On Diophantine quintuples

by

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1. Introduction. The Greek mathematician Diophantus of Alexandria noted that the set $\left\{ \frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16} \right\}$ has the following property: the product of any two of its distinct elements increased by 1 is a square of a rational number (see [5]). Fermat first found a set of four positive integers with the above property, and it was $\{1, 3, 8, 120\}$.

Let n be an integer. A set of positive integers $\{x_1, \ldots, x_m\}$ is said to have the *property $D(n)$* if for all $1 \leq i < j \leq m$ the following holds: $x_i x_j + n = y_{ij}^2$, where y_{ij} is an integer. Such a set is called a *Diophantine m-tuple*.

Davenport and Baker [4] showed that if d is a positive integer such that the set $\{1, 3, 8, d\}$ has the property of Diophantus, then d has to be 120. This implies that the Diophantine quadruple $\{1, 3, 8, 120\}$ cannot be extended to the Diophantine quintuple with the property $D(1)$. Analogous result was proved for the Diophantine quadruple $\{2, 4, 12, 420\}$ with the property $D(1)$ [17], for the Diophantine quadruple $\{1, 5, 12, 96\}$ with the property $D(4)$ [15] and for the Diophantine quadruples $\{k-1, k+1, 4k, 16k^3 - 4k\}$ with the property $D(1)$ for almost all positive integers k [9].

Euler proved that every Diophantine pair $\{x_1, x_2\}$ with the property $D(1)$ can be extended in infinitely many ways to the Diophantine quadruple with the same property (see [12]). In [6] it was proved that the same conclusion is valid for the pair with the property $D(l^2)$ if the additional condition that $x_1 x_2$ is not a perfect square is fulfilled.

Arkin, Hoggatt and Straus [3] proved that every Diophantine triple with the property $D(1)$ can be extended to the Diophantine quadruple. More precisely, if $x_i x_j + 1 = y_{ij}^2$, then we can set $x_4 = x_1 + x_2 + x_3 + 2x_1 x_2 x_3 + 2y_{12} y_{13} y_{23}$. For the Diophantine quadruple obtained in this way, they proved the existence of a positive rational number x_5 with the property that $x_i x_5 + 1$ is a square of a rational number for $i = 1, \ldots, 4$.

Using this construction, in [2, 7, 8, 11] some formulas for Diophantine quintuples in terms of polynomials, Fibonacci, Lucas, Pell and Pell–Lucas numbers were obtained.

In the present paper we prove that for all positive rational numbers q, x_1, x_2, x_3, x_4 such that $x_i x_j + q^2 = y_{ij}^2$, $y_{ij} \in \mathbb{Q}$, for $1 \le i < j \le 4$, and $x_1 x_2 x_3 x_4 \ne q^4$, there exists a positive rational number x_5 such that $x_i x_5 + q^2$ is a square of a rational number for $i = 1, \ldots, 4$. As a corollary we get the result that for *all* Diophantine quadruples $\{x_1, x_2, x_3, x_4\}$ with the property $D(1)$ there exists a rational number x_5 such that $x_i x_5 + 1$ is a square of a rational number for $i = 1, \ldots, 4$.

2. Extension of Diophantine quadruples

THEOREM 1. *Let* q, x_1, x_2, x_3, x_4 *be rational numbers such that* $x_i x_j + q^2 = y_{ij}^2$, $y_{ij} \in \mathbb{Q}$, *for all* $1 \le i < j \le 4$. *Assume that* $x_1 x_2 x_3 x_4 \ne q^4$. *Then the rational number* $x_5 = A/B$, *where*

$$A = q^3[2 y_{12} y_{13} y_{14} y_{23} y_{24} y_{34} + q x_1 x_2 x_3 x_4 (x_1 + x_2 + x_3 + x_4)$$

$$+ 2 q^3 (x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4) + q^5 (x_1 + x_2 + x_3 + x_4)],$$

$$B = (x_1 x_2 x_3 x_4 - q^4)^2,$$

has the property that $x_i x_5 + q^2$ *is a square of a rational number for* $i = 1, \ldots, 4$. *To be more precise, for* $i \in \{1, 2, 3, 4\}$ *we have*

$$x_i x_5 + q^2 = \left(q \frac{x_i y_{jk} y_{jl} y_{kl} + q y_{ij} y_{ik} y_{il}}{x_1 x_2 x_3 x_4 - q^4} \right)^2, \quad \text{where } \{i, j, k, l\} = \{1, 2, 3, 4\}.$$

Proof. Let $i \in \{1, 2, 3, 4\}$ and $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then

$$(x_1 x_2 x_3 x_4 - q^4)^2 (x_i x_5 + q^2)$$

$$= 2 q^3 x_i y_{12} y_{13} y_{14} y_{23} y_{24} y_{34} + q^4 x_1 x_2 x_3 x_4 x_i (x_1 + x_2 + x_3 + x_4)$$

$$+ 2 x_i q^6 (x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4)$$

$$+ x_i q^8 (x_1 + x_2 + x_3 + x_4) + q^2 x_1^2 x_2^2 x_3^2 x_4^2 - 2 q^6 x_1 x_2 x_3 x_4 + q^{10}$$

$$= q^2 [2 q x_i y_{12} y_{13} y_{14} y_{23} y_{24} y_{34} + q^2 x_i^2 x_j x_k x_l (x_i + x_j + x_k + x_l)$$

$$+ 2 q^4 x_i^2 (x_j x_k + x_j x_l + x_k x_l) + 2 q^4 x_i x_j x_k x_l + q^6 x_i^2$$

$$+ q^6 (x_i x_j + x_i x_k + x_i x_l) + x_i^2 x_j^2 x_k^2 x_l^2 - 2 q^4 x_i x_j x_k x_l + q^8]$$

$$= q^2 [2 q x_i y_{12} y_{13} y_{14} y_{23} y_{24} y_{34} + x_i^2 (x_j x_k + q^2)(x_j x_l + q^2)(x_k x_l + q^2)$$

$$+ q^2 (x_i x_j + q^2)(x_i x_k + q^2)(x_i x_l + q^2)]$$

$$= q^2 (2 q x_i y_{12} y_{13} y_{14} y_{23} y_{24} y_{34} + x_i^2 y_{jk}^2 y_{jl}^2 y_{kl}^2 + q^2 y_{ij}^2 y_{ik}^2 y_{il}^2)$$

$$= [q(x_i y_{jk} y_{jl} y_{kl} + q y_{ij} y_{ik} y_{il})]^2,$$

which proves the theorem. ∎

Since the signs of y_{ij} are arbitrary, we have two choices for x_5. Let x_5^+ and x_5^- denote these two numbers, and let x_5^+ be the number which corresponds to the case where all y_{ij} are nonnegative.

COROLLARY 1. *Let $\{x_1, x_2, x_3, x_4\} \subset \mathbb{N}$ be the set with the property $D(1)$. Then there exists a rational number x_5, $0 < x_5 < 1$, such that $x_i x_5 + 1$ is a square of a rational number for $i = 1, \ldots, 4$.*

Proof. We claim that the number x_5^+, obtained by applying the construction from Theorem 1 to the set $\{x_1, x_2, x_3, x_4\}$ has the desired property. Indeed, it is sufficient to prove that $x_5^+ < 1$. Let us introduce the following notation:

$$\sigma_1 = x_1 + x_2 + x_3 + x_4,$$
$$\sigma_2 = x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4,$$
$$\sigma_3 = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4,$$
$$\sigma_4 = x_1 x_2 x_3 x_4,$$
$$X = \sigma_1 \sigma_4 + 2\sigma_3 + \sigma_1,$$
$$Y = y_{12} y_{13} y_{14} y_{23} y_{24} y_{34}.$$

The proof that

$$x_5^+ = \frac{2Y + X}{(\sigma_4 - 1)^2} < 1$$

is completed by showing that

(1) $$2X < (\sigma_4 - 1)^2 \quad \text{and} \quad 4Y < (\sigma_4 - 1)^2.$$

Without loss of generality we can assume that $x_1 < x_2 < x_3 < x_4$. If $x_1 = 1$, then $x_2 \neq 2$. Therefore, $x_2 \geq 3$, $x_3 \geq 4$ and $x_4 \geq 5$. Hence $\sigma_4 \geq 60$. Furthermore, from

$$\frac{1}{x_1 x_2 x_3} + \frac{1}{x_1 x_2 x_4} + \frac{1}{x_1 x_3 x_4} + \frac{1}{x_2 x_3 x_4} \leq \frac{13}{60} < \frac{1}{4}$$

it follows that $52 \leq 4\sigma_1 < \sigma_4$. In the same manner we can see that $59 \leq \sigma_2 < \sigma_4$ and $107 \leq \sigma_3 < 2\sigma_4$ (see also [12]). Hence

$$(\sigma_4 - 1)^2 - 2X > \sigma_4^2 - 2\sigma_4 + 1 - \frac{\sigma_4^2}{2} - 8\sigma_4 - \frac{\sigma_4}{2} = \frac{1}{2}(\sigma_4^2 - 21\sigma_4 + 2) > 0$$

(since $\sigma_4 \geq 60$). To get the second inequality from (1), we note that

$$Y^2 = (x_1 x_2 + 1)(x_1 x_3 + 1)(x_1 x_4 + 1)(x_2 x_3 + 1)(x_2 x_4 + 1)(x_3 x_4 + 1)$$
$$= \sigma_4^3 + \sigma_2 \sigma_4^2 - \sigma_4^2 + \sigma_1 \sigma_3 \sigma_4 + \sigma_1^2 \sigma_4 - 2\sigma_2 \sigma_4 + \sigma_3^2 - \sigma_4 + \sigma_1 \sigma_3 + \sigma_2 + 1$$
$$< \sigma_4^3 + \sigma_4^3 - \sigma_4^2 + \frac{\sigma_4^3}{2} + \frac{\sigma_4^3}{16} - 118\sigma_4 + 4\sigma_4^2 - \sigma_4 + \frac{\sigma_4^2}{2} + \sigma_4 + 1$$
$$= \frac{41}{16}\sigma_4^3 + \frac{7}{2}\sigma_4^2 - 118\sigma_4 + 1.$$

Therefore,

$$(\sigma_4 - 1)^4 - 16Y^2 > \sigma_4^4 - 4\sigma_4^3 + 6\sigma_4^2 - 4\sigma_4 + 1 - 41\sigma_4^3 - 56\sigma_4^2 + 1888\sigma_4 - 16$$
$$= \sigma_4^4 - 45\sigma_4^3 - 50\sigma_4^2 + 1884\sigma_4 - 15 > 0$$

(since $\sigma_4 \geq 60$), which completes the proof. ∎

COROLLARY 2. *Let* q, x_1, x_2, x_3 *be rational numbers such that* $x_i x_j + q^2 = y_{ij}^2$, $y_{ij} \in \mathbb{Q}$ *for all* $1 \leq i < j \leq 3$. *Let*

$$x_4 = [2y_{12}y_{13}y_{23} + 2x_1x_2x_3 + q^2(x_1 + x_2 + x_3)]/q^2,$$
$$x_5 = \frac{4y_{12}y_{13}y_{23}(x_1y_{23} + y_{12}y_{13})(x_2y_{13} + y_{12}y_{23})(x_3y_{12} + y_{13}y_{23})}{(x_1x_2x_3x_4 - q^4)^2}.$$

Then the set $\{x_1, x_2, x_3, x_4, x_5\}$ *has the property that the product of any two distinct elements of it increased by* q^2 *is equal to the square of a rational number. In the notation of Theorem 1, we have*

$$x_5 = \frac{4q^3 y_{12}y_{13}y_{14}y_{23}y_{24}y_{34}}{(x_1x_2x_3x_4 - q^4)^2}.$$

P r o o f. Let $z_1 = x_1$, $z_2 = x_2$, $z_3 = x_3$, $z_4 = 0$. Then the rational numbers z_1, z_2, z_3, z_4 satisfy the conditions of Theorem 1, and its application gives us the number

$$z_5 = [2y_{12}y_{13}y_{23} + 2x_1x_2x_3 + q^2(x_1 + x_2 + x_3)]/q^2.$$

Set $x_4 = z_5$. We can now apply Theorem 1 to the numbers x_1, x_2, x_3, x_4. Let x_5 be the number which is obtained by this construction. Observe that, by Theorem 1, for all $i \in \{1, 2, 3\}$,

(2) $$qy_{i4} = x_i y_{jk} + y_{ij}y_{ik},$$

where $\{i, j, k\} = \{1, 2, 3\}$. Let us introduce the following notation:

$$\Sigma_1 = x_1 + x_2 + x_3, \quad \Sigma_2 = x_1x_2 + x_1x_3 + x_2x_3, \quad \Sigma_3 = x_1x_2x_3,$$
$$V = y_{12}y_{13}y_{23}, \quad W = y_{14}y_{24}y_{34}.$$

We have

$$V^2 = (x_1x_2 + q^2)(x_1x_3 + q^2)(x_2x_3 + q^2) = \Sigma_3^2 + q^2\Sigma_1\Sigma_3 + q^4\Sigma_2 + q^6.$$

From (2) it follows that

$$q^3W = (x_1y_{23} + y_{12}y_{13})(x_2y_{13} + y_{12}y_{23})(x_3y_{12} + y_{13}y_{23})$$
$$= 4\Sigma_3^2 + 3q^2\Sigma_1\Sigma_3 + 2q^4\Sigma_2 + q^6 + V(4\Sigma_3 + q^2\Sigma_1).$$

Now it is easy to check that, in the notation of Corollary 1,

(3) $$q^4\sigma_1\sigma_4 + 2q^6\sigma_3 + q^8\sigma_1 = 2q^3VW.$$

Consequently,

$$x_5 = \frac{4q^3VW}{(x_1x_2x_3x_4 - q^4)^2} = \frac{4q^3y_{12}y_{13}y_{14}y_{23}y_{24}y_{34}}{(x_1x_2x_3x_4 - q^4)^2}$$

$$= \frac{4y_{12}y_{13}y_{23}(x_1y_{23} + y_{12}y_{13})(x_2y_{13} + y_{12}y_{23})(x_3y_{12} + y_{13}y_{23})}{(x_1x_2x_3x_4 - q^4)^2}. \quad \blacksquare$$

Let us now consider the question when one or both (since x_5^+ and x_5^- can be equal) of the numbers x_5^+ and x_5^- will be equal to zero. For obvious reasons, such an extension of a Diophantine quadruple will be called trivial. We will see that the answer to this question is closely connected to the construction of Corollary 2. From now on, we assume that $q \neq 0$.

PROPOSITION 1. *In the notation of Theorem 1, we have $x_5^+ = x_5^- = 0$ if and only if there exist $1 \leq i < j \leq 4$ such that $x_i x_j = -q^2$ and $x_i + x_j = x_k + x_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.*

P r o o f. From $x_5^+ = x_5^-$ we conclude that there exist $1 \leq i < j \leq 4$ such that $y_{ij} = 0$, i.e. $x_i x_j = -q^2$. Substituting this into the expression for x_5 we obtain

(4) $$x_5 = \frac{q^2(x_i + x_j - x_k - x_l)}{x_k x_l + q^2}.$$

Consequently, the condition $x_5 = 0$ implies that $x_i + x_j = x_k + x_l$.

Conversely, suppose that x_1, x_2, x_3, x_4 satisfy the condition of the proposition. Then $y_{ij} = 0$, and (4) implies that $x_5^+ = x_5^- = 0$. $\blacksquare$

PROPOSITION 2. *In the notation of Theorem 1, we have $0 \in \{x_5^+, x_5^-\}$ if and only if there exists $i \in \{1, 2, 3, 4\}$ such that*

$$x_i = [2y_{jk}y_{jl}y_{kl} + 2x_j x_k x_l + q^2(x_j + x_k + x_l)]/q^2,$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

P r o o f. We can assume that $y_{ij} \neq 0$ since otherwise the assertion of the proposition follows from Proposition 1. If $x_5 = 0$, then $x_i x_5 + q^2 = q^2$ for $i = 1, \ldots, 4$. Hence, if $0 \in \{x_5^+, x_5^-\}$, then Theorem 1 implies that for an appropriate choice of the sign of y_{ij} we have

$$x_i y_{jk} y_{jl} y_{kl} + q y_{ij} y_{ik} y_{il} = \pm(x_1 x_2 x_3 x_4 - q^4),$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Hence, there is no loss of generality in assuming that

$$x_1 y_{23} y_{24} y_{34} + q y_{12} y_{13} y_{14} = x_2 y_{13} y_{14} y_{34} + q y_{12} y_{23} y_{24}.$$

This gives $(x_1 y_{34} - q y_{12})y_{23}y_{24} = (x_2 y_{34} - q y_{12})y_{13}y_{14}$. Set $x_1 y_{34} - q y_{12} =$

$\alpha y_{13} y_{14}$. Then $x_2 y_{34} - q y_{12} = \alpha y_{23} y_{24}$, and so

$$\alpha(x_1 y_{23} y_{24} y_{34} + q y_{12} y_{13} y_{14}) = x_1 y_{34}(x_2 y_{34} - q y_{12}) + q y_{12}(x_1 y_{34} - q y_{12})$$
$$= x_1 x_2 y_{34}^2 - q^2 y_{12}^2 = x_1 x_2 x_3 x_4 - q^4.$$

We thus get $\alpha = \pm 1$ and $x_1 y_{34} - q y_{12} = \pm y_{13} y_{14}$. Squaring this relation we obtain

$$x_1^2 x_3 x_4 + q^2 x_1^2 + q^2 x_1 x_2 + q^4 - 2 q x_1 y_{12} y_{34} = x_1^2 x_3 x_4 + q^2 x_1 x_3 + q^2 x_1 x_4 + q^4,$$

and (if $x_1 \neq 0$) $2 y_{12} y_{34} = q(x_1 + x_2 - x_3 - x_4)$. Squaring again we obtain the quadratic equation in x_4:

$$q^2 x_4^2 - 2 x_4 [q^2(x_1 + x_2 + x_3) + 2 x_1 x_2 x_3]$$
$$+ q^2(x_1^2 + x_2^2 + x_3^2 - 2 x_1 x_2 - 2 x_1 x_3 - 2 x_2 x_3 - 4 q^2) = 0,$$

with the solutions

(5) $$x_4 = [q^2(x_1 + x_2 + x_3) + 2 x_1 x_2 x_3 \pm 2 y_{12} y_{13} y_{23}]/q^2.$$

We have been working under the assumption that $x_1 \neq 0$. Now suppose that $x_1 = 0$. In the same manner, using Corollary 2, it can be proved that $x_1 = 0$ implies

$$x_4 = x_2 + x_3 \pm y_{23},$$

which is exactly the relation (5) for $x_1 = 0$.

This proves one implication of the proposition. The opposite implication is a direct consequence of (3). ∎

3. Examples

EXAMPLE 1. Let us first show that the condition $x_1 x_2 x_3 x_4 \neq q^4$ from Theorem 1 is not superfluous. Indeed, the set $\{25600, 50625, 82944, 518400\}$ has the property $D(86400^2)$ and

$$25600 \cdot 50625 \cdot 82944 \cdot 518400 = 86400^4.$$

As an illustration of the situation from Proposition 1 let us consider the set $\{-25, 25, -24, 24\}$ with the property $D(625)$ and the set $\{-1, 64, 48, 15\}$ with the property $D(64)$. In both cases the construction from Theorem 1 gives $x_5^+ = x_5^- = 0$.

From [10, (13)], for $a = 2$ and $k = 3$, we obtain the Diophantine quadruple $\{2, 20, 44, 72\}$ with the property $D(81)$. It is easy to check that this quadruple does not satisfy the conditions of Proposition 2. Therefore the numbers x_5^+ and x_5^- are different from 0. Indeed, $x_5^+ = \frac{4860}{169}$ and $x_5^- = -\frac{1156680}{1054729}$. Using x_5^+, we obtain the Diophantine quintuple $\{338, 3380, 4860, 7436, 12168\}$ with the property $D(39^4)$.

If we apply the construction from Theorem 1 to Diophantus' original set $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$, we obtain $x_5^+ = \frac{1557225246720}{42521^2}$ and $x_5^- = -\frac{4387246080}{42521^2}$.

The definition of a Diophantine m-tuple can be extended to subsets of $\mathbb{Q}$. Let q be a rational number. We call a set $A = \{x_1, \ldots, x_m\} \subset \mathbb{Q} \setminus \{0\}$ a (*rational*) *Diophantine m-tuple with the property* $D(q)$ if the product of any two distinct elements of A increased by q is equal to the square of a rational number. The construction of the rational Diophantine quintuple with the property $D(1)$ which extends a given Diophantine triple was described in [3]. That construction is equivalent to the construction from Corollary 2. But Theorem 1 enables extension of Diophantine quadruples which are not of the form $\{x_1, x_2, x_3, x_4\}$ from Corollary 2. One such quadruple is the set $\{2, 20, 44, 72\}$ from Example 1. Let us now examine two ways for generation of such Diophantine quadruples.

EXAMPLE 2. Let $\{x_1, x_2, x_3, x_4\} \subset \mathbb{Q}$ be an arbitrary set with the property $D(q^2)$ and let $x_5 \in \mathbb{Q}$ be the number which is obtained by applying Theorem 1 to this set. Then $\{x_2, x_3, x_4, x_5\}$ also has the property $D(q^2)$, and we can apply Theorem 1 again. In this way we obtain $x_6 \in \mathbb{Q}$ such that $\{x_2, x_3, x_4, x_5, x_6\}$ has the property $D(q^2)$.

For example, if $x_1 = k - 1$, $x_2 = k + 1$, $x_3 = 4k$ and $x_4 = 16k^3 - 4k$, then $\{x_1, x_2, x_3, x_4\}$ has the property $D(1)$ ([6, p. 22]) and we obtain

$$x_5 = \frac{4k(2k-1)(2k+1)(4k^2 - 2k - 1)(4k^2 + 2k - 1)(8k^2 - 1)}{(64k^6 - 80k^4 + 16k^2 - 1)^2},$$

and $x_6 = P(k)/Q(k)$, where

$$P(k) = (8k^3 - 4k^2 + 1)(8k^3 + 4k^2 - 4k - 1)(8k^3 - 12k^2 + 1)$$
$$\times (8k^4 + 4k^3 - 8k^2 - k + 1)(32k^4 - 8k^3 + 28k^2 + 3)$$
$$\times (32k^4 + 8k^3 - 12k^2 + 1)(32k^4 + 24k^3 - 12k^2 - 4k + 1)$$
$$\times (32k^4 + 40k^3 + 4k^2 - 4k + 1),$$
$$Q(k) = (131072k^{14} + 131072k^{13} - 184320k^{12} - 180224k^{11} + 96256k^{10}$$
$$+ 86016k^9 - 26880k^8 - 18432k^7 + 4480k^6 + 1792k^5 - 480k^4$$
$$- 64k^3 + 32k^2 - 1)^2.$$

It turns out that this factorization of the numerator of x_6 is not accidental. Namely, it can be checked that, in the notation of Theorem 1, $x_6 = P/Q$, where

$$P = q^3(y_{12}y_{13}y_{14} + qy_{12}y_{13} + qy_{12}y_{23} + qy_{13}y_{23})$$
$$\times (y_{12}y_{13}y_{14} + qy_{12}y_{13} - qy_{12}y_{23} - qy_{13}y_{23})$$
$$\times (y_{12}y_{13}y_{14} - qy_{12}y_{13} + qy_{12}y_{23} - qy_{13}y_{23})$$
$$\times (y_{12}y_{13}y_{14} - qy_{12}y_{13} - qy_{12}y_{23} + qy_{13}y_{23})(y_{23}y_{24} + y_{23}y_{34} + y_{24}y_{34})$$

$$\times (y_{23}y_{24} + y_{23}y_{34} - y_{24}y_{34})(y_{23}y_{24} - y_{23}y_{34} + y_{24}y_{34})$$
$$\times (-y_{23}y_{24} + y_{23}y_{34} + y_{24}y_{34}),$$
$$Q = x_1^4(4x_2x_3x_4y_{12}y_{13}y_{14}y_{23}y_{24}y_{34} - qx_1^2x_2^2x_3^2x_4^2 + 2q^5x_1x_2x_3x_4 - q^9)^2.$$

PROPOSITION 3. *Let x_1, x_2 and x_3 be rational numbers such that the denominator of*

$$x_4 = \frac{8(x_3 - x_1 - x_2)(x_1 + x_3 - x_2)(x_2 + x_3 - x_1)}{(x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3)^2}$$

is different from 0. Then $x_1x_4 + 1$, $x_2x_4 + 1$ and $x_3x_4 + 1$ are squares of rational numbers.

Proof. It follows immediately that

$$x_1x_4 + 1 = \left(\frac{x_2^2 - 2x_2x_3 + x_3^2 - 3x_1^2 + 2x_1x_2 + 2x_1x_3}{x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3} \right)^2,$$

and analogous relations hold for $x_2x_4 + 1$ and $x_3x_4 + 1$. ∎

EXAMPLE 3. Observe that the set $\{x_1, x_2, x_3\}$ in Proposition 3 does not need to have the property $D(1)$. Take for example $x_1 = F_{2n+1}$, $x_2 = F_{2n+3}$ and $x_3 = F_{2n+5}$. Then $\{x_1, x_2, x_3\}$ has the property $D(-1)$ for every positive integer n (see [13, 14]). Proposition 3 implies that there exists a rational number x_4 with the property that $x_ix_4 + 1$, $i = 1, 2, 3$, are squares of rational numbers. We will show that in this case x_4 is an integer. Indeed,

$$x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$$
$$= (x_1 - x_2 + x_3)^2 - 4x_1x_3$$
$$= [F_{2n+1} - F_{2n+3} + (3F_{2n+3} - F_{2n+1})]^2 - 4F_{2n+1}F_{2n+5}$$
$$= 4(F_{2n+3}^2 - F_{2n+1}F_{2n+5}) = -4.$$

Hence,

$$x_4 = \frac{8}{16} \cdot 2F_{2n+2} \cdot 2F_{2n+3} \cdot 2F_{2n+4} = 4F_{2n+2}F_{2n+3}F_{2n+4}.$$

EXAMPLE 4. If $x_1x_2 + 1 = y_{12}^2$ and $x_3 = x_1 + x_2 + 2y_{12}$, then $\{x_1, x_2, x_3\}$ has the property $D(1)$. If we apply the construction from Proposition 3 to this set we obtain

$$x_4 = 4y_{12}(x_1 + y_{12})(x_2 + y_{12}).$$

If we apply the construction from Corollary 2 we obtain exactly the same result.

EXAMPLE 5. Let $x_1 = 1$, $x_2 = 3$ and $x_3 = 120$. Then Proposition 3 gives $x_4 = \frac{834968}{3361^2}$. The set $\{x_1, x_2, x_3, x_4\}$ has the property $D(1)$ and we can apply

the construction from Theorem 1. We obtain

$$x_5^+ = \frac{3985166705520 \cdot 481^2}{601439^2 \cdot 481^2}, \qquad x_5^- = \frac{426360 \cdot 601439^2}{481^2 \cdot 601439^2}.$$

It turns out that this cancellation is not accidental. Namely, let $\{x_1, x_2, x_3\}$ be an arbitrary set with the property $D(1)$, let x_4 be obtained by applying Proposition 3 to this set, and let x_5^+ and x_5^- be obtained by applying Theorem 1 to $\{x_1, x_2, x_3, x_4\}$. Then

$$\sqrt{x_1 x_5^+ + 1} \cdot \sqrt{x_1 x_5^- + 1} = \left| \frac{(a+b)(a-b)cd}{c^2 d^2} \right|,$$

where

$a = x_1 y_{23}[x_1^2(4x_2x_3 + 1) - 2x_1(x_2 + x_3)(2x_2x_3 - 1) - (3x_2^2 + 2x_2x_3 + 3x_3^2)],$

$b = y_{12}y_{13}[x_1^2(-4x_2x_3 - 3) + 2x_1(x_2 + x_3)(2x_2x_3 + 1) + (x_2 - x_3)^2],$

$c = (x_1 + x_2 + x_3)^2 - 4(x_1x_2x_3 - y_{12}y_{13}y_{23})^2 + 4,$

$d = 4(x_1x_2x_3 + y_{12}y_{13}y_{23})^2 - (x_1 + x_2 + x_3)^2 - 4.$

For $x_1 = 1$, $x_2 = 3$ and $x_3 = 120$, we get $c = 4 \cdot 481$ and $d = 4 \cdot 601439$.

4. Some open problems. One question still unanswered is whether there exists a (positive integer) Diophantine quintuple with the property $D(1)$. Corollary 1 shows that if such a quintuple exists it cannot be obtained by the construction from Theorem 1. Let us mention that the analogous result for the sets with the property $D(l^2)$, where $l > 1$, does not hold. For example, if we apply the construction from Theorem 1 to the quadruples $\{4, 21, 69, 125\}$ and $\{7, 12, 63, 128\}$ with the property $D(400)$, we obtain $x_5^+ = 384$, $x_5^- = -\frac{4032000}{1129^2}$ and $x_5^+ = 375$, $x_5^- = -\frac{11856000}{2021^2}$, respectively. Hence, the sets $\{4, 21, 69, 125, 384\}$ and $\{7, 12, 63, 128, 375\}$ are Diophantine quintuples with the property $D(400)$.

One may ask what is the least positive integer n_1, and what is the greatest negative integer n_2, for which there exists a Diophantine quintuple with the property $D(n_i)$, $i = 1, 2$. Certainly $n_1 \leq 256$ and $n_2 \geq -255$, since the sets $\{1, 33, 105, 320, 18240\}$ and $\{5, 21, 64, 285, 6720\}$ have the property $D(256)$, and the set $\{8, 32, 77, 203, 528\}$ has the property $D(-255)$.

In the present paper we have considered quintuples with the property $D(q)$, where q was a square of a rational number. However, the last set with the property $D(-255)$ indicates that there exist quintuples with the property $D(q)$, where q is not a perfect square (see also [9, 16]). Thus we come to the following open problem: For what rational numbers q does there exist a rational Diophantine quintuple with the property $D(q)$? It follows easily from [6, Theorem 5] that for every rational number q there exists a rational Diophantine quadruple with the property $D(q)$.

At present it is not known whether there exists a rational number $q \neq 0$ such that there exists a rational Diophantine sextuple with the property $D(q)$. In [1], some rational "sextuples" with the property $D(1)$ were obtained, but all of them have two equal elements. Thus, they are actually quintuples with the additional property that $x_1^2 + 1$ is a perfect square. There exists also a rational Diophantine quintuple $\{x_1, \ldots, x_5\}$ with the property $D(1)$ such that $x_1^2 + 1$, $x_2^2 + 1$ and $x_3^2 + 1$ are perfect squares. However, the question of the existence of Diophantine sextuples is still open.

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