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Limitation to the asymptotic formula in Waring's problem

by

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1. Introduction. In 1920's, Hardy and Littlewood introduced an analytic method for solving Waring's problem: That is, they showed that every sufficiently large natural number can be expressed as a sum of at most s kth powers, where s depends only on k. Let $R_s(n)$ denote the number of representations of n as the sum of s kth powers. The idea of the Hardy–Littlewood method is to show that there is an asympotic formula for $R_s(n)$ when n is sufficiently large, i.e.

$$R_s(n) = (\mathfrak{S}_s(n) + o(1))\Gamma\left(1+\frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} n^{s/k-1}, \tag{1}$$

where $\mathfrak{S}_s(n)$ is called the *singular series* and defined by

$$\mathfrak{S}_s(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (S(q,a)/q)^s e(-an/q), \tag{2}$$

with

$$S(q,a) = \sum_{m=1}^{q} e(am^k/q).$$

Let $\widetilde{G}(k)$ denote the least integer t such that (1) holds for all $s \ge t$. Hardy and Littlewood [3] also obtained $\widetilde{G}(k) \le (k-2)2^{k-1}+5$ for $k \in \mathbb{N}$. Hua [5] obtained $\widetilde{G}(k) \le 2^k + 1$ for small k, and Vaughan [10, 11] improved this to $\widetilde{G}(k) \le 2^k$ for $k \ge 3$. In 1988, Heath-Brown [4] showed that $\widetilde{G}(k) \le 7 \cdot 2^{k-3} + 1$ for $k \ge 6$ and Boklan [1] recently obtained $\widetilde{G}(k) \le 7 \cdot 2^{k-3}$. For large k Vinogradov [12] proved that $\widetilde{G}(k) \le 183k^9(\log k + 1)^2$ and then Hua [6] showed that $\widetilde{G}(k) \le (4+o(1))k^2 \log k$ as $k \to \infty$. Recently, Wooley [13] obtained $\widetilde{G}(k) \le (2+o(1))k^2 \log k$ as $k \to \infty$ by using an improved form of

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Vinogradov's Mean Value Theorem. It seems likely that $\widetilde{G}(k) = O(k)$, and Vaughan has conjectured that (1) holds whenever $s \ge \max(k+1, \Gamma_0(k))$ where $\Gamma_0(k)$ is the least s such that for every n and q the congruence $x_1^k + \ldots + x_s^k \equiv n \pmod q$ has a solution with $(x_1, q) = 1$.

In this paper, we wish to show that the usual approximation to $R_s(n)$ cannot always be very precise. We will obtain some analogues of the theorems in [7].

First of all, we restrict ourselves to $k > 2$.

THEOREM 1. *Suppose that $1/2 \le r < 1$ and $k+1 \le s < 2k$. Then*

$$\sum_{n=1}^{\infty} \left(R_s(n) - \Gamma\left(1+\frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} \mathfrak{S}_s(n) n^{s/k-1} \right)^2 r^n \gg R^{s/k}, \tag{3}$$

where $R = (1-r)^{-1}$.

COROLLARY 1. *Suppose that $k+1 \le s < 2k$. As $x \to \infty$, we have*

$$\sum_{n \le x} \left(R_s(n) - \Gamma\left(1+\frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} \mathfrak{S}_s(n) n^{s/k-1} \right)^2 = \Omega(x^{s/k}). \tag{4}$$

THEOREM 2. *Suppose that $s \ge k+2$ is fixed and $1/2 \le r < 1$. Then*

$$\begin{aligned} \sum_{n=1}^{\infty} \left(R_s(n) - \Gamma\left(1+\frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} \mathfrak{S}_s(n) n^{s/k-1} \right) r^n \\ = -\frac{s}{2}\Gamma\left(1+\frac{1}{k}\right)^{s-1} R^{(s-1)/k} + O(R^{(s-2)/k}), \end{aligned} \tag{5}$$

where $R = (1-r)^{-1}$.

COROLLARY 2. *Suppose that $s \ge k+2$ is fixed and $1/2 \le r < 1$. Then*

$$\begin{aligned} \sum_{n=1}^{\infty} \left(R_s(n) - \Gamma\left(1+\frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} \mathfrak{S}_s(n) n^{s/k-1} \right)^2 r^n \\ \ge \frac{s^2}{4}\Gamma\left(1+\frac{1}{k}\right)^{2s-2} R^{(2s-2)/k-1} + O(R^{(2s-3)/k-1}). \end{aligned}$$

COROLLARY 3. *Suppose that s is fixed and $s \ge k+2$. As $x \to \infty$, we have*

$$\sum_{n \le x} \left(R_s(n) - \Gamma\left(1+\frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} \mathfrak{S}_s(n) n^{s/k-1} \right)^2 = \Omega(x^{(2s-2)/k-1}).$$

Remark. Note that when $k = 2$, Theorem 2 and Corollaries 2 and 3 hold for $s \ge 5$. The proofs of these results are exactly the same as in the case $k > 2$, except that the condition $s \ge k+2$ is replaced by $s \ge 5$.

The following corollary shows that the approximation of $R_s(n)$ by the asymptotic formula cannot be very precise.

COROLLARY 4. *For* $k \geq 3$,

$$R_{k+1}(n) - \Gamma\left(1+\frac{1}{k}\right)^k \mathfrak{S}_{k+1}(n) n^{1/k} = \Omega(n^{1/(2k)}),$$

and for $s \geq k+2$ *and* $k \geq 3$,

$$R_s(n) - \Gamma\left(1+\frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} \mathfrak{S}_s(n) n^{s/k-1} = \Omega_-(n^{(s-1)/k-1}).$$

When $k = 2$, the analogue of Theorem 2 cannot apply for $s = 4$. However, we can use some elementary arguments to obtain a similar result.

THEOREM 3. *For* $k = 2$,

$$R_4(n) - \frac{\pi^2}{16}\mathfrak{S}_4(n) n = \Omega_-(n^{1/2}),$$

and for $k = 2$ *and* $s \geq 5$,

$$R_s(n) - \frac{\pi^{s/2}}{2^s}\Gamma\left(\frac{s}{2}\right)^{-1} \mathfrak{S}_s(n) n^{s/2-1} = \Omega_-(n^{s/2-3/2}).$$

Note that $r_4(n) = \operatorname{card}\{(x_1, \ldots, x_4) \in \mathbb{Z}^4 : x_1^2 + \ldots + x_4^2 = n\}$ satisfies

$$r_4(n) = \pi^2 \mathfrak{S}_4(n) n.$$

2. Preliminary lemmas

LEMMA 1. *Suppose that* $1/2 \leq r < 1$ *and* $R = (1-r)^{-1}$. *Then, as* $r \to 1-$,

$$f(r) \sim L(r), \tag{6}$$

where $f(r) = \sum_{n=1}^{\infty} r^{n^k}$ *and*

$$L(r) = \Gamma\left(1+\frac{1}{k}\right)(1-r)^{-1/k}. \tag{7}$$

In addition,

$$f(r) - L(r) = -1/2 + O((1-r)^{1/k}), \tag{8}$$

where $k \geq 2$.

Proof. Suppose that Φ has a continuous second derivative on $[0, \infty)$. Then, by the Euler–Maclaurin summation formula, we have

$$\sum_{1 \leq n \leq x} \Phi(n) = \int_1^x \Phi(y)\,dy + \tfrac{1}{2}\Phi(1) - B_1(x)\Phi(x) + \int_1^x B_1(y)\Phi'(y)\,dy \tag{9}$$

$$= \int_1^x \Phi(y)\,dy + \tfrac{1}{2}\Phi(1) - B_1(x)\Phi(x) + [B_2(y)\Phi'(y)]_1^x$$
$$- \int_1^x B_2(y)\Phi''(y)\,dy,$$

where $B_j(x) = b_j(\{x\})$, $b_1(y) = y - \frac{1}{2}$, $b_2(y) = \frac{1}{2}y^2 - \frac{1}{2} + \frac{1}{12}$. Put $\Phi(y) = r^{y^k}$. Then

$$\Phi'(y) = -ky^{k-1}r^{y^k}\left(\log\frac{1}{r}\right), \tag{10}$$

$$\Phi''(y) = -k(k-1)y^{k-2}r^{y^k}\left(\log\frac{1}{r}\right) + (ky^{k-1})^2 r^{y^k}\left(\log\frac{1}{r}\right)^2, \tag{11}$$

and $\Phi(1) = r$.

Let $y_0 = \left(\frac{k-1}{k\log(1/r)}\right)^{1/k}$. Then, by (11), $\Phi''(y) \le 0$ for $y \le y_0$, and $\Phi''(y) \ge 0$ for $y \ge y_0$. Hence, assuming $r \ge 1/\sqrt{e}$,

$$\begin{aligned}\Big|\int_1^\infty B_2(y)\Phi''(y)\,dy\Big| &\le \frac{1}{12}\int_1^{y_0} -\Phi''(y)\,dy + \frac{1}{12}\int_{y_0}^\infty \Phi''(y)\,dy \\ &= \frac{1}{12}\Phi'(1) - \frac{1}{6}\Phi'(y_0) \\ &= \frac{-kr}{12}\log\frac{1}{r} + \frac{1}{6}ky_0^{k-1}r^{y_0^k}\left(\log\frac{1}{r}\right) \qquad \text{(by (10))} \\ &= \frac{-kr}{12}\log\frac{1}{r} + \frac{1}{6}y_0^{-1}\frac{k-1}{\log(1/r)}r^{y_0^k}\left(\log\frac{1}{r}\right) \\ &= \frac{-kr}{12}\log\frac{1}{r} + \frac{k-1}{6}r^{y_0^k}\left(\frac{k\log(1/r)}{k-1}\right)^{1/k}.\end{aligned} \tag{12}$$

Put $\Phi(y) = r^{y^k}$ in (9). By (12), we have

$$\sum_{n=1}^\infty r^{n^k} = \int_1^\infty r^{y^k}\,dy + \frac{r}{2} + O\left(\left(\log\frac{1}{r}\right)^{1/k}\right). \tag{13}$$

By changing variable $u = y^k\log(1/r)$, this is

$$\int_{\log(1/r)}^\infty \left(\log\frac{1}{r}\right)^{-1/k}\frac{1}{k}u^{1/k-1}e^{-u}\,du + \frac{r}{2} + O\left(\left(\log\frac{1}{r}\right)^{1/k}\right). \tag{14}$$

We will extend the range of the integral, so we need to estimate the value of the integral from 0 to $\log(1/r)$, and note that then $e^{-y} = 1 + O(y)$. Thus

$$\int_0^{\log(1/r)} \left(\log\frac{1}{r}\right)^{-1/k} \frac{1}{k} y^{1/k-1} e^{-y}\,dy$$
$$= \left(\log\frac{1}{r}\right)^{-1/k} \int_0^{\log(1/r)} \frac{1}{k} y^{1/k-1} e^{-y}\,dy$$
$$= \left(\log\frac{1}{r}\right)^{-1/k} \int_0^{\log(1/r)} \frac{1}{k} y^{1/k-1} (1+O(y))\,dy$$
$$= \left(\log\frac{1}{r}\right)^{-1/k} \left(\log\frac{1}{r}\right)^{1/k} + O\left(\log\frac{1}{r}\right)$$
$$= 1 + O\left(\log\frac{1}{r}\right).$$

Combine this with (14). Then we have

$$\sum_{n=1}^{\infty} r^{n^k} = \int_0^{\infty} \left(\log\frac{1}{r}\right)^{-1/k} \frac{1}{k} y^{1/k-1} e^{-y}\,dy - 1 + r/2 + O\left(\left(\log\frac{1}{r}\right)^{1/k}\right). \tag{15}$$

Obviously,

$$\log\frac{1}{r} = \log\frac{1}{1-(1-r)}.$$

By Taylor's expansion, this is $(1-r) + O((1-r)^2)$. Hence

$$\left(\log\frac{1}{r}\right)^{-1/k} = (1-r)^{-1/k}(1+O(1-r)) = (1-r)^{-1/k} + O((1-r)^{1/k}),$$

provided that $k \ge 2$. Combine this with (15) to get

$$\sum_{n=1}^{\infty} r^{n^k} = (1-r)^{-1/k}\Gamma\left(1+\frac{1}{k}\right) - \frac{1}{2} + O((1-r)^{1/k}) \tag{16}$$

as $r \to 1-$.

LEMMA 2. *Suppose that* $s \ge k+1$. *Then*

$$\sum_{q \le Q} q^{1/k} |S_n(q)| \ll (nQ)^{\varepsilon},$$

where

$$S_n(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (S(q,a)/q)^s e(-an/q).$$

Proof. See Lemma 4.8 of [9].

Lemma 3. *Suppose $y \geq 1$, $\varepsilon > 0$ and $s \geq k+1$. Let*
$$\mathfrak{S}_s(n,y) = \sum_{q\leq y} \sum_{\substack{a=1\\(a,q)=1}}^{q} (S(q,a)/q)^s e(-an/q),$$
and
$$E_s(n,y) = \mathfrak{S}_s(n) - \mathfrak{S}_s(n,y).$$
Then $E_s(n,y) \ll n^\varepsilon y^{\varepsilon-1/k}$.

Proof. By Lemma 2, we have
$$\sum_{R<q\leq 2R} q^{1/k}|S_n(q)| \ll n^\varepsilon R^\varepsilon.$$
Also
$$\sum_{R<q\leq 2R} |S_n(q)| \leq \left(\frac{1}{R}\right)^{1/k} \sum_{R<q\leq 2R} q^{1/k}|S_n(q)| \ll n^\varepsilon R^{\varepsilon-1/k}.$$
Sum over $R = y, 2y, 4y, 8y, \ldots$ to get
$$\sum_{q>y} |S_n(q)| \ll n^\varepsilon y^{\varepsilon-1/k}.$$

Lemma 4. *Suppose that $1/2 \leq r < 1$, $R = (1-r)^{-1}$ and $\alpha > -1$. Then*
$$\sum_{n=2}^{\infty} n^\alpha (\log n)^\beta r^n \ll R^{\alpha+1}(\log R)^\beta.$$
The implicit constant may depend on α and β.

Proof. See Lemma 2 of [7].

Lemma 5. *Let $\alpha > 0$. Then for every t, we have*
$$(-1)^n \binom{-\alpha}{n} = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\Big\{1 + \sum_{j=1}^{t} b_j(\alpha) n^{-j}\Big\} + O(n^{\alpha-t-2}),$$
as $n \to \infty$, where the coefficients $b_k(\alpha)$ are real numbers which depend at most on k and α.

Proof. See Lemma 4.1 of [8].

Lemma 6. *Let $\mathfrak{S}_s(n)$ be given by (2) and $s \geq k+2$. Then*
$$\sum_{n\leq x} \mathfrak{S}_s(n) = x + O(1). \tag{17}$$

Proof. The term with $q = 1$ in the definition of $\mathfrak{S}_s(n)$ contributes $[x]$ when summed. Thus, we need to show that the terms with $q \ge 2$ contribute $O(1)$ when summed. By Lemma 4.4 of [9], if $p \nmid a$ and $l > \gamma$, then

$$S(p^l, a) = \begin{cases} p^{k-1} S(p^{l-k}, a) & \text{when } l > k, \\ p^{l-1} & \text{when } l \le k, \end{cases} \tag{18}$$

where γ is defined by

$$\gamma = \begin{cases} \tau + 2 & \text{when } p = 2 \text{ and } \tau = 0, \\ \tau + 1 & \text{when } p > 2 \text{ or } p = 2 \text{ and } \tau > 0, \end{cases}$$

and τ is the largest t such that p^t divides k. Note that $\gamma \le k$ unless $k = p = 2$ in which case $\gamma = 3$. Suppose that $2 \le l \le \gamma$. Then

$$|S(p^l, a)| \le p^l \le kp^{l-1}, \tag{19}$$

since $l \le k$ and $p \mid k$. For $l = 1$, by (3.54) of Hardy and Littlewood [3], we have

$$|S(p, a)| \le (k-1)p^{1/2}. \tag{20}$$

Let $q = \prod_p p^{\alpha_p}$. Rewrite q as $q_1 q_2^2 q_3^3 \ldots q_k^k$, where $q_1, q_2, \ldots, q_{k-1}$ are square-free and pairwise coprime. By Lemma 2.10 of [9],

$$S(q, a) = \prod_{p^{\alpha_p} \| q} S(p^{\alpha_p}, a_{p^{\alpha_p}}),$$

where $a_{p^{\alpha_p}} \equiv a \pmod{p}$. By (18), we have

$$S(q, a) = \prod_{u=1}^{k-1} \prod_{\substack{p \mid q_u \\ p>2}} S(p^u, a_{p^{\alpha_p}}) \prod_{\substack{p \mid q_k \\ p>2}} p^{v_p(k-1)} S(2^{\alpha_2}, a_{2^{\alpha_2}}). \tag{21}$$

Therefore,

$$\begin{aligned} |S(q, a)| &\le \prod_{u=2}^{k-1} \prod_{\substack{p \mid q_u,\, p>2 \\ (p,k)=1}} p^{u-1} \prod_{\substack{p \mid q_u,\, p>2 \\ (p,k)>1}} kp^{u-1} \\ &\quad \times \prod_{\substack{p \mid q_1 \\ p>2}} kp^{1/2} \prod_{\substack{p \mid q_k \\ p>2}} p^{v_p(k-1)} \, (4 \cdot 2^{\alpha_2/2}) \\ &\ll \Big(\prod_{u=2}^{k-1} q_u^{u-1} \Big) \Big(\prod_{p \le k} k \Big) q_1^{1/2} \Big(\prod_{p \mid q} k \Big) (q_k^{k-1}) \\ &\ll q^{\varepsilon} q_1^{1/2} q_2^1 q_3^2 \ldots q_k^{k-1}. \end{aligned} \tag{22}$$

If $q>1$ and $(a,q)=1$, then

$$\sum_{n\le x} e(-an/q) \ll |\sin(\pi a/q)|^{-1} \ll \|a/q\|,$$

where $\|y\|$ is the distance of y from the nearest integer. So the terms with $q\ge 2$ in (17) contribute

$$\ll \sum_{q=2}^{\infty}\sum_{a=1}^{q-1}(q^{\varepsilon}q_1^{1/2}q_2^1q_3^2\dots q_k^{k-1})^s q^{-s}\|a/q\|^{-1}$$
$$\ll \sum_{q=2}^{\infty}(q_1^{1/2}q_2^1q_3^2\dots q_k^{k-1})^s q^{1-s}q^{\eta},$$

where $\eta=\varepsilon(s+1)$. The last sum is

$$\le \sum_{q_1=1}^{\infty}\sum_{q_2=1}^{\infty}\dots\sum_{q_k=1}^{\infty} q_1^{1+\eta-s/2}q_2^{2+2\eta+s-2s}q_3^{3+3\eta+2s-3s}\dots q_k^{k+k\eta+(k-1)s-ks}$$
$$= \sum_{q_1=1}^{\infty}\sum_{q_2=1}^{\infty}\dots\sum_{q_k=1}^{\infty} q_1^{1+\eta-s/2}q_2^{2+2\eta-s}q_3^{3+3\eta-s}\dots q_k^{k+k\eta-s}.$$

When $s\ge k+2$, it is convergent. Hence, the lemma follows.

LEMMA 7. *Let $1/2\le r<1$ and $L(r)$ be as in Lemma 1 and suppose that $s\ge \max(5,k+2)$. Then*

$$\sum_{n=1}^{\infty}\mathfrak{S}_s(n)\Gamma\left(1+\frac{1}{k}\right)^s\Gamma\left(\frac{s}{k}\right)^{-1} n^{s/k-1}r^n = L^s(r)+O(R^{s/k-1}). \tag{23}$$

Proof. Clearly,

$$L^s(r)=\Gamma\left(1+\frac{1}{k}\right)^s(1-r)^{-s/k}.$$

By the binomial expansion, we have

$$L^s(r)=\Gamma\left(1+\frac{1}{k}\right)^s\sum_{n=0}^{\infty}(-1)^n\binom{-s/k}{n}r^n.$$

Hence, by Lemma 5, we have

$$L^s(r)=\Gamma\left(1+\frac{1}{k}\right)^s\sum_{n=1}^{\infty}\Gamma\left(\frac{s}{k}\right)^{-1}(n^{s/k-1})r^n+O\Big(1+\sum_{n=1}^{\infty}n^{s/k-2}r^n\Big).$$

By Lemma 4, this is

$$\Gamma\left(1+\frac{1}{k}\right)^s\Gamma\left(\frac{s}{k}\right)^{-1}\sum_{n=1}^{\infty}n^{s/k-1}r^n+O(R^{s/k-1}). \tag{24}$$

The difference between the main terms in (23) is

$$\sum_{n=1}^{\infty}(\mathfrak{S}_s(n)-1)\Gamma\left(1+\frac{1}{k}\right)^s\Gamma\left(\frac{s}{k}\right)^{-1}n^{s/k-1}r^n,$$

which by partial summation is

$$\sum_{n=1}^{\infty}\Big(\sum_{m\le n}\mathfrak{S}_s(m)-n\Big)\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}(n^{s/k-1}r^n-(n+1)^{s/k-1}r^{n+1}). \tag{25}$$

From Lemma 6, we see that the first factor $\ll 1$. By the binomial expansion, the last factor is

$$\begin{aligned}&(n^{s/k-1}-(n+1)^{s/k-1})r^n+(1-r)(n+1)^{s/k-1}r^n\\&\quad=-\left(\frac{s}{k}-1\right)n^{s/k-2}r^n+(1-r)(n+1)^{s/k-1}r^n+O(n^{s/k-3}r^n).\end{aligned}$$

Thus, by Lemma 4, (25) becomes $\ll R^{s/k-1}$. Combining this with (24) gives the lemma.

3. Proof of theorems

Proof of Theorem 2. We have to show that

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(R_s(n)-\Gamma\left(1+\frac{1}{k}\right)^s\Gamma\left(\frac{s}{k}\right)^{-1}\mathfrak{S}_s(n)n^{s/k-1}\right)r^n\\&\qquad=-\frac{s}{2}\Gamma\left(1+\frac{1}{k}\right)^{s-1}(1-r)^{-(s-1)/k}+O((1-r)^{-(s-2)/k}).\end{aligned}$$

From Lemma 7 we see that this is simply a matter of establishing that

$$f^s(r)-L^s(r)=-\frac{s}{2}\Gamma\left(1+\frac{1}{k}\right)^{s-1}R^{(s-1)/k}+O(R^{(s-2)/k}),$$

where $R=(1-r)^{-1}$. By Lemma 1, it follows that

$$\begin{aligned}f^s(r)-L^s(r)&=(s+O(r^{-1/k}))(f(r)-L(r))L^{s-1}(r)\\&=-\frac{s}{2}\Gamma\left(1+\frac{1}{k}\right)^{s-1}R^{(s-1)/k}+O(R^{(s-2)/k}),\end{aligned}$$

as required.

Proof of Theorem 1. Choose $y=R^k$. First of all, we show that it suffices to prove

$$\sum_{n=1}^{\infty}\left(R_s(n)-\Gamma\left(1+\frac{1}{k}\right)^s\Gamma\left(\frac{s}{k}\right)^{-1}\mathfrak{S}_s(n,y)n^{s/k-1}\right)^2r^{2n}\gg R^{s/k}, \tag{26}$$

where $\mathfrak{S}_s(n,y)$ is as in Lemma 3.

By definition of $\mathfrak{S}_s(n,y)$, the left hand side is

$$\ll \sum_{n=1}^{\infty}\left(R_s(n)-\Gamma\left(1+\frac{1}{k}\right)^{s}\Gamma\left(\frac{s}{k}\right)^{-1}\mathfrak{S}_s(n)n^{s/k-1}\right)^{2}r^{2n} + \sum_{n=1}^{\infty}(E_s(n,y))^2 n^{2(s/k-1)}r^{2n}.$$

By Lemma 3, the second sum is

$$\ll \sum_{n=1}^{\infty} n^{2\varepsilon}y^{2\varepsilon-2/k}n^{2(s/k-1)}r^{2n}.$$

By Lemma 4, this is $\ll y^{2\varepsilon-2/k}R^{2s/k-1+2\varepsilon}$. Since $y = R^k$, this is $\ll R^{2s/k-3+\varepsilon'}$. For $k+1 \le s < 2k$, this is $o(R^{s/k})$.

Now, we prove (26). By Parseval's identity, we may write the left hand side of (26) as

$$\int_0^1 \sum_{n=1}^{\infty}\left|\left(R_s(n)-\Gamma\left(1+\frac{1}{k}\right)^{s}\Gamma\left(\frac{s}{k}\right)^{-1}\mathfrak{S}_s(n,y)n^{s/k-1}\right)r^n e(n\alpha)\right|^2 d\alpha.$$

By the Cauchy–Schwarz inequality, this is at least T^2, where

$$T=\int_0^1\left|\sum_{n=1}^{\infty}R_s(n)r^n e(n\alpha)-\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}\sum_{n=1}^{\infty}\mathfrak{S}_s(n,y)n^{s/k-1}r^n e(n\alpha)\right|d\alpha.$$

Clearly,

$$T \ge \int_1 - \int_2, \tag{27}$$

where

$$\int_1 = \int_0^1\left|\sum_{n=1}^{\infty}R_s(n)r^n e(n\alpha)\right|d\alpha, \tag{28}$$

$$\int_2 = \int_0^1\left|\sum_{n=1}^{\infty}\Gamma\left(1+\frac{1}{k}\right)^{s}\Gamma\left(\frac{s}{k}\right)^{-1}\mathfrak{S}_s(n,y)n^{s/k-1}r^n e(n\alpha)\right|d\alpha. \tag{29}$$

By Parseval's identity, we have

$$\sum_{n=1}^{\infty} r^{2n^k} = \int_0^1\left|\sum_{n=1}^{\infty} r^{n^k}e(n^k\alpha)\right|^2 d\alpha.$$

By Hölder's inequality, this is

$$\leq \Big(\int_0^1 \Big|\sum_{n=1}^{\infty} r^{n^k} e(n^k\alpha)\Big|^s d\alpha\Big)^{2/s}\Big(\int_0^1 1\, d\alpha\Big)^{1-2/s}$$
$$= \Big(\int_0^1 \Big|\sum_{n=1}^{\infty} r^{n^k} e(n^k\alpha)\Big|^s d\alpha\Big)^{2/s}.$$

By Lemma 1 with r replaced by r^2, we have

$$\Big(\int_0^1 \Big|\sum_{n=1}^{\infty} r^{n^k} e(n^k\alpha)\Big|^s d\alpha\Big)^{2/s} \gg \frac{1}{(1-r)^{1/k}}$$

as $r \to 1-$. Since $R = (1-r)^{-1}$, therefore,

$$\int_1 \gg R^{s/(2k)}. \tag{30}$$

Finally, we estimate the integral $\int_2$. By definition of $\mathfrak{S}_s(n,y)$ and (29), we have

$$\begin{aligned}\int_2 &= \int_0^1 \Big|\sum_{n=1}^{\infty} \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \sum_{q\leq y}\sum_{(a,q)=1,a=1}^{q} \Big(\frac{S(q,a)}{q}\Big)^s \\ &\qquad \times n^{s/k-1} r^n e\Big(n\Big(\alpha - \frac{a}{q}\Big)\Big)\Big|\, d\alpha \\ &\leq \Gamma\Big(1+\frac{1}{k}\Big)^s \sum_{q\leq y}\sum_{(a,q)=1,a=1}^{q} \Big|\frac{S(q,a)}{q}\Big|^s \\ &\qquad \times \int_0^1 \Big|\sum_{n=1}^{\infty} \frac{n^{s/k-1}}{\Gamma(s/k)} r^n e\Big(n\Big(\alpha-\frac{a}{q}\Big)\Big)\Big|\, d\alpha.\end{aligned} \tag{31}$$

Now, our task is to estimate the integral in (31). Suppose that $|\beta| \leq 1/2$ and $|\beta| > 1-r$. By Lemma 5, we may write

$$\frac{N^\gamma}{\Gamma(\gamma+1)} = \sum_{j=1}^{t} f_j(-1)^N \binom{-\gamma-2+j}{N} + O(N^{\gamma-t}),$$

where the f_i depend at most on γ and t. This enables us to write

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n^{s/k-1}}{\Gamma(s/k)} r^n e(n\beta) &= \sum_{n=1}^{\infty}\sum_{j=1}^{t} f_j(-1)^n \binom{-s/k-1+j}{n} r^n e(n\beta) \\ &\quad + \sum_{n=1}^{\infty} (O(n^{s/k-1-t})) r^n e(n\beta).\end{aligned} \tag{32}$$

Put $t = 2$. Since $s < 2k$, the last sum is

$$\ll \sum_{n=1}^{\infty} n^{s/k-3} \ll 1. \tag{33}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^{s/k-1}}{\Gamma(s/k)} r^n e(n\beta) = \sum_{n=0}^{\infty} \sum_{j=1}^{2} f_j (-1)^n \binom{-s/k-1-j}{n} r^n e(n\beta) + O(1).$$

Hence, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^{s/k-1}}{\Gamma(s/k)} r^n e(n\beta) \\ = f_1(1-re(n\beta))^{-s/k} + f_2(1-re(n\beta))^{-s/k+1} + O(1). \end{aligned} \tag{34}$$

Since $|1-re(\beta)|^2 = (1-r)^2 + 4r(\sin \pi\beta)^2$, we have

$$\begin{aligned} \left| \frac{1}{1-re(\beta)} \right|^{s/k} &= \left(\frac{1}{\sqrt{(1-r)^2 + 4r(\sin \pi\beta)^2}} \right)^{s/k} \\ &\ll \min((1-r)^{-s/k}, |\beta|^{-s/k}). \end{aligned} \tag{35}$$

Replace $\alpha - a/q$ by β in the integral of right hand side of (31) and by periodicity replace the interval $[-a/q, 1-a/q]$ by $[-1/2, 1/2]$. Then the integral becomes

$$\int_{-1/2}^{1/2} \sum_{n=1}^{\infty} n^{s/k-1} r^n e(n\beta)\, d\beta.$$

Hence, by (34) and (35), this is

$$\begin{aligned} &\ll \int_{-1/2}^{1/2} \min((1-r)^{-s/k}, |\beta|^{-s/k})\, d\beta \\ &= \int_{|\beta| \le 1-r} (1-r)^{-s/k}\, d\beta + \int_{1-r}^{1/2} \beta^{-s/k}\, d\beta + \int_{-1/2}^{-(1-r)} (-\beta)^{-s/k}\, d\beta \\ &\ll (1-r)^{1-s/k}. \end{aligned}$$

By (31), we have

$$\int_2 \ll \sum_{q \le y} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \frac{S(a,q)}{q} \right|^s (1-r)^{1-s/k}.$$

By Lemma 4.9 of [9] with $s \ge k+1$ and since $R = (1-r)^{-1}$, we have

$\int_2 \ll y^{\varepsilon} R^{s/k-1}$. Since $y = R^k$ and $s < 2k$, we have

$$\int_2 = o(R^{s/(2k)}). \tag{36}$$

By (27)–(29) and noting that $s < 2k$, we obtain $T \gg R^{s/(2k)}$. Hence, the theorem follows.

Proof of Theorem 3. We divide the solutions counted by $r_4(n)$ according to how many of the x_i are non-zero. Let

$$\varrho_j(n) = \operatorname{card}\{x_i \in \mathbb{Z}/\{0\} : x_1^2 + \ldots + x_j^2 = n\}.$$

Then

$$r_4(n) = \varrho_4(n) + 4\varrho_3(n) + 6\varrho_2(n) + 4\varrho_1(n) + \varrho_0(n).$$

Now we have

$$\varrho_4(n) = 2^{-4} R_4(n) \quad \text{and} \quad r_4(n) = \pi^2 \mathfrak{S}_4(n) n$$

(see Hardy [2], Section 3.11) and $4\varrho_3(n) + 6\varrho_2(n) + 4\varrho_1(n) + \varrho_0(n)$ is readily seen to be $\Omega_+(n^{1/2})$, which gives the first part of the theorem. The second part of the theorem follows at once from Theorem 2.

4. Proof of corollaries

Proof of Corollary 1. Multiply both sides of (3) by

$$R = (1-r)^{-1} = \sum_{l=0}^{\infty} r^l.$$

Then the left hand side of (3) becomes

$$\sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \left(R_s(n) - \Gamma\left(1 + \frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} \mathfrak{S}_s(n) n^{s/k-1} \right)^2 r^{n+l}.$$

Obviously, this is

$$\sum_{n=1}^{\infty} \sum_{m \le n} \left(R_s(m) - \Gamma\left(1 + \frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} \mathfrak{S}_s(m) m^{s/k-1} \right)^2 r^{n}.$$

The right hand side of (3) becomes $R^{s/k+1}$. Hence, we have

$$\sum_{n=1}^{\infty} \sum_{m \le n} \left(R_s(m) - \Gamma\left(1 + \frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} \mathfrak{S}_s(m) m^{s/k-1} \right)^2 r^{n} \gg R^{s/k+1}. \tag{37}$$

If (4) were false, then we would have

$$\sum_{m \le n} \left(R_s(m) - \Gamma\left(1 + \frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} \mathfrak{S}_s(m) m^{s/k-1} \right)^2 = o(n^{s/k}).$$

Multiply both sides by r^n and sum over n. Then

$$\sum_{n=1}^{\infty}\sum_{m\le n}\left(R_s(m)-\Gamma\left(1+\frac{1}{k}\right)^s\Gamma\left(\frac{s}{k}\right)^{-1}\mathfrak{S}_s(m)m^{s/k-1}\right)^2 r^n = o(R^{s/k+1}).$$

This contradicts (37), and hence (4) is true.

Proof of Corollary 2. By Cauchy's inequality,

$$\left(\sum_{n=1}^{\infty}\left(R_s(n)-\Gamma\left(1+\frac{1}{k}\right)^s\Gamma\left(\frac{s}{k}\right)^{-1}\mathfrak{S}_s(n)n^{s/k-1}\right)^2 r^n\right)\left(\sum_{n=1}^{\infty}r^n\right)$$
$$\ge\left(\sum_{n=1}^{\infty}\left(R_s(n)-\Gamma\left(1+\frac{1}{k}\right)^s\Gamma\left(\frac{s}{k}\right)^{-1}\mathfrak{S}_s(n)n^{s/k-1}\right)r^n\right)^2.$$

By Theorem 2, the right hand side is

$$\frac{s^2}{4}\Gamma\left(1+\frac{1}{k}\right)^{2s-2}R^{(2s-2)/k}+O(R^{(2s-3)/k})$$

and the second sum on the left hand side is $rR = R+O(1)$. Hence, the result follows.

Proof of Corollary 3. This is similar to the proof of Corollary 1.

Proof of Corollary 4. The first part of the corollary is immediate from Corollary 1 and the second part from Theorem 2.

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