I}\frac{x_s}{n_s}=c\ \&\ \Bigg\{\sum_{s\in I}\frac{a_sx_s}{n_s}\Bigg\}=w\Bigg\}\Bigg|$$
$$-\sum_{n=0}^{|I|-m}(-1)^n\binom{|I|-m}{n}\sum_{\substack{J\subseteq\bar I\\ \sum_{s\in J}1/n_s=c-n\\ \{\sum_{s\in J}a_s/n_s\}=w}}(-1)^{|J|}$$

for $w\in W_2$.

Case 1: $|I| \le m$. In this case (22) and (23) are obvious if $W_1 = \emptyset$. Suppose that W_1 is nonempty. If the inequality

$$\bar{I}_*(c) + \sum_{n=0}^{m-|I|} I^*(c-n) \ge |W_1| \ge p(N)$$

fails or $N = 1$, then $\sum_{w \in W_1} B_1(c,w)e^{2\pi i d w} = 0$ for every $d = 1, \dots, |W_1|$. Since

$$|(e^{2\pi i d w})_{1 \le d \le |W_1|,\, w \in W_1}| / \prod_{w \in W_1} e^{2\pi i w}$$

is a determinant of Vandermonde's type, $B_1(c,w) = 0$ for all $w \in W_1$ and hence (22) follows. When $|S|, |T| \le 1$ and $S \cap T = \emptyset$ where S and T are as in Theorem 3.2, there is an $\varepsilon \in \{1, -1\}$ such that

$$\begin{aligned}
\varepsilon B_1(c,w) &= |B_1(c,w)| \\
&= \left|\left\{J \subseteq \bar{I} : \sum_{s \in J} \frac{1}{n_s} = c \ \&\ \left\{\sum_{s \in J} \frac{a_s}{n_s}\right\} = w\right\}\right| \\
&\quad + \sum_{n=0}^{m-|I|} \binom{m-|I|}{n} \left|\left\{\langle x_s \rangle_{s \in I} : x_s \in R(n_s) \text{ for } s \in I,\right.\right. \\
&\qquad\qquad \left.\left.\sum_{s \in I} \frac{x_s}{n_s} = c - n \ \&\ \left\{\sum_{s \in I} \frac{a_s x_s}{n_s}\right\} = w\right\}\right|
\end{aligned}$$

for every $w \in W_1$. If $N \ne 1$ then $\sum_{w \in W_1} |B_1(c,w)|(e^{2\pi i w})^d = 0$ for all positive integers d divisible by none of prime divisors of N and therefore by the Lemma

$$\bar{I}_*(c) + \sum_{n=0}^{m-|I|} \binom{m-|I|}{n} I^*(c-n) = \sum_{w \in W_1} |B_1(c,w)| \in D(N).$$

If $N = 1$ then the last equality also holds because $B_1(c,w) = 0$ for every $w \in W_1$.

Case 2: $|I| \ge m$. Apparently (25) and (26) are valid if $W_2 = \emptyset$. Now assume $|W_2| \ge 1$. If the equality

$$I^*(c) + \sum_{n=0}^{|I|-m} \bar{I}_*(c-n) \ge |W_2| \ge p(N)$$

fails or N equals one, then $\sum_{w \in W_2} B_2(c,w)e^{2\pi i d w} = 0$ for every $d = 1, \dots, |W_2|$, hence $B_2(c,w) = 0$ for all $w \in W_2$ and so we have (25). If $c \ne n + \sum_{s \in J} 1/n_s$ for each $n = 0, 1, \dots, |I| - m$ and $J \subseteq \bar{I}$ with $n \equiv |J| \pmod 2$, then

$$\begin{aligned} &B_2(c,w) \\ &\quad = \Big|\Big\{\langle x_s\rangle_{s\in I} : x_s \in R(n_s) \text{ for } s \in I,\ \sum_{s\in I}\frac{x_s}{n_s} = c\ \&\ \Big\{\sum_{s\in I}\frac{a_s x_s}{n_s}\Big\} = w\Big\}\Big| \\ &\qquad + \sum_{n=0}^{|I|-m}\binom{|I|-m}{n}\Big|\Big\{J \subseteq \bar I : \sum_{s\in J}\frac{1}{n_s} = c-n\ \&\ \Big\{\sum_{s\in J}\frac{a_s}{n_s}\Big\} = w\Big\}\Big| \end{aligned}$$

for all $w \in W_2$, so with the help of the Lemma, whether $N = 1$ or not, (26) always holds.

P r o o f o f T h e o r e m 3.3. (i) First suppose $|I| = m$. Let $r \in R([n_s]_{s\in I})$. In the light of Theorem 4.2,

$$\sum_{\substack{J\subseteq \bar I\\ \{\sum_{s\in J}1/n_s\}=r/[n_s]_{s\in I}}} (-1)^{|J|}e^{2\pi i\sum_{s\in J}a_s/n_s} = \sum_{\substack{x_s\in R(n_s)\text{ for } s\in I\\ \{\sum_{s\in I}x_s/n_s\}=r/[n_s]_{s\in I}}} e^{2\pi i\sum_{s\in I}a_s x_s/n_s}.$$

As $(n_s, n_t) \mid a_s - a_t$ for all $s, t \in I$, by Corollary 2.3 the absolute value of the right hand side is $\prod_{s\in I} n_s/[n_s]_{s\in I}$. So

$$\begin{aligned} \sum_{n=0}^{m-1}\bar I_*\Big(n + \frac{r}{[n_s]_{s\in I}}\Big) &= \Big|\Big\{J \subseteq \bar I : \Big\{\sum_{s\in J}\frac{1}{n_s}\Big\} = \frac{r}{[n_s]_{s\in I}}\Big\}\Big| \\ &\ge \Big|\sum_{\substack{J\subseteq \bar I\\ \{\sum_{s\in J}1/n_s\}=r/[n_s]_{s\in I}}} (-1)^{|J|}e^{2\pi i\sum_{s\in J}a_s/n_s}\Big| \\ &= \frac{\prod_{s\in I} n_s}{[n_s]_{s\in I}}. \end{aligned}$$

Next we handle the case where $|I| \ne m$. Choose an integer x such that $x \in \bigcap_{s\in I} a_s + n_s\mathbb{Z}$ if $I \ne \emptyset$. Let

$$I' = \{1 \le s \le k : x \equiv a_s \pmod{n_s}\}.$$

Then $|I'| = m$ and $I' \supset I$. By the previous argument,

$$\Big|\Big\{J \subseteq \{1,\dots,k\}\setminus I' : \Big\{\sum_{s\in J}\frac{1}{n_s}\Big\} = \frac{a}{[n_s]_{s\in I'}}\Big\}\Big| \ge \frac{\prod_{s\in I'} n_s}{[n_s]_{s\in I'}}$$

for every $a \in R([n_s]_{s\in I'})$. So, for any $r \in R([n_s]_{s\in I})$, we have

$$\begin{aligned}&\bigg|\bigg\{J\subseteq \bar I:\bigg\{\sum_{s\in J}\frac{1}{n_s}\bigg\}=\frac{r}{[n_s]_{s\in I}}\bigg\}\bigg|\\
&\qquad\ge\bigg|\bigg\{J\subseteq\{1,\dots,k\}\setminus I':\bigg\{\sum_{s\in J}\frac{1}{n_s}\bigg\}=\frac{r[n_s]_{s\in I'}/[n_s]_{s\in I}}{[n_s]_{s\in I'}}\bigg\}\bigg|\\
&\qquad\ge\frac{\prod_{s\in I'}n_s}{[n_s]_{s\in I'}}=\frac{\prod_{s\in I}n_s\cdot\prod_{s\in I'\setminus I}n_s}{[[n_s]_{s\in I},[n_s]_{s\in I'\setminus I}]}\ge\frac{\prod_{s\in I}n_s}{[n_s]_{s\in I}}.\end{aligned}$$

(ii) If $[n_s]_{s\in I}\theta\notin\mathbb{Z}$, then $\{\sum_{s\in I}x_s/n_s\}\ne\theta$ whenever $x_s\in R(n_s)$ for all $s\in I$, and thus by Theorem 4.2,

$$(*)\qquad \sum_{w\in W}e^{2\pi i w}\sum_{\substack{J\subseteq\bar I\\ \{\sum_{s\in J}1/n_s\}=\theta\\ \{\sum_{s\in J}a_s/n_s\}=w}}(-1)^{|J|}=\sum_{\substack{J\subseteq\bar I\\ \{\sum_{s\in J}1/n_s\}=\theta}}(-1)^{|J|}e^{2\pi i\sum_{s\in J}a_s/n_s}=0$$

where

$$W=\bigg\{\bigg\{\sum_{s\in J}\frac{a_s}{n_s}\bigg\}:J\subseteq\bar I\text{ and }\bigg\{\sum_{s\in J}\frac{1}{n_s}\bigg\}=\theta\bigg\}.$$

If $(n_{s_1},n_{s_2})\nmid a_{s_1}-a_{s_2}$ for some $s_1,s_2\in I$, then $\{a_s+n_s\mathbb{Z}\}_{s\in I}$ covers each integer at most $m-1$ times because $a_{s_1}+n_{s_1}\mathbb{Z}\cap a_{s_2}+n_{s_2}\mathbb{Z}=\emptyset$, therefore system $\{a_s+n_s\mathbb{Z}\}_{s\in\bar I}$ forms a cover of $\mathbb{Z}$ and $(*)$ holds by Theorem 1 of [Su3]. For each integer a prime to $[n_s]_{s\in\bar I}$, by applying the automorphism σ_a of the cyclotomic field $\mathbb{Q}(e^{2\pi i/[n_s]_{s\in\bar I}})$ with $\sigma_a(e^{2\pi i/[n_s]_{s\in\bar I}})=e^{2\pi i a/[n_s]_{s\in\bar I}}$ we obtain from $(*)$ the equality

$$(*_a)\qquad \sum_{w\in W}(e^{2\pi i w})^a\sum_{\substack{J\subseteq\bar I\\ \{\sum_{s\in J}1/n_s\}=\theta\\ \{\sum_{s\in J}a_s/n_s\}=w}}(-1)^{|J|}=0.$$

Observe that

$$|W|\le\bigg|\bigg\{J\subseteq\bar I:\bigg\{\sum_{s\in J}\frac{1}{n_s}\bigg\}=\theta\bigg\}\bigg|=\sum_{n=0}^{m-1}\bar I_*(n+\theta).$$

If $0<|W|<p([n_s]_{s\in\bar I})$, then $(*_a)$ holds for every $a=1,\dots,|W|$, hence

$$\sum_{\substack{J\subseteq\bar I\\ \{\sum_{s\in J}1/n_s\}=\theta\\ \{\sum_{s\in J}a_s/n_s\}=w}}(-1)^{|J|}=0\quad\text{for all }w\in W$$

and in particular

$$\sum_{\substack{J\subseteq \bar I,\, 2\,|\,|J| \\ \{\sum_{s\in J}1/n_s\}=\theta}} 1 - \sum_{\substack{J\subseteq \bar I,\, 2\nmid |J| \\ \{\sum_{s\in J}1/n_s\}=\theta}} 1 = \sum_{\substack{J\subseteq \bar I \\ \{\sum_{s\in J}1/n_s\}=\theta}} (-1)^{|J|} = 0,$$

for the determinant of the matrix $((e^{2\pi i w})^a)_{1\le a\le |W|,\, w\in W}$ is nonzero. In the case $W=\emptyset$ we obviously have the last equality and (32). Assume $W\ne\emptyset$ below. Provided that all the $|J| \bmod 2$ with $J\subseteq \bar I$ and $\{\sum_{s\in J}1/n_s\}=\theta$ are the same, if $[n_s]_{s\in\bar I}=1$ then $\theta=0$ and we must have $\bar I=\emptyset$, i.e. $k=|I|=m$, which contradicts the fact that $(n_i,n_j)\nmid a_i-a_j$ for some $i,j\in I$; if $[n_s]_{s\in\bar I}>1$, then (32) follows from the Lemma and the validity of $(*_a)$ for all integers a prime to $[n_s]_{s\in\bar I}$. This ends the proof.

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