ACTA ARITHMETICA
LXXVI.2 (1996)

On Waring's problem with polynomial summands

by

HONG BING YU (Hefei)

1. Introduction. Let $f_k(x)$ be an integral-valued polynomial of degree k with positive leading coefficient. Let $G(f_k(x))$ be the least s such that the Diophantine equation

$$f_k(x_1) + \ldots + f_k(x_s) = n, \quad x_i \ge 0, \tag{1.1}$$

is solvable for all sufficiently large integers n. Then $f_k(x)$ must satisfy the condition that there do not exist integers c and $q > 1$ such that $f_k(x) \equiv c \pmod q$ identically. This condition is equivalent ([5]) to $f_k(x)$ being of the form

$$f_k(x) = a_k F_k(x) + \ldots + a_1 F_1(x) \tag{1.2}$$

(without loss of generality we have supposed that $f_k(0) = 0$), where $a_1, \ldots, a_k$ are integers satisfying

$$(a_1, \ldots, a_k) = 1 \quad \text{and} \quad a_k > 0 \tag{1.3}$$

and

$$F_i(x) = \frac{x(x-1)\ldots(x-i+1)}{i!} \quad (1 \le i \le k). \tag{1.4}$$

The above problem was investigated by many authors (see [11] and the references therein). The best results were obtained by L. K. Hua and V. I. Nechaev. In [8, 9] Hua proved that

$$G(f_3(x)) \le 8 \quad \text{and} \quad G(f_k(x)) \le (k-1)2^{k+1} \quad \text{for } k \ge 4.$$

He also announced [7] that $G(f_4(x)) \le 2^4 + 1$ and $G(f_5(x)) \le 2^5 - 1$, but the proof seems never to be published (cf. [10, §27]). For the case $k = 6$ Nechaev [11] improved Hua's result to $G(f_6(x)) \le 2^6 + 1$.

In [8] Hua also proved that whenever $k \ge 4$, if

$$H_k(x) = 2^{k-1}F_k(x) - 2^{k-2}F_{k-1}(x) + \ldots + (-1)^{k-1}F_1(x), \tag{1.5}$$

Project supported by the National Natural Science Foundation of China.

then $G(H_k(x)) = 2^k - 1$ for odd k and 2^k for even k. Then he conjectured further (see also [10, §27]) that generally

$$G(f_k(x)) \leq \begin{cases} 2^k - 1 & \text{for odd } k \geq 3, \\ 2^k & \text{for even } k \geq 4. \end{cases}$$

The purpose of this paper is to prove that the above conjecture is true for $k = 4, 5$ and 6 (see Corollary 1 below). The difficulty of the work is arithmetical rather than analytical. In fact, let $G^*(f_k(x))$ be the least number such that if $s \geq G^*(f_k(x))$ and if the singular series corresponding to the equation (1.1) (see [6]) is positive for every n, then (1.1) has solutions in integers $x_i \geq 0$. Then by a standard application of Davenport's iteration method we have (cf. [10, §27]):

Theorem 1A. *$G^*(f_4(x)) \leq 14$, $G^*(f_5(x)) \leq 24$ and $G^*(f_6(x)) \leq 37$.*

Furthermore, we define $\mathfrak{S}^*(f_k(x))$ to be the least number such that if $s \geq \mathfrak{S}^*(f_k(x))$ then the singular series corresponding to the equation (1.1) is positive for every n. Hua [9, §4] actually proved that $\mathfrak{S}^*(f_3(x)) \leq 2^3 - 1$. In this paper, we prove:

Theorem 1. *$\mathfrak{S}^*(f_4(x)) \leq 2^4$, $\mathfrak{S}^*(f_5(x)) \leq 2^5 - 1$ and $\mathfrak{S}^*(f_6(x)) \leq 2^6$.*

Combining this with Theorem 1A we have:

Corollary 1. *$G(f_4(x)) \leq 2^4, G(f_5(x)) \leq 2^5 - 1$ and $G(f_6(x)) \leq 2^6$.*

In the case $k = 5$, we prove a slightly more precise result which may be of independent interest:

Theorem 2. *Let $H_5(x)$ be as in (1.5). If*

$$2 \nmid f_5(1) \quad \text{and} \quad f_5(x) \equiv f_5(1) H_5(x) \pmod{2^5} \quad \text{for all } x, \tag{1.6}$$

then $G(f_5(x)) = 2^5 - 1$; otherwise, we have

$$\mathfrak{S}^*(f_5(x)) \leq 2^4 \quad \text{and} \quad \max_{f_5} G(f_5(x)) \geq 2^4. \tag{1.7}$$

In view of the first assertion of (1.7), the methods of Davenport [2] and [3] are readily adapted to give the following result.

Corollary 2. *If $f_5(x)$ does not satisfy (1.6), then almost all positive integers are representable as the sum of 16 positive values of $f_5(x)$.*

Remark. By the second inequality of Lemma 5.3(i), the result in Corollary 2 is the best possible, in the sense that the number 16 cannot be replaced by a smaller one.

Our results mentioned above pose two obvious questions. First, can we establish the asymptotic formula for the number of solutions of the equation (1.1) when $s = 31$ (for $k = 5$) or $s = 2^k$ (for $k = 4$ or 6)? (Cf. Theorem 1 of Hua [8].) Second, is it true that $G^*(f_3(x)) \leq 7$ and $G^*(f_5(x)) \leq 2^4$? By adapting the method of Vaughan [14], G. Yu and the author have proved, among other things, that $G^*(f_5(x)) \le 21$. On the other hand, for the classical Waring problem many achievements have recently been made by Boklan [1], Heath-Brown [4], Vaughan [13, 15–17], and Vaughan and Wooley [18]. However, their methods do not appear to be applicable to the present problems.

2. Notation and preliminary results. The following notation will be used throughout.

Let $f_k(x)$ be as in (1.2), and let d be the least common denominator of the coefficients of $f_k(x)$. Then $d \mid k!$. For each prime p, we define p^t to be the highest power of p dividing d, and write $p^t f_k(x) = \varphi_k(x)$. Then the denominators of the coefficients of $\varphi_k(x)$ are not divisible by p. Let $\theta^{(i)}$ be the greatest integer such that the ith derivative of $\varphi_k(x)$ satisfies

$$\varphi_k^{(i)}(x) \equiv 0 \pmod{p^{\theta^{(i)}}}$$

for all x, and let $f_k^*(x) = p^{-\theta'}\varphi_k'(x)$. Let

$$\delta = \max_{1\le i\le k-1}(\theta^{(i)} - \theta^{(i+1)}). \tag{2.1}$$

We note that $p^\delta \le k-1$ (see [6, Lemma 7.4]). Let

$$\gamma = \begin{cases} \theta' - t + \delta + 2 & \text{for } p = 2, \\ \theta' - t + \delta + 1 & \text{for } p > 2. \end{cases} \tag{2.2}$$

Of course, γ depends on both p and $f_k(x)$. We define $\Gamma^*(f_k(x), p^\gamma)$ to be the least s such that the congruence

$$f_k(x_1) + \ldots + f_k(x_s) \equiv n \pmod{p^\gamma}$$

has a primitive solution, that is, a solution with the $f_k^*(x_i)$ not all divisible by p, for every n. Also, for any $l > 0$ we define $\Gamma(f_k(x), p^l)$ to be the least s for which the congruence

$$f_k(x_1) + \ldots + f_k(x_s) \equiv n \pmod{p^l}$$

has a solution for every n. It follows from the definition that (cf. [6, Lemma 7.8])

$$\Gamma(f_k(x), p^\gamma) \le \Gamma^*(f_k(x), p^\gamma) \le \Gamma(f_k(x), p^\gamma) + 1 \tag{2.3}$$

and

$$G(f_k(x)) \ge \max_{p,l} \Gamma(f_k(x), p^l). \tag{2.4}$$

By Theorem 2 of Hua [8], Theorem 1A (with $k = 5$) and (2.4), we see that in order to establish Theorems 1 and 2, it will suffice to prove the following results.

Theorem 3. (i) *For* $k = 4$ *and* 6 *we have* $\Gamma^*(f_k(x), p^\gamma) \le 2^k$.

(ii) *If* $f_5(x)$ *satisfies* (1.6), *then*

$$\Gamma^*(f_5(x), p^\gamma) \le 2^5 - 1 \quad \textit{and} \quad \Gamma(f_5(x), 2^\gamma) = 2^5 - 1;$$

otherwise

$$\Gamma^*(f_5(x), p^\gamma) \le 2^4 \quad \textit{and} \quad \max_{f_5} \Gamma(f_5(x), 2^5) \ge 2^4.$$

It is easily seen that the first assertion of (i) (i.e. for $k = 4$) is a straightforward consequence of the second one of (ii). Moreover, we note that the case $p > k$ of Theorem 3 follows readily from Lemma 2.1 below.

Lemma 2.1 (Hua [8]). *For* $p > k$ *we have* $\Gamma^*(f_k(x), p^\gamma) \le 2k$.

Therefore, to prove Theorem 3 it will suffice to consider the cases when $k = 5$ and 6 and $p \le k$.

The proof of Theorem 3 (see Sections 3 to 6) is elementary but very delicate. The main difficulty of the argument lies in that when $p \le k$, in particular when $p = 2$, we generally lack in understanding the behaviour of the value set $\{f_k(x) \bmod p^\gamma\}$ which depends on $\theta^{(i)}$ ($i \ge 1$) defined previously. This makes it very difficult and complicated to compute $\Gamma^*(f_k(x), p^\gamma)$, even if k is fairly small.

Before proceeding further we record some results that will be useful later. Firstly, we need the following well-known result (cf. [8, Lemma 2.1]).

Lemma 2.2. *Let* $\alpha_1, \ldots, \alpha_r$ *be* r *different residue classes* $\bmod h$, *and* $\beta_1, \ldots, \beta_s$ *be* s *different residue classes* $\bmod h$, *and* $(\beta_1, \ldots, \beta_s, h) = 1$. *Then the number of different residue classes represented by*

$$\alpha_i \ \textit{or}\ \alpha_i + \beta_j \qquad (1 \le i \le r, 1 \le j \le s)$$

is greater than or equal to $\min(r + s, h)$.

Secondly, let p be prime. For integers $x_1, \ldots, x_r$ with $(x_1, \ldots, x_r, p) = 1$ and $l > 0$, we denote by $R(x_1, \ldots, x_r; p^l)$ the least number of summands $x_1, \ldots, x_r$ sufficient to represent every residue class $\bmod p^l$. The following result is obvious (see [11, Lemma 2.5]).

Lemma 2.3. *If* $u \ge v > 0$, *and* $(\alpha_1, \ldots, \alpha_r, p) = (\beta_1, \ldots, \beta_s, p) = 1$, *then*

$$R(\alpha_1, \ldots, \alpha_r, \beta_1 p^v, \ldots, \beta_s p^v; p^u) \le R(\alpha_1, \ldots, \alpha_r; p^v) + R(\beta_1, \ldots, \beta_s; p^{u-v}).$$

Finally, we have (see the proof of Hua [8, Lemma 3.2])

Lemma 2.4. *The derivatives of* $f_6(x)$ *are given by*

$$\begin{aligned} f_6'(x) &= a_6 F_5(x) + \left(-\frac{a_6}{2} + a_5\right) F_4(x) + \left(\frac{a_6}{3} - \frac{a_5}{2} + a_4\right) F_3(x) \\ &\quad + \left(-\frac{a_6}{4} + \frac{a_5}{3} - \frac{a_4}{2} + a_3\right) F_2(x) \end{aligned} \tag{2.5}$$

$$+\left(\frac{a_6}{5}-\frac{a_5}{4}+\frac{a_4}{3}-\frac{a_3}{2}+a_2\right)F_1(x)$$
$$+\left(-\frac{a_6}{6}+\frac{a_5}{5}-\frac{a_4}{4}+\frac{a_3}{3}-\frac{a_2}{2}+a_1\right),$$

$$f_6''(x)=a_6F_4(x)+(-a_6+a_5)F_3(x)+\left(\frac{11}{12}a_6-a_5+a_4\right)F_2(x) \tag{2.6}$$
$$+\left(-\frac{5}{6}a_6+\frac{11}{12}a_5-a_4+a_3\right)F_1(x)$$
$$+\left(\frac{137}{180}a_6-\frac{5}{6}a_5+\frac{11}{12}a_4-a_3+a_2\right),$$

$$f_6'''(x)=a_6F_3(x)+\left(-\frac{3}{2}a_6+a_5\right)F_2(x)+\left(\frac{7}{4}a_6-\frac{3}{2}a_5+a_4\right)F_1(x) \tag{2.7}$$
$$+\left(-\frac{15}{8}a_6+\frac{7}{4}a_5-\frac{3}{2}a_4+a_3\right),$$

$$f_6^{(4)}(x)=a_6F_2(x)+(-2a_6+a_5)F_1(x)+\left(\frac{17}{6}a_6-2a_5+a_4\right), \tag{2.8}$$

$$f_6^{(5)}(x)=a_6F_1(x)+\left(-\frac{5}{2}a_6+a_5\right). \tag{2.9}$$

3. Proof of Theorem 3(i) for $k=6$ and $p=2$. From Section 2 we have

$$0\le t\le 4 \quad\text{and}\quad 0\le\delta\le 2. \tag{3.1}$$

First of all, it is easy to see that $\theta'\le 3$ when $t=1$ or 2 and that $\theta'\le 4$ when $t=3$ or 4. Thus, by (2.2) we have $\gamma\le 6$ for the case $t>0$. Since $f_6(x)$ assumes both odd and even values modulo 2^γ, therefore, by (2.3) and repeated application of Lemma 2.2 we have

$$\Gamma^*(f_6(x),2^\gamma)\le\Gamma(f_6(x),2^6)+1\le 2^6.$$

Henceforward we assume that $t=0$. Then $f_6(x)=\varphi_6(x)$, $2^4\mid a_6$, $2^3\mid(a_4,a_5)$ and $2\mid(a_2,a_3)$. For convenience we put

$$\frac{a_i}{i!}\equiv b_i \pmod{2^\gamma}\qquad(i=2,\ldots,6). \tag{3.2}$$

Now a_1 must be odd; without loss of generality we may assume that $a_1=1$ (see the remarks following Lemma 16.3 of Hua [9]). Moreover, it is easy to see that

$$0\le\theta'\le 5\qquad\text{when } t=0. \tag{3.3}$$

Lemma 3.1. *If $t=0$ and $0\le\theta'\le 3$, then $\Gamma^*(f_6(x),2^\gamma)\le 2^6$.*

Proof. See the proof of Nechaev [11, Lemma 2.6].

LEMMA 3.2. *If $t = 0$ and $\theta' = 4$, then $\Gamma^*(f_6(x), 2^\gamma) \le 2^6$.*

Proof. Clearly $\gamma \le 8$. By (2.5) and (3.2) we can deduce that

$$2 \mid b_6, \quad 2 \parallel b_5, \quad 2 \nmid b_4, \quad b_3 \equiv -2 \pmod{2^3}, \quad b_2 \equiv -1 \pmod{2^2}. \tag{3.4}$$

Moreover, we record for future use that

$$2^3 \mid (2b_6 + 2b_4 - b_3) \quad \text{and} \quad 2^4 \mid (-6b_4 + 2b_3 - b_2 + 1), \tag{3.5}$$

which are easily seen from (2.5), (3.2) and (3.4). Let $b_i = 2b_i'$ $(i = 5, 6)$. We consider two cases.

(I) $2 \mid b_6'$. Then by (3.4) and (3.5), $b_4 \equiv -1 \pmod{2^2}$ and $b_2 \equiv 3 \pmod{2^3}$. Thus $f_6(2) \equiv 2^3 c \pmod{2^8}$ with $2 \nmid c$. It follows from Lemma 2.3 that

$$\Gamma(f_6(x), 2^\gamma) \le R(f_6(0), f_6(1), f_6(2); 2^8) \le R(0, 1; 2^5) + R(0, c; 2^3) \le 2^5 + 2^3,$$

which is more than is required.

(II) $2 \nmid b_6'$. Then $b_4 \equiv 1 \pmod{2^2}$ and $b_2 \equiv -1 \pmod{2^3}$. Further, in view of $\gamma \le 8$, we may suppose that

$$b_2 \equiv -1 \pmod{2^5}, \quad \text{i.e.} \quad f_6(2) \equiv 0 \pmod{2^6}, \tag{3.6}$$

for in the contrary case the lemma follows as above. Then, by (3.4)–(3.6), $b_4 \equiv 5 \pmod{2^3}$. Now, by Lemma 2.4, we find that

$$f_6''(x) \equiv -2^2(b_5' + 1)x + 2^3 \pmod{2^4}, \tag{3.7}$$

$$f_6'''(x) \equiv 2^3 x + 2^2(b_5' + b_6') \pmod{2^4}, \tag{3.8}$$

$$f_6^{(4)}(x) \equiv 2^3 \pmod{2^4}, \quad \text{and} \quad \theta^{(5)} = \theta^{(6)} = 5. \tag{3.9}$$

It follows from (2.1) that $\delta = 1$ and so $\gamma = 7$. Finally, by Taylor's expansion we have, for any x,

$$f_6'(x+2) - f_6'(x) \equiv 2^3(b_5' - 1)x + 2^3(b_5' + b_6' - 2) \pmod{2^5}. \tag{3.10}$$

We are now in a position to prove the lemma. When $4 \mid (b_5' - 1)$, we have

$$f_6(3) \equiv \sum_{i=0}^{4} \frac{f_6^{(i)}(1)2^i}{i!} \equiv 1 + \frac{f_6^{(4)}(1)2^4}{4!} \equiv 1 + 2^4 \pmod{2^5}. \tag{3.11}$$

From this it is easily seen that $\Gamma(f_6(x), 2^7) \le 2^4 + 2^5$, and the lemma thus follows. Hence, recalling that $2 \nmid b_5'$, we may assume from now on that $2 \parallel (b_5' - 1)$. Then, by (3.7) to (3.9) and Taylor's expansion, we have

$$\text{either } 2 \nmid f_6^*(x) \quad \text{or } f_6(x+4) \equiv f_6(x) + 2^6 \pmod{2^7} \quad \text{for any } x. \tag{3.12}$$

Suppose first that $2 \parallel (b_6' - 1)$. Then (3.10) becomes

$$f_6'(x+2) - f_6'(x) \equiv 2^4 x \pmod{2^5} \quad \text{for any } x.$$

It follows that either $2^5 \mid f_6'(1)$ or $2^5 \mid f_6'(3)$. If $2^5 \mid f_6'(1)$, then $2^4 \parallel f_6'(3)$. Also, we may suppose now that

$$f_6(3) \equiv 1 \text{ or } 1 + 2^6 \pmod{2^7},$$

for in the contrary case, in view of (3.12) with $x = 1$, $f_6(x)$ takes at least three distinct odd values modulo 2^7, and then the lemma follows from $\gamma = 7$ and Lemma 2.2. Therefore, it is now easily seen that one of the following four cases holds:

$$\begin{aligned}
&f_6(0) \equiv 0, \quad f_6(3) \equiv 1, \quad f_6(5) \equiv 1 + 2^6 \pmod{2^7}, \quad 2 \nmid f_6^*(0) f_6^*(3);\\
&f_6(0) \equiv 0, \quad f_6(1) \equiv 1, \quad f_6(3) \equiv 1 + 2^6 \pmod{2^7}, \quad 2 \nmid f_6^*(0) f_6^*(3);\\
&f_6(3) \equiv 1, \quad f_6(4) \equiv 2^6, \quad f_6(5) \equiv 1 + 2^6 \pmod{2^7}, \quad 2 \mid f_6^*(0),\ 2 \nmid f_6^*(3);\\
&f_6(1) \equiv 1, \quad f_6(4) \equiv 2^6, \quad f_6(3) \equiv 1 + 2^6 \pmod{2^7}, \quad 2 \mid f_6^*(0),\ 2 \nmid f_6^*(3);
\end{aligned}$$

and the lemma can be verified directly. When $2^5 \mid f_6'(3)$, by the same argument, the lemma also follows.

Suppose now that $4 \mid (b_6' - 1)$. Then (3.10) becomes

$$f_6'(x+2) - f_6'(x) \equiv 2^4 x + 2^4 \pmod{2^5} \quad \text{for any } x.$$

From this, (3.6) and (3.12), the lemma follows in a similar manner to the above.

The proof of Lemma 3.2 is now complete.

LEMMA 3.3. *If $t = 0$ and $\theta' = 5$, then $\Gamma^*(f_6(x), 2^\gamma) \le 2^6$.*

Proof. Clearly $\gamma \le 9$ and (3.4) still holds. Further, by the hypothesis of the lemma and (2.5), we have (retaining the notation of the proof of Lemma 3.2), in particular,

$$2 \nmid b_6', \quad 4 \mid (b_5' - b_4). \tag{3.13}$$

Hence $b_2 \equiv -1 \pmod{2^3}$ and $b_4 \equiv 1 \pmod{2^2}$ (see the beginning of Lemma 3.2(II)), so that, by (3.13),

$$4 \mid (b_5' - 1). \tag{3.14}$$

Moreover, in view of $\gamma \le 9$ and $b_2 \equiv -1 \pmod{2^3}$, we may suppose now that $b_2 \equiv -1 \pmod{2^5}$ (see (3.6)), thus (3.7) to (3.9) are valid in the present situation. Therefore, on noting that (3.14), $2 \nmid b_6'$ and $2^5 \mid f_6'(x)$, we have (cf. (3.11))

$$f_6(3) \equiv 1 + 2^4 \pmod{2^5} \quad \text{and} \quad f_6(4) \equiv 2^6 \pmod{2^7},$$

and the lemma follows from Lemmas 2.2 and 2.3 easily.

In view of (3.3), the proof of Theorem 3(i) for $k = 6$ and $p = 2$ is now complete.

4. Proof of Theorem 3(i) for $k = 6$. In view of the remark following Lemma 2.1 and the result of Section 3, we see that to complete the proof of Theorem 3(i) for $k = 6$ we need only prove the following two lemmas.

LEMMA 4.1. $\Gamma^*(f_6(x), 3^\gamma) \le 41$.

Proof. We have $0 \le t \le 2$ and $\delta \le 1$. When $t > 0$ the lemma is trivial. If $t = 0$, then $3^2 \mid a_6, 3 \mid (a_3, a_4, a_5)$ and $\theta' \le 2$. If $\theta' \le 1$, the lemma is again trivial. Hence, it remains to consider the case of $\theta' = 2$. We then have $\gamma \le 4$ and (using (2.5))

$$3^2 \mid a_5, \quad 3^2 \,\Big|\, \left(\frac{a_6}{3} + a_4\right), \quad 3 \,\Big|\, \left(\frac{a_4}{3} + a_2\right),$$

which, together with Lemma 2.4, implies that $\theta^{(i)} \ge 1$ $(2 \le i \le 6)$.

If $3^3 \mid a_6$, then $3 \mid a_2$ and so $3 \nmid a_1$ by (1.3). Thus

$$f_6(x) \equiv a_1 x \pmod 3 \quad \text{for any } x.$$

From this and Lemma 2.2 the lemma follows easily.

If $3^2 \parallel a_6$, then by contradiction it is easy to prove that there exists x_0 such that

$$f_6(x_0 + 3) \not\equiv f_6(x_0) \pmod{3^4}. \tag{4.1}$$

On the other hand, by Taylor's expansion we have

$$f_6(x_0 + 3) \equiv f_6(x_0) \pmod{3^2}. \tag{4.2}$$

Thus, if $3 \nmid f_6(x_0)$ then $3 \nmid f_6(x_0 + 3)$, and the lemma follows from $\gamma \le 4$, (4.1) and Lemma 2.2. If $3 \mid f_6(x_0)$ then $3 \mid f_6(x_0 + 3)$. Also, from (4.1) we see that at least one of $f_6(x_0)$ and $f_6(x_0 + 3)$ is not divisible by 3^4, and then the lemma follows from Lemma 2.3.

LEMMA 4.2. $\Gamma^*(f_6(x), 5^\gamma) \le 32$.

Proof. Clearly, $t \le 1$ and $\delta \le 1$. If $t = 1$, the result is trivial. If $t = 0$, then $5 \mid (a_5, a_6)$ and $\theta' \le 1$. We may assume that $\theta' = 1$; then $\gamma \le 3$ and

$$5 \mid (a_3, a_4), \quad 5 \,\Big|\, \left(\frac{a_6}{5} + a_2\right). \tag{4.3}$$

If $5 \mid a_2$, then $5 \nmid a_1$ and the lemma follows as in the proof of Lemma 4.1. If $5 \nmid a_2$, then it is easily seen by (4.3) that $5 \nmid f_6''(0)$. Moreover, we have

$$f_6'(5x) - f_6'(5(x-1)) \equiv 5 f_6''(0) \pmod{5^2}, \quad x = 1, \ldots, 4.$$

From this we deduce that there exists l $(0 \le l \le 4)$ such that $5^2 \mid f_6'(5l)$. Therefore $f_6(5) \equiv 5^2 c \pmod{5^3}$ with $5 \nmid c$, and the lemma follows.

5. Proof of Theorem 3(ii) for $p = 2$. We have

$$0 \le t \le 3 \quad \text{and} \quad 0 \le \delta \le 2. \tag{5.1}$$

When $t > 0$, our result can be proved easily (see the beginning of Section 3).

Henceforward we assume that $t = 0$. Then a_1 must be odd, and we may assume that $a_1 = 1$. We again put

$$\frac{a_i}{i!} \equiv b_i \pmod{2^\gamma} \qquad (i = 2, \ldots, 5). \tag{5.2}$$

Also, it is easy to see that

$$0 \le \theta' \le 4 \qquad \text{when } t = 0. \tag{5.3}$$

LEMMA 5.1. *If $t = 0$ and $\theta' = 1$, then $\Gamma^*(f_5(x), 2^\gamma) \le 2^4$.*

P r o o f. Clearly, $\gamma \le 5$ and $\theta^{(i)} \ge 1$ $(i = 2, \ldots, 5)$. By Taylor's expansion we have

$$f_5'(x+2) - f_5'(x) \equiv 0 \pmod{2^2} \qquad \text{for any } x.$$

Thus, if $2^2 \mid f_5'(0)$, then $2^2 \mid f_5'(x)$ for any even x. It follows that there exists an odd x_0 such that $2 \parallel f_5'(x_0)$, which implies $2 \parallel f_5'(1)$, and therefore

$$f_5(5) \equiv f_5(1) + 4f_5'(1) \equiv 1 + 2^3 \pmod{2^4} \quad \text{and} \quad f_5(9) \equiv 1 + 2^4 \pmod{2^5}.$$

The lemma follows from $\gamma \le 5$ and Lemma 2.2 immediately.

If $2 \parallel f_5'(0)$, then $f_5(4) \equiv 2^3 \pmod{2^4}$, and the lemma also follows.

LEMMA 5.2. *If $t = 0$ and $\theta' = 2$, then $\Gamma^*(f_5(x), 2^\gamma) \le 2^4$.*

P r o o f. By (2.5) and (5.2), we have

$$2 \mid b_3, \quad 2^2 \mid (2b_5 + b_3 + 2b_2), \quad 2^2 \mid (2b_4 - b_2 + 1). \tag{5.4}$$

When $2 \mid b_4$, it is easily verified that $\gamma = 5$ and $f_5(2) \equiv 2^2 \pmod{2^3}$, and then the lemma follows at once. Hence we may assume from now on that $2 \nmid b_4$. Then, by (5.4),

$$b_2 \equiv -1 \pmod{2^2}, \qquad \text{i.e.} \quad f_5(2) \equiv 0 \pmod{2^3}. \tag{5.5}$$

Suppose first that $2 \mid b_5$. Then $2 \parallel b_3$ by (5.4). By Lemma 2.4 we now have

$$2 \le \theta'' \le 3, \qquad 2 \le \theta''' \le 3 \le \theta^{(4)} \le \theta^{(5)}. \tag{5.6}$$

Thus $\gamma \le 5$ and (by using Taylor's expansion)

$$f_5'(x+2) - f_5'(x) \equiv 0 \pmod{2^3} \qquad \text{for any } x.$$

Hence, if $2^3 \mid f_5'(1)$, then $2^2 \parallel f_5'(x)$ for any even x, and so

$$f_5(4) \equiv f_5(0) + 4f_5'(0) \equiv 2^4 \pmod{2^5}.$$

If $2^2 \parallel f_5'(1)$, then $2^2 \parallel f_5'(5)$ and $f_5(5) \equiv 1 + 2^4 \pmod{2^5}$. In both cases the lemma can be verified directly.

Suppose now that $2 \nmid b_5$. Then it is easily seen that $\gamma = 5$. Also, we have $2^2 \mid f_5''(x)$ and $2 \parallel f_5'''(x)$ for any even x, and therefore,

$$f_5'(x+2) - f_5'(x) \equiv 2^2 \pmod{2^3} \quad \text{for any even } x.$$

From this and (5.5), the lemma follows in the same way as above.

Lemma 5.3. (i) *Suppose that* $t = 0$ *and* $\theta' = 3$. *If* $f_5(x)$ *does not satisfy* (1.6), *then*

$$\Gamma^*(f_5(x), 2^\gamma) \le 2^4 \quad \text{and} \quad \max_{f_5} \Gamma(f_5(x), 2^5) \ge 2^4.$$

(ii) *If* $f_5(x)$ *satisfies* (1.6), *then*

$$\Gamma^*(f_5(x), 2^\gamma) = \Gamma(f_5(x), 2^\gamma) = 2^5 - 1.$$

Proof of (i). From (2.5) we can deduce that $2 \nmid b_2 b_4$, $2 \mid b_5$ and $2 \parallel b_3$. Hence (5.5) and (5.6) still hold (see the proof of Lemma 5.2). Thus $\gamma \le 6$. Moreover, if $b_2 \equiv 3 \pmod{2^3}$, then the lemma follows easily. Hence by (5.5) we may assume from now on that

$$b_2 \equiv -1 \pmod{2^3}, \quad \text{i.e.} \quad f_5(2) \equiv 0 \pmod{2^4}. \tag{5.7}$$

We divide into cases:

(I) $4 \mid b_5$. Then, from the hypothesis of the lemma, (2.5) and (5.7), we further have $b_3 \equiv 2 \pmod{2^3}$ and $b_4 \equiv 1 \pmod{2^2}$. Now it is easily verified that $\theta''' = 3$ and $2^2 \parallel f_5''(x)$ for any odd x. Thus, by using Taylor's expansion and (5.6), we have

$$f_5(3) \equiv 1 + 2^3 \pmod{2^4} \tag{5.8}$$

and

$$f_5'(x+2) - f_5'(x) \equiv 2^3 \pmod{2^4} \quad \text{for any odd } x. \tag{5.9}$$

We will show that the congruence

$$f_5(x_1) + \ldots + f_5(x_s) \equiv m \pmod{2^6}, \quad 0 \le m \le 2^6 - 1, \tag{5.10}$$

has a solution for $s = 15$, and then, in view of (2.3), the first assertion of (i) follows.

We write $m = 2^4 u + v$ with $0 \le u \le 3$ and $0 \le v \le 2^4 - 1$. When $v \ne 2^3$, by (5.8) we see that 7 summands $f_5(0), f_5(1)$ and $f_5(3)$ are sufficient for representing $v \bmod 2^4$. Hence, in order to establish the desired result, it will suffice to verify that 8 summands $f_5(1)$, $f_5(3)$, $f_5(5)$ and $f_5(7)$ are sufficient for representing $2^4 u$ and $m = 2^4 u + 2^3$ $(1 \le u \le 3) \bmod 2^6$.

Indeed, if $2^3 \parallel f_5'(1)$, then $2^4 \mid f_5'(3)$ by (5.9) and therefore (noting that $2^2 \parallel f_5''(3)$)

$$f_5(7) \equiv f_5(3) + 4 f_5'(3) + \frac{f_5''(3) 4^2}{2!} \equiv f_5(3) + 2^5 \pmod{2^6}. \tag{5.11}$$

From this and (5.8) we may suppose that $f_5(3) \equiv 1 + 2^3$ or $1 + 2^3 + 2^4 \pmod{2^6}$. It follows that $7f_5(1) + f_5(7)$ or $5f_5(1) + 3f_5(7)$ is congruent modulo 2^6 to $3 \cdot 2^4$. Furthermore, it is easy to check that $7f_5(1) + f_5(3)$, $6f_5(1) + 2f_5(3)$, $5f_5(1) + 3f_5(3)$, $4f_5(1) + 4f_5(3)$ and $6f_5(1) + f_5(3) + f_5(7)$ are congruent modulo 2^6 to 2^4u ($u = 1, 2$) and $2^4u + 2^3$ ($u = 1, 2, 3$). Hence the desired result follows.

If $2^4 \mid f_5'(1)$, then $f_5(5) \equiv 1 + 2^5 \pmod{2^6}$ and so

$$f_5(5) + f_5(3) - f_5(1) \equiv f_5(3) + 2^5 \pmod{2^6}.$$

Hence, we can replace $f_5(7)$ by $f_5(5) + f_5(3) - f_5(1)$ in the above argument (see (5.11)), and then the desired result follows easily.

(II) $2 \parallel b_5$. Similar to case (I), we have

$$b_3 \equiv -2 \pmod{2^3} \quad \text{and} \quad b_4 \equiv 1 \pmod{2^2}. \tag{5.12}$$

Also, it is easily verified that $\theta'' = 3$ and $2^2 \parallel f_5'''(x)$ for any x. Then

$$f_5(3) \equiv 1 \pmod{2^4}, \tag{5.13}$$

$$f_5'(x+2) - f_5'(x) \equiv 2^3 \pmod{2^4} \quad \text{for any } x, \tag{5.14}$$

and

$$f_5(x+4y) \equiv f_5(x) + 4yf_5'(x) \pmod{2^6} \quad \text{for any } x \text{ and } y. \tag{5.15}$$

Because $f_5(x)$ does not satisfy (1.6) (note that we have supposed that $f_5(1) = a_1 = 1$), we see from (1.6), (5.2), (5.12) and $2 \parallel b_5$ that at least one of $b_2 \equiv -1 \pmod{2^4}$ and $b_3 \equiv 6 \pmod{2^4}$ cannot be satisfied, or equivalently, the following two congruences:

$$f_5(2) \equiv 0 \pmod{2^5} \quad \text{and} \quad f_5(3) \equiv 1 \pmod{2^5} \tag{5.16}$$

cannot both hold. We will show that when $s = 16$ the congruence (5.10) has a primitive solution.

In fact, if $f_5(2) \not\equiv 0 \pmod{2^5}$, then by (5.7), $f_5(2) \equiv 2^4$ or $3 \cdot 2^4 \pmod{2^6}$. From this, (5.14) and (5.15), the following is easily seen:

There are x_i ($1 \le i \le 4$), $0 \le x_i \le 7$, such that $2 \nmid f_5^*(x_i)$ and that the values of $f_5(x_i)$ are congruent modulo 2^6 to either $1, 2^4, 1 + 2^5, 3 \cdot 2^4$ or $2^4, 1 + 2^4, 3 \cdot 2^4, 1 + 3 \cdot 2^4$ or $0, 1, 2^5, 1 + 2^5$ or $0, 1 + 2^4, 2^5, 1 + 3 \cdot 2^4$.

Hence, recalling that $f_5(2) \equiv 2^4$ or $3 \cdot 2^4 \pmod{2^6}$, the first assertion of (i) can now be verified directly.

If $f_5(3) \not\equiv 1 \pmod{2^5}$, then $f_5(3) \equiv 1 + 2^4$ or $1 + 3 \cdot 2^4 \pmod{2^6}$ by (5.13). In this case we have the same result as above, and the first assertion of (i) also follows.

Furthermore, when $f_5(2) \equiv 0$ and $f_5(3) \not\equiv 1 \pmod{2^5}$, it is easy to see that (using (5.15)) $f_5(x)$ takes only three different values, 0, 1 and $1 + 2^4$, $\bmod 2^5$. Thus $\Gamma(f_5(x), 2^5) \ge 2^4$. This proves the second assertion of (i).

The proof of (i) is now complete.

Proof of (ii). If $f_5(x)$ satisfies (1.6), it is easily seen that $t=0$, $\theta'=3$ and $\gamma=6$. Further, (5.14)–(5.16) hold. Then, by an argument similar to the above, the desired results can be verified directly.

LEMMA 5.4. *If $t=0$ and $\theta'=4$, then $\Gamma^*(f_5(x),2^\gamma)\le 2^4$.*

Proof. From the proof of Lemma 3.2 (taking $b_6=0$), we have

$$2\parallel b_5,\quad b_4\equiv -1 \pmod{2^2},\quad b_3\equiv -2,\quad b_2\equiv 3 \pmod{2^3}.$$

It follows by Lemma 2.4 that

$$2^2\parallel f_5''(x)\quad\text{for any } x,\quad \theta'''=2\quad\text{and}\quad 3\le\theta^{(4)}\le\theta^{(5)}.$$

Thus $\gamma=8$. Further, on applying Taylor's expansion, we have

$$f_5(x+4)\equiv f_5(x),\quad f_5'(x+4)-f_5'(x)\equiv 2^4 \pmod{2^5}\quad\text{for any } x. \tag{5.17}$$

Similarly,

$$f_5(2)\equiv 2^3,\quad f_5(3)\equiv 1+2^3 \pmod{2^4}\quad\text{and}\quad f_5(4)\equiv 2^5 \pmod{2^6}. \tag{5.18}$$

Let $f_5(2)\equiv 2^3c_1$, $f_5(3)\equiv 1+2^3c_2 \pmod{2^5}$ and $f_5(4)\equiv 2^5c_3 \pmod{2^8}$, where $c_1,c_2=1$ or 3 and $2\nmid c_3$. It is easily verified that 9 summands $0,1,2^3c_1$ and $1+2^3c_2$ are sufficient for representing every residue classes mod 2^5. Thus

$$\begin{aligned}\Gamma(f_5(x),2^8)&\le R(f_5(0),f_5(1),f_5(2),f_5(3),f_5(4);2^8)\\&\le R(0,1,2^3c_1,1+2^3c_2;2^5)+R(0,c_3;2^3)\\&\le 9+7=2^4.\end{aligned} \tag{5.19}$$

On the other hand, replacing $f_5(l)$ by $f_5(l+4)$ (see (5.17)) if necessary, we may suppose that $2\nmid f_5^*(l)$ ($l=0,1,2,3$). Then the lemma follows from this and (5.19) immediately.

In view of (5.3), the proof of Theorem 3(ii) for $p=2$ is now complete.

6. Proof of Theorem 3(ii). By Lemma 2.1 and the result of Section 5, we see that to complete the proof of Theorem 3(ii), it suffices to prove the following two lemmas.

LEMMA 6.1. $\Gamma^*(f_5(x),3^\gamma)\le 2^4$.

Proof. Clearly, $t\le 1$ and $\delta\le 1$. When $t=1$ the result is trivial. If $t=0$ then $\theta'\le 2$. For the case $\theta'=1$ the lemma can be proved by an argument similar to that used in Lemma 4.2. If $\theta'=2$, then we have

$$3^2\parallel a_5,\quad 3^2\mid a_4,\quad 3\parallel a_3,\quad 3\mid a_2,\quad 3\nmid a_1, \tag{6.1}$$

$$3^2\,\Big|\,\Big(\frac{a_5}{3}+a_3\Big),\quad 3^2\,\Big|\,\Big(\frac{a_4}{3}-\frac{a_3}{2}+a_2\Big),\quad 3^2\,\Big|\,\Big(\frac{a_3}{3}-\frac{a_2}{2}+a_1\Big). \tag{6.2}$$

Without loss of generality we may assume that $a_1=1$, so that

$$f_5(x)\equiv x \pmod 3\quad\text{for any } x. \tag{6.3}$$

From (6.1) and (6.2) we have $\theta^{(i)} \geq 1$ $(2 \leq i \leq 5)$ and $\gamma = 4$. Also, for any l,

$$f_5''(3l) \equiv 2(a_2 - a_3), \quad f_5''(3l) + f_5'''(3l) \equiv 2(a_2 + a_3) \pmod{3^2}, \tag{6.4}$$

$$f_5''(3l+1) \equiv 2a_2, \quad f_5''(3l+1) + f_5'''(3l+1) \equiv 2(a_2 - a_3) \pmod{3^2}, \tag{6.5}$$

and

$$f_5''(3l+2) \equiv 2(a_2 + a_3), \quad f_5''(3l+2) + f_5'''(3l+2) \equiv 2a_2 \pmod{3^2}. \tag{6.6}$$

We divide into cases:

(I) $3 \parallel (a_2 - a_3)$. By (6.4) and an argument similar to that used in Lemma 4.2, we infer that there exist l_1 and l_2 $(0 \leq l_1, l_2 \leq 2)$ such that $3^2 \parallel f_5'(3l_1)$ and $3^3 \mid f_5'(3l_2)$. Therefore, by using Taylor's expansion and (6.4), we find that either $f_5(3)$ or $f_5(6)$ is congruent $\bmod 3^4$ to $3^3 c$ with $3 \nmid c$, and the lemma follows from (6.3) easily.

(II) $3^2 \mid (a_2 - a_3)$. Then by (6.2) we have (noting that $a_1 = 1$)

$$a_2 \equiv 6 \pmod{3^2}. \tag{6.7}$$

Moreover, in view of $3 \parallel a_3$, we have $3 \parallel a_2$ and $3 \parallel (a_2 + a_3)$. Hence, similar to case (I), we deduce that there exist l_3 and l_4 $(1 \leq l_3,\ l_4 \leq 2)$ such that

$$f_5(3l_3 + 1) \equiv f_5(1) + 3^3 c_1 \equiv 1 + 3^3 c_1 \pmod{3^4}, \quad c_1 = 1 \text{ or } 2, \tag{6.8}$$

and

$$f_5(3l_4 + 2) \equiv f_5(2) + 3^3 c_2 \pmod{3^4}, \quad c_2 = 1 \text{ or } 2. \tag{6.9}$$

We now complete the proof of the lemma by showing that the congruence

$$f_5(x_1) + \ldots + f_5(x_{15}) \equiv m \pmod{3^4}, \quad 0 \leq m \leq 3^4 - 1, \tag{6.10}$$

has a solution.

We write $m = 3^3 u + v$ with $0 \leq u \leq 2$ and $0 \leq v \leq 3^3 - 1$. We note first that, by (6.3) and Lemma 2.2, 13 summands $f_5(0)$, $f_5(1)$ and $f_5(2)$ are sufficient for representing every residue class $\bmod 3^3$, and 2 summands $f_5(1)$ and $f_5(3l_3 + 1)$ are sufficient for representing $3^3 + 2$ and $2 \cdot 3^3 + 2 \bmod 3^4$. Thus, when $v \geq 2$ the congruence (6.10) has a solution.

Next we verify the solubility of (6.10) when $m = 3^3 u + v$ $(0 \leq u \leq 2$, $v = 0, 1)$. From $a_1 = 1$ and (6.7) we see that

$$f_5(2) \equiv 3^3 i + 3^2 j - 1 \pmod{3^4} \quad (0 \leq i \leq 2,\ 1 \leq j \leq 3).$$

If $i = 0$ the result is trivial. If $i = 1$, without loss of generality we may assume that $c_2 = 1$ in (6.9). Then

$$f_5(3l_3 + 1) + f_5(2) \equiv 3^2 j \quad \text{or} \quad f_5(3l_3 + 1) + f_5(3l_4 + 2) \equiv 3^2 j \pmod{3^4}.$$

Now the desired result can be verified directly. If $i = 2$, the argument is similar. This completes the proof of Lemma 6.1.

LEMMA 6.2. *$\Gamma^*(f_5(x), 5^\gamma) \leq 7$.*

Proof. Clearly, $t \leq 1$ and $\delta = 0$. It is easily seen that we need only consider the case $t = 0$. Then $\theta' \leq 1$ and so $\gamma \leq 2$. Further, from (2.5) we have $5 \mid (a_2, a_3, a_4, a_5)$, so that $5 \nmid a_1$. The lemma follows at once.

Acknowledgements. The author is grateful to Professor M. G. Lu for suggesting this problem and for his considerable help. He also thanks the referee for the critical comments and calling his attention to Nechaev's work.

References

[1] K. D. Boklan, *The asymptotic formula in Waring's problem*, Mathematika 41 (1994), 329–347.
[2] H. Davenport, *On Waring's problem for cubes*, Acta Math. 71 (1939), 123–143.
[3] —, *On sums of positive integral k-th powers*, Amer. J. Math. 64 (1942), 189–198.
[4] D. R. Heath-Brown, *Weyl's inequality, Hua's inequality, and the Waring's problem*, J. London Math. Soc. (2) 38 (1988), 216–230.
[5] L. K. Hua, *On Waring's problem with polynomial summands*, Amer. J. Math. 58 (1936), 553–562.
[6] —, *On a generalized Waring problem*, Proc. London Math. Soc. (2) 43 (1937), 161–182.
[7] —, *Some results on Waring's problem for smaller powers*, C. R. Acad. Sci. URSS (2) 18 (1938), 527–528.
[8] —, *On a generalized Waring problem, II*, J. Chin. Math. Soc. 2 (1940), 175–191.
[9] —, *On a Waring's problem with cubic polynomial summands*, J. Indian Math. Soc. 4 (1940), 127–135.
[10] —, *Die Abschätzung von Exponentialsummen und ihre anwendung in der Zahlentheorie*, Enzyklopädie der Math. Wiss. Band I,2. Heft 13, Teil 1, Teubner, Leipzig, 1959.
[11] V. I. Nechaev, *Waring's problem for polynomials*, Trudy Mat. Inst. Steklov 38 (1951), 190–243.
[12] R. C. Vaughan, *The Hardy–Littlewood Method*, Cambridge Univ. Press, 1981.
[13] —, *On Waring's problem for cubes*, J. Reine Angew. Math. 365 (1986), 122–170.
[14] —, *On Waring's problem for smaller exponents*, Proc. London Math. Soc. (3) 52 (1986), 445–463.
[15] —, *On Waring's problem for smaller exponents, II*, Mathematika 33 (1986), 6–22.
[16] —, *A new iterative method in Waring's problem*, Acta Math. 162 (1989), 1–70.
[17] —, *The use in additive number theory of numbers without large prime factors*, Philos. Trans. Roy. Soc. London Ser. A 345 (1993), 363–376.
[18] R. C. Vaughan and T. D. Wooley, *On Waring's problem: some refinements*, Proc. London Math. Soc. 63 (1991), 35–68.

Department of Mathematics
University of Science and Technology of China
Hefei, 230026, Anhui
The People's Republic of China

Received on 18.1.1995
and in revised form on 2.10.1995 (2731)