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On equal values of power sums

by

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Introduction. There are several classical diophantine problems related to the power values and arithmetical properties of the sum $S_k(x) = 1^k + \ldots + (x-1)^k$ (cf. [3], [7]–[9], [13], [15]–[17]).

The purpose of this paper is to investigate the equation

$$S_k(x) = S_l(y), \tag{1}$$

where k, l are given distinct positive integers. Unfortunately, there seems to be no way to treat it in its full generality. One would start with $l = 1$, therefore,

$$8S_k(x) + 1 = (2y-1)^2. \tag{2}$$

The known general results on the equation

$$sS_k(x) + r = y^z$$

(see [8], [9], [17]) do not cover it, the special cases $k = 2, 3$ of (2) are resolved in [1], [5], [10], [14].

Theorem 1. *If $k > 1$ then all the solutions of the equation*

$$S_k(x) = S_1(y) \quad \textit{in positive integers } x, y$$

satisfy $\max(x, y) < c_1$, *where c_1 is an effectively computable constant depending only on k.*

A similar statement can be obtained if $l = 3$, that is, $S_3(y)$ is a complete square (cf. [12]). The remaining cases are strongly related to the irreducibility of Bernoulli polynomials.

Let I denote the set of positive integers k such that the kth Bernoulli polynomial denoted by $B_k(x)$ is irreducible (over $\mathbb{Q}$). Most likely $B_k(x)$ is irreducible for almost every even k (see the known cases for $k \le 200$ in

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[11]); for instance, if p is an odd prime and $1 \le m \le p$ then $B_{m(p-1)}(X)$ is irreducible (see [4]).

Theorem 2. *If $k, l \in I$ with $k > 2, (k, l) = 2$, then equation* (1) *in positive integers x, y has only finitely many solutions.*

Auxiliary results. Let f, g be polynomials having degrees $n > 1$ and $m > 1$, respectively. For a $\lambda \in \mathbb{C}$ we write $D(\lambda) = \text{discriminant}(f(x) + \lambda)$ and $E(\lambda) = \text{discriminant}(g(x) + \lambda)$.

Lemma 1. *If there are at least $[n/2]$ distinct roots of $D(\lambda) = 0$ for which $E(\lambda) \neq 0$ and $m > 3, n > 3$, then the equation*

$$f(x) = g(y) \qquad \text{in rational integers } x, y$$

has at most a finite number of solutions.

Proof. See Theorem 1 of [6].

In the next lemma we summarize some classical properties of Bernoulli polynomials. For the proofs of these results we refer to [12].

Lemma 2. *Let $B_n(X)$ denote the nth Bernoulli polynomial and $B_n = B_n(0)$, $n = 1, 2, \ldots$ Further, let D_n be the denominator of B_n. Then we have*

(A) $B_n(X) = X^n + \sum_{i=1}^{n} \binom{n}{i} B_i X^{n-i}$,
(B) $1^k + 2^k + \ldots + (x-1)^k = \frac{1}{k+1}(B_{k+1}(x) - B_{k+1})$,
(C) $B_n(X) = (-1)^n B_n(1 - X)$,
(D) $B_{2n+1} = 0, n = 1, 2, \ldots$,
(E) (von Staudt–Clausen) $D_{2n} = \prod_{p-1 \mid 2n,\, p \text{ prime}} p$,
(F) $B'_{n+1}(X) = (n+1) B_n(X)$,
(G) $B_{2n}\left(\frac{1}{2}\right) = (2^{1-2n} - 1) B_{2n}, n = 1, 2, \ldots$,
(H) $X(X-1)\left(X - \frac{1}{2}\right) \mid B_{2n-1}(X)$ (*in* $\mathbb{Q}[X]$), $n = 1, 2, \ldots$

Lemma 3. *Let $f(X) \in \mathbb{Q}[X]$ be a polynomial having at least three zeros of odd multiplicities. Then the equation*

$$f(x) = y^2 \qquad \text{in integers } x, y$$

implies $\max(|x|, |y|) < c$, *where c is an effectively computable constant depending only on the coefficients of f.*

Proof. Lemma 3 is a special case of the Theorem of [2].

Lemma 4. *Let $P(X) = a_n X^n + \ldots + a_1 X + a_0$ be a polynomial with integral coefficients, for which a_0 is odd, $4 \mid a_i$, $i = 1, \ldots, n$, and the dyadic order of a_n is* 3. *Then every zero of P* (*in* $\mathbb{C}$) *is simple.* (*P is not necessarily irreducible, e.g.* $8X^3 + 8X^2 + 8X + 3$ *is divisible by* $2X + 1$.)

P r o o f. If the polynomial $P(X)$ has a multiple zero, then it can be written as $P_1^2P_2$, where $P_1, P_2 \in \mathbb{Z}[X]$, and P_1 and P_2 are not necessarily relatively prime polynomials. By taking the natural homomorphism $\mathbb{Z}[X] \to \mathbb{Z}_2[X]$ we have $P_i = 2Q_i+1$ with some $Q_i \in \mathbb{Z}[X]$, $i = 1, 2$. The degree of Q_2 is certainly at least one, otherwise the dyadic order of the leading coefficient of $P_1^2P_2$ would not be equal to 3. Every coefficient, apart from the constant term, of the polynomial

$$(2Q_1 + 1)^2(2Q_2 + 1) = 8Q_1^2Q_2 + 8Q_1Q_2 + 2Q_2 + 4Q_1^2 + 4Q_1 + 1$$

is divisible by 4, therefore, the leading coefficient of Q_2 is even; however, a_n is not divisible by 16.

Proofs

P r o o f o f T h e o r e m 1. Let d be the smallest positive integer for which

$$8d(B_{k+1}(X) - B_{k+1}) \in \mathbb{Z}[X].$$

By Lemma 3 it suffices to prove that the polynomial

$$P(X) = 8d(B_{k+1}(X) - B_{k+1}) + d(k + 1)$$

has at least three zeros of odd multiplicities. We distinguish some cases. If $k + 1$ is odd then the above statement is a simple consequence of Lemma 4. Since P is not a complete square (in $\mathbb{Z}[X]$) we just have to exclude the remaining case

$$P(X) = (aX^2 + bX + c)R^2(X),$$

where $aX^2 + bX + c$, $R(X) \in \mathbb{Z}[X]$ and $aX^2 + bX + c$ has two distinct zeros. If $k + 1$ is even, but not divisible by 4, then $\frac{1}{2}P(X)$ is a polynomial in $\mathbb{Z}[X]$ having odd constant term. Hence it can be factorized as

$$P(X)/2 = (2S_1(X) + 1)^2(2S_2(X) + 1);$$

however, the leading coefficient of $\frac{1}{2}P(X)$ is not divisible by 8. In the amusing last case when $4 \mid k + 1$, the degree of R is odd and the relation $P(X) = P(1-X)$ implies $R^2(X) = R^2(1-X)$, therefore, $R(X) = -R(1-X)$ and $0 = R\left(\frac{1}{2}\right) = P\left(\frac{1}{2}\right)$ yields

$$B_{k+1} = \frac{2^{k-3}(k + 1)}{2^{k+1} - 1} \qquad (k + 1 \geq 4),$$

which is impossible, since the denominator of B_{k+1} should be divisible by 2.

P r o o f o f T h e o r e m 2. Put

$$B^{[j]} = \left\{ \frac{1}{j+1} B_{j+1}(\alpha) \;\middle|\; B_j(\alpha) = 0 \right\}, \qquad j = 1, 2, \dots$$

Since $D(\lambda) = C \cdot \prod_{f'(x)=0}(f(x)+\lambda)$, where C is a non-zero numerical constant (cf. [4]) it is enough to show that the sets $B^{[k]}$ and $B^{[l]}$ are disjoint. Supposing the contrary we have

$$\gamma = \frac{1}{k+1} B_{k+1}(\alpha) = \frac{1}{l+1} B_{l+1}(\beta)$$

with some α and β. The polynomials $B_k(X)$ and $B_l(X)$ are irreducible and $\gamma \in \mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta)$, therefore, the degree of γ is at most $(k,l) = 2$. Every zero of $B_{k+1}(X)$ is simple ($k+1$ is odd and $B'_{k+1}(X) = (k+1)B_k(X)$), hence $\gamma \neq 0$. If γ is rational then α is a zero of the polynomial

$$B_{k+1}(X) - \gamma(k+1) \in \mathbb{Q}[X]$$

and $(X-\alpha_1)B_k(X) = B_{k+1}(X) - \gamma(k+1)$ with some rational α_1. By differentiating both sides we obtain

$$(X-\alpha_1)B_{k-1}(X) = B_k(X),$$

which contradicts the irreducibility of $B_k(X)$. If the degree of γ is 2 over $\mathbb{Q}$ and $\overline{\gamma}$ denotes the algebraic conjugate of γ then α is a zero of the polynomial

$$\begin{aligned}(B_{k+1}(X) - \gamma(k+1))&(B_{k+1}(X) - \overline{\gamma}(k+1)) \\ &= B_{k+1}^2(X) + r_1 B_{k+1}(X) + r_2 \in \mathbb{Q}[X],\end{aligned}$$

therefore,

$$B_k(X) \mid B_{k+1}^2(X) + r_1 B_{k+1}(X) + r_2.$$

Substituting $1-X$ instead of X a simple subtraction implies

$$B_k(X) \mid 2r_1 B_{k+1}(X) \qquad (\text{in } \mathbb{Q}[x]),$$

which is impossible in case of $r_1 \neq 0$, since $X(X-1)\left(X-\frac{1}{2}\right) \mid B_{k+1}(X)$ and $B_k(X)$ is irreducible. In the remaining case $r_1 = 0$ we obtain

$$B_k(X)F(X) = B_{k+1}^2(X) + r_2$$

with an $F(X) \in \mathbb{Q}[X]$. Differentiation yields

$$B_k(X) \mid B_{k-1}(X) \cdot F(X),$$

that is, $B_k(X) \mid F(X)$. Then there is a quadratic polynomial $M(X) \in \mathbb{Q}[X]$ for which

$$M(X)B_k^2(X) = B_{k+1}^2(X) + r_2,$$

hence,

$$M'(X)B_k(X) = 2(k+1)B_{k+1}(X) - 2kM(X)B_{k-1}(X).$$

The right-hand side is divisible by $X(X-1)\left(X-\frac{1}{2}\right)$; however, the other one is not.

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