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On sums involving reciprocals of the largest prime factor of an integer II

by

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1. Introduction. Let $P(n)$ denote the largest prime factor of an integer $n \ge 2$ and let $P(1) = 1$. Sums involving reciprocals of $P(n)$ have been investigated in many works (see, for example, [3], [5], [8]–[10], [13] and [14], where additional references may be found). In particular it was proved by Erdős, Pomerance and the author [5] that

$$\sum_{n\le x} \frac{1}{P(n)} = x\delta(x)\biggl(1 + O\biggl(\biggl(\frac{\log\log x}{\log x}\biggr)^{1/2}\biggr)\biggr), \tag{1.1}$$

where

$$\delta(x) = \int_2^x \varrho\biggl(\frac{\log x}{\log t}\biggr)\frac{dt}{t^2}, \tag{1.2}$$

and $\varrho(u)$ is the continuous solution (the so-called Dickman function) to the differential delay equation

$$u\varrho'(u) = -\varrho(u-1) \tag{1.3}$$

with the initial condition $\varrho(u) = 1$ for $0 \le u \le 1$. One also defines $\varrho(u) = 0$ for $u < 0$. It is well known that

$$\varrho(u) = \exp\biggl\{-u\biggl(\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + O\biggl(\biggl(\frac{\log_2 u}{\log u}\biggr)^2\biggr)\biggr)\biggr\}, \tag{1.4}$$

and from [10] it follows that

$$\delta(x) = \exp\biggl\{-(2\log x\log_2 x)^{1/2}\biggl(1 + g_0(x) + O\biggl(\biggl(\frac{\log_3 x}{\log_2 x}\biggr)^3\biggr)\biggr)\biggr\}, \tag{1.5}$$

where $\log_k x = \log(\log_{k-1} x)$ is the k-fold iterated natural logarithm of x

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and, for $r > -1$,

$$g_r(x) = \frac{\log_3 x + \log(1+r) - 2 - \log 2}{2\log_2 x}\left(1 + \frac{2}{\log_2 x}\right) - \frac{(\log_3 x + \log(1+r) - \log 2)^2}{8\log_2^2 x}.$$

It is possible to obtain more precise expressions for both functions appearing in the exponentials in (1.4) and (1.5). K. Alladi [1] (see also G. Tenenbaum [12]) proved an asymptotic formula for $\varrho(u)$:

$$\varrho(u) = \left(1 + O\left(\frac{1}{u}\right)\right)\left(\frac{\xi'(u)}{2\pi}\right)^{1/2} \exp\left(\gamma - \int_1^u \xi(t)\,dt\right), \tag{1.6}$$

where $\gamma = 0.57721\ldots$ is Euler's constant and $\xi = \xi(u)$ is the unique positive solution of the equation

$$e^\xi - 1 = u\xi \qquad (u > 1). \tag{1.7}$$

By using standard methods of asymptotic analysis one finds (see Hildebrand–Tenenbaum [7]) that, for $u \ge u_0 > 0$, $\xi(u)$ is given by the convergent series

$$\xi(u) = \log u + \log_2 u + \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} c_{m,k}\left(\frac{1}{\log u}\right)^m \left(\frac{1 + u\log_2 u}{u\log u}\right)^k, \tag{1.8}$$

where

$$c_{m,k} = \binom{m+k}{m} \operatorname*{Res}_{z=0}\left\{\frac{z^m}{(e^z-1)^{m+k}}\left(\frac{ze^z}{e^z-1} - \frac{m}{m+k}\right)\right\}.$$

Computing the first few values of $c_{m,k}$ one obtains, from (1.8),

$$\xi(u) = \log u + \log_2 u + \frac{\log_2 u}{\log u} - \frac{\frac12 \log_2^2 u - \log_2 u}{\log^2 u} + O\left(\left(\frac{\log_2 u}{\log u}\right)^3\right), \tag{1.9}$$

and from (1.6) and (1.8) one obtains a sharpening of (1.4).

The main aim of this paper is to study the asymptotic behaviour of the sums

$$S_r(x) := \sum_{n\le x,\, P(n)\equiv l\ (\mathrm{mod}\ k)} \frac{1}{P^r(n)} \qquad (r \ge 0) \tag{1.10}$$

and

$$T_r(x) := \sum_{n\le x,\, P(n)\equiv l\ (\mathrm{mod}\ k),\, P^2(n)\mid n} \frac{1}{P^r(n)} \qquad (r \ge -1), \tag{1.11}$$

where r is a fixed real number, $1 \le l \le k$, $(k, l) = 1$ are fixed integers. In the special case $k = 1$ it was shown by Ivić–Pomerance [10] that

$$S_r(x) = x \exp\left\{ -(2r \log x \log_2 x)^{1/2}\left(1 + g_{r-1}(x) + O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^3\right)\right)\right\} \quad (r > 0) \tag{1.12}$$

and

$$T_r(x) = x \exp\left\{ -(2r+2)^{1/2}(\log x \log_2 x)^{1/2}\left(1 + g_r(x) + O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^3\right)\right)\right\} \quad (r > -1). \tag{1.13}$$

The asymptotic behaviour of $S_r(x)$ for $r < 0$ and of $T_r(x)$ for $r < -1$ is substantially different, and is less difficult to determine than in the cases $r \geq 0$ and $r \geq -1$, respectively. For instance we have, if $r < 0$ and $k = 1$,

$$\begin{aligned} S_r(x) &= \sum_{p \le x} p^{-r} \Psi\left(\frac{x}{p}, p\right) + O(1) = \sum_{\sqrt{x} < p \le x} p^{-r}\left[\frac{x}{p}\right] + O(x^{1-r/2}) \\ &= \sum_{pn \le x} p^{-r} + O(x^{1-r/2}) = x^{1-r}\left(\sum_{j=1}^{J} \frac{c_j(r)}{\log^j x} + O\left(\frac{1}{\log^{J+1} x}\right)\right), \end{aligned}$$

where $J \geq 1$ is any fixed integer, $[t]$ is the integer part of t, $\Psi(x, y)$ denotes the number of positive integers $\leq x$ all of whose prime factors are $\leq y$, and p denotes primes. The constants $c_j(r)$ are effectively computable, and in particular $c_1(r) = \zeta(1-r)/(1-r)$. This is obtained analogously to De Koninck–Ivić [4], where the case $r = -1$ was treated.

One can generalize (1.1) by using the methods of [5] and obtain

$$S_r(x) = \frac{x}{\varphi(k)}\delta_r(x)\left(1 + O\left(\left(\frac{\log\log x}{\log x}\right)^{1/2}\right)\right) \quad (r > 0), \tag{1.14}$$

where

$$\delta_r(x) := \int_2^x \varrho\left(\frac{\log x}{\log t}\right)\frac{dt}{t^{r+1}} \quad (x \geq 2,\ r > 0), \tag{1.15}$$

so that $\delta_1(x) = \delta(x)$, as defined by (1.2). This is a non-elementary function which can be well approximated by elementary functions. A comparison of (1.12) and (1.14) (with $k = 1$) yields

$$\delta_r(x) = \exp\left\{ -(2r \log x \log_2 x)^{1/2}\left(1 + g_{r-1}(x) + O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^3\right)\right)\right\}. \tag{1.16}$$

Note that (1.16) is an asymptotic formula not for $\delta_r(x)$, but for $\log\delta_r(x)$. We shall evaluate $\delta_r(x)$ directly and prove an asymptotic formula for it, which contains the simpler function $\xi(u)$, defined by (1.7). In view of (1.6), $\xi(u)$ appears to be the function well-suited to approximate expressions containing $\varrho(u)$. We shall treat $S_r(x)$ for $r\ge 0$ and $T_r(x)$ for $r\ge -1$. Our aim is to sharpen the existing results, and in particular (1.14). This research is a continuation of several works on sums of $1/P(n)$ mentioned at the beginning, especially of [5]. It was instigated by a question of P. Erdős, who asked me for the asymptotic evaluation of $S_0(x)$, and whom I thank for valuable remarks.

2. Statement of results. Our first result gives an asymptotic formula for $\delta_r(x)$, defined by (1.15). The formula will be given in terms of $\xi=\xi(u)$, defined by (1.7), and $u_r=u_r(x)$. The latter denotes the solution of $f_r'(y) = 0$, where

$$f_r(y) := ry + \int_1^{y^{-1}\log x} \xi(t)\,dt \qquad (r>0,\ 1\le y\le \log x). \tag{2.1}$$

Thus u_r satisfies

$$u_r^2 = \xi\Big(\frac{\log x}{u_r}\Big)\frac{\log x}{r}, \tag{2.2}$$

and using (1.9) it follows that

$$u_r = \bigg\{\frac{\log x}{2r}\bigg(\log_2 x + \log_3 x + \log(r/2) + \frac{\log_3 x + \log(r/2)}{\log_2 x} + O\Big(\frac{\log_3^2 x}{\log_2^2 x}\Big)\bigg)\bigg\}^{1/2}. \tag{2.3}$$

It is clear that by using (1.8) and iteration one can obtain an asymptotic expansion of u_r. With this notation we can formulate

THEOREM 1. *For $r>0$ a fixed number we have*

$$\delta_r(x) = e^{\gamma}\bigg\{\frac{\xi'((\log x)/u_r)}{f_r''(u_r)}\bigg\}^{1/2}\bigg(1+O\bigg(\Big(\frac{\log_2 x}{\log x}\Big)^{1/4}\bigg)\bigg)e^{-f_r(u_r)}. \tag{2.4}$$

The main contribution in (2.4) comes from $e^{-f_r(u_r)}$, and by evaluating this term one can obtain a sharpening of (1.16). Comparing (2.4) with (1.6) we obtain

COROLLARY 1. *For $r>0$ a fixed number we have*

$$\delta_r(x) = \Big(\frac{2\pi}{f_r''(u_r)}\Big)^{1/2}\bigg(1+O\bigg(\Big(\frac{\log_2 x}{\log x}\Big)^{1/4}\bigg)\bigg)\varrho\Big(\frac{\log x}{u_r}\Big)e^{-ru_r}. \tag{2.5}$$

One can evaluate $f_r''(u_r)$ by noting that (2.1) gives

$$f_r''(y) = \xi'\left(\frac{\log x}{y}\right)\frac{\log^2 x}{y^4} + 2\xi\left(\frac{\log x}{y}\right)\frac{\log x}{y^3} = \frac{\log x(\log_2 x + O(\log_3 x))}{y^3} \tag{2.6}$$

for $u_r \ll y \ll u_r$. But since

$$\xi'(u) = \frac{\xi(u)}{1 + u\xi(u) - u} = \frac{1}{u}\left(1 + \frac{1}{\log u} + \frac{1 - \log_2 u}{\log^2 u} + O\left(\frac{\log_2^3 u}{\log^3 u}\right)\right) \tag{2.7}$$

and (2.2) holds, from (2.6) we obtain

$$f_r''(u_r) = \frac{2r}{u_r}\left(1 + \frac{\log x}{2ru_r^2}\left(1 + \frac{1}{\log_2 x - \log u_r}\right) + O\left(\frac{\log_3 x}{\log_2^3 x}\right)\right). \tag{2.8}$$

Hence we obtain

Corollary 2. *For $r > 0$ a fixed number we have*

$$\delta_r(x) = \left(\frac{\pi}{r}u_r\right)^{1/2}\left(1 - \frac{\log x}{4ru_r^2}\left(1 + \frac{1}{\log_2 x - \log u_r}\right) + O\left(\frac{1}{\log_2^2 x}\right)\right)\varrho\left(\frac{\log x}{u_r}\right)e^{-ru_r}. \tag{2.9}$$

For the sum $S_r(x)$ defined by (1.10), where $r \geq 0$ is fixed and $1 \leq l \leq k$, $(k, l) = 1$, are fixed integers, we shall prove the asymptotic expansion given by

Theorem 2. *For any given $\varepsilon > 0$,*

$$S_0(x) = \frac{x}{\varphi(k)} + O(x\exp(-\log^{3/8-\varepsilon} x)). \tag{2.10}$$

For fixed $r > 0$ and any fixed integer $J \geq 0$,

$$S_r(x) = \frac{x}{\varphi(k)}\int_2^x \varrho\left(\frac{\log x}{\log t}\right)\left(\sum_{j=0}^{J}\frac{Q_{j,r}(\log t)}{\log^j x} + O\left(\left(\frac{\log t}{\log x}\right)^{J+1}\right)\right)\frac{dt}{t^{r+1}} \tag{2.11}$$

for suitable polynomials $Q_{j,r}(x)$ $(j = 0, 1, 2, \ldots)$ of degree j in x whose coefficients depend on r. In particular,

$$Q_{0,r}(x) = r, \quad Q_{1,r}(x) = (r - r\gamma)(rx - 1). \tag{2.12}$$

From (2.11) we obtain

Corollary 3. *If $r > 0$ is fixed and u_r is given by* (2.2), *then*

$$S_r(x) = \left(r + \frac{r^2 - r^2\gamma}{\log x} u_r + O\left(\left(\frac{\log_2 x}{\log x}\right)^{3/4}\right)\right) \frac{x}{\varphi(k)} \delta_r(x). \tag{2.13}$$

One can use (2.3) to express the right-hand side of (2.13) as a function of x times $\delta_r(x)$. Corollary 3 in the special case $r = k = 1$ sharpens (1.1), showing incidentally that the error term in (1.1) is of the correct order of magnitude.

A result analogous to Theorem 2 holds for $T_r(x)$ defined by (1.11):

Theorem 3. *For any given $\varepsilon > 0$,*

$$\begin{aligned} T_{-1}(x) &= \frac{Cx}{\varphi(k)} + O(x \exp(-\log^{3/8-\varepsilon} x)), \\ C &= \int_0^\infty \frac{\varrho(v)}{v+2}\, dv < 1. \end{aligned} \tag{2.14}$$

For fixed $r > -1$ and any fixed integer $J \ge 1$,

$$T_r(x) = \frac{x}{\varphi(k)} \int_2^x \varrho\left(\frac{\log x}{\log t}\right) \times \left(\sum_{j=0}^{J} \frac{R_{j+1,r}(\log t)}{\log^j x} + O\left(\frac{\log^{J+2} t}{\log^{J+1} x}\right)\right) \frac{dt}{t^{r+2}} \tag{2.15}$$

for suitable polynomials $R_{j,r}(x)$ $(j = 1, 2, \ldots)$ of degree j in x whose coefficients depend on r. In particular,

$$R_{1,r}(x) = (r+1)^2 x - r - 1. \tag{2.16}$$

From (2.11) and (2.15) it follows that, for $r > -1$, one should compare $T_r(x)$ to $S_{r+1}(x)$. Indeed, we obtain

Corollary 4. *For fixed $r > -1$ and u_r defined by* (2.2) *we have*

$$T_r(x) = (r+1) S_{r+1}(x)(u_{r+1} + O((\log x)^{1/4} (\log_2 x)^{3/4})). \tag{2.17}$$

In the most interesting case $r = 0$ we obtain, from (2.17) and (2.3):

Corollary 5.

$$\sum_{n \le x,\, P^2(n) \mid n} 1 = \left(\frac{\log x}{2}\left(\log_2 x + \log_3 x - \log 2 + \frac{\log_3 x - \log 2}{\log_2 x} + O\left(\frac{\log_3^2 x}{\log_2^2 x}\right)\right)\right)^{1/2} \sum_{n \le x} \frac{1}{P(n)}.$$

Like earlier works on sums of $1/P(n)$, our proofs of Theorems 2 and 3 use results on $\Psi(x, y)$. In the next section the necessary material on $\Psi(x, y)$ and

$\varrho(u)$ will be presented. In Section 4 we shall use the classical Laplace method from asymptotic analysis to prove (2.4). In Section 5 we shall prove (2.10), and (2.11) in Section 6. Finally, the proof of Theorem 3 will be given in Section 7.

3. Some results on $\Psi(x,y)$. The basic tool in the proofs of Theorems 2 and 3 is the asymptotic formula

$$\Psi(x,y) = \Lambda(x,y)(1+O(\exp(-\log^{3/5-\varepsilon} y))), \tag{3.1}$$

proved by E. Saias [11]. This formula is a substantial sharpening of an older result of N. G. de Bruijn [2], who introduced $\Lambda(x,y)$ as a good approximation to $\Psi(x,y)$. One defines

$$\Lambda(x,y) = x \int_{1-0}^{\infty} \varrho\left(\frac{\log x - \log t}{\log y}\right) d\left(\frac{[t]}{t}\right) \tag{3.2}$$

for $x, y \ge 1$, if x is not an integer. If x is an integer, then $\Lambda(x,y) = \Lambda(x+0,y)$. Saias proved (3.1) in the wide range

$$\exp\{(\log\log x)^{5/3+\varepsilon}\} \le y \le x, \qquad x \ge x_0(\varepsilon). \tag{3.3}$$

The formula (3.1) is very sharp, as it gives

$$\Psi(x,y) = x\varrho(u)\left(1+O\left(\frac{\log(u+2)}{\log y}\right)\right) \qquad \left(u = \frac{\log x}{\log y}\right) \tag{3.4}$$

in the range (3.3), which is a result of A. Hildebrand [6]. The proof of (1.1) depended on (3.4), while the sharper (3.1) will be used for the proofs of Theorems 2 and 3. Note, however, that $\Lambda(x,y)$ is not readily approximated by elementary functions. From (3.2) it follows that it has discontinuities at natural numbers, which requires it to be treated with caution. Its properties, as well as those of $\Psi(x,y)$ and $\varrho(u)$ are extensively discussed by G. Tenenbaum [12]. For

$$\begin{gathered} x \ge 2, \qquad (\log x)^{1+\varepsilon} \le y \le x, \\ \min_{0\le j\le k,\, j\le y} \frac{u-j}{k+1-j} \ge \frac{\log\log y}{\log y}, \qquad u = \frac{\log x}{\log y}, \end{gathered} \tag{3.5}$$

one has the asymptotic formula

$$\Lambda(x,y) = x\sum_{j=0}^{k} a_j \varrho^{(j)}(u) \log^{-j} y + O(x|\varrho^{(k+1)}(u)|\log^{-k-1} y) \tag{3.6}$$

for any fixed integer $k \ge 0$. Note that $\varrho^{(k)}(u)$ for $k \ge 1$ is defined on $\mathbb{R}\setminus\{0,1,\ldots,k\}$, and has discontinuities of the first kind at the exceptional points $u = 0,1,\ldots,k$. In (3.6) the constants a_j are the Taylor coefficients of $s\zeta(s+1)/(s+1)$. Hence $a_0 = 1$, $a_1 = \gamma_0 - 1$, $a_2 = 1 - \gamma_0 + \gamma_1, \ldots$, where for $j \geq 0$,

$$\gamma_j = \frac{(-1)^j}{j!} \lim_{N\to\infty} \left(\sum_{n\leq N} \frac{1}{n} \log^j n - \frac{\log^{j+1} N}{j+1} \right),$$

and in particular $\gamma_0 = \gamma$ is Euler's constant. Another way to write explicitly a_j is to note that, for $\operatorname{Re} s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \int_{1-0}^{\infty} t^{-s}\, d[t] = s \int_1^{\infty} [t] t^{-s-1}\, dt.$$

By analytic continuation we then have, for $\operatorname{Re} s > -1$,

$$\begin{aligned}\frac{\zeta(s+1)}{s+1} &= \int_1^{\infty} t^{-s-1}\, dt - \int_1^{\infty} \{t\} t^{-s-2}\, dt \\ &= \frac{1}{s} - \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} \int_1^{\infty} \{t\} t^{-2} (\log t)^n\, dt,\end{aligned}$$

where $\{t\} = t - [t]$ is the fractional part of t. This shows that $a_0 = 1$, and for $j \geq 1$,

$$a_j = \frac{(-1)^j}{(j-1)!} \int_1^{\infty} \{t\} t^{-2} (\log t)^{j-1}\, dt. \tag{3.7}$$

From (1.6) and (1.7) one obtains

$$\varrho(u-1) = u\xi(u)\varrho(u)(1 + O(1/u)). \tag{3.8}$$

Hence by using (1.3), (3.8) and induction on k it follows that, for $k \geq 0$ a fixed integer and $u \geq k+2$,

$$\varrho^{(k)}(u) = (-\xi(u))^k \varrho(u)(1 + O(1/u)). \tag{3.9}$$

We also need the elementary estimate

$$\Psi(x,y) \ll x \exp\left(-\frac{\log x}{2\log y} \right) \quad (2 \leq y \leq x), \tag{3.10}$$

which is useful because it holds uniformly for all relevant y.

4. Asymptotic evaluation of $\delta_r(x)$. In this section we shall prove the asymptotic formula (2.4). We start from the definition (1.15), writing

$$\begin{aligned}\delta_r(x) &= \left(\int_2^{\exp(L_r(x)/3)} + \int_{\exp(L_r(x)/3)}^{\exp(3L_r(x))} + \int_{\exp(3L_r(x))}^{x} \right) \varrho\left(\frac{\log x}{\log t} \right) \frac{dt}{t^{r+1}} \\ &= I_1 + I_2 + I_3,\end{aligned}$$

say, where

$$L_r(x) := \left(\frac{\log x \log\log x}{2r}\right)^{1/2} \qquad (r > 0). \tag{4.1}$$

Using (1.4) and (1.16) we obtain

$$\begin{aligned} I_1 &= \int_2^{\exp(L_r(x)/3)} \varrho\left(\frac{\log x}{\log t}\right)\frac{dt}{t^{r+1}} \\ &\le \exp\left\{-(1+o(1))\frac{3\log x}{L_r(x)}\log\left(\frac{3\log x}{L_r(x)}\right)\right\}\int_2^{\infty}\frac{dt}{t^{r+1}} \\ &\ll \delta_r(x)\log^{-A} x \end{aligned}$$

for any given $A > 0$, and using the trivial $0 < \varrho(u) \le 1$ we have

$$\begin{aligned} I_3 &= \int_{\exp(3L_r(x))}^{x} \varrho\left(\frac{\log x}{\log t}\right)\frac{dt}{t^{r+1}} \le \int_{\exp(3L_r(x))}^{\infty} \frac{dt}{t^{r+1}} \\ &\ll e^{-3rL_r(x)} \ll \delta_r(x)\log^{-A} x. \end{aligned}$$

Hence for any given $A > 0$ we have

$$\delta_r(x) = (1 + O(\log^{-A} x))\int_{\exp(L_r(x)/3)}^{\exp(3L_r(x))} \varrho\left(\frac{\log x}{\log t}\right)\frac{dt}{t^{r+1}}. \tag{4.2}$$

Now we use the asymptotic formula (1.6) to obtain from (4.2), after the change of variable $\log t = y$,

$$\begin{aligned} \delta_r(x) = \left((2\pi)^{-1/2}e^{\gamma} + O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right) \\ \times \int_{L_r(x)/3}^{3L_r(x)} \left(\xi'\left(\frac{\log x}{y}\right)\right)^{1/2} \exp(-f_r(y))\, dy, \end{aligned} \tag{4.3}$$

where $f_r(y)$ is given by (2.1). The key step in the proof is to further restrict the range of integration in (4.3). To this end let

$$D_r = D_r(x) := 20(\log x)^{1/4}(r^{-1}\log_2 x)^{3/4} \tag{4.4}$$

and write the integral on the right-hand side of (4.3) as

$$\int_{u_r-D_r}^{u_r+D_r} + \int_{L_r(x)/3}^{u_r-D_r} + \int_{u_r+D_r}^{3L_r(x)} = J_1 + J_2 + J_3,$$

say, where u_r is defined by (2.2). We shall show that the contribution of J_2 and J_3 is small. Since both integrals are estimated similarly it suffices to consider only J_3. We have by Taylor's formula, since $f_r'(u_r)=0$ and (2.6) holds,

$$
\begin{aligned}
(4.5)\ J_3 = &\int\limits_{D_r}^{3L_r(x)-u_r}\Big(\xi'\Big(\frac{\log x}{y+u_r}\Big)\Big)^{1/2}\exp(-f_r(y+u_r))\,dy\\
&\ll \exp(-f_r(u_r))\int\limits_{D_r}^{3L_r(x)-u_r}\exp\Big(-\int\limits_{u_r}^{y+u_r}f_r''(t)(y+u_r-t)\,dt\Big)\,dy\\
&\ll \exp(-f_r(u_r))\int\limits_{D_r}^{3L_r(x)}\exp\Big(-\frac{\log x(\log_2 x+O(\log_3 x))}{(3L_r(x))^3}\\
&\qquad\qquad\times\int\limits_{u_r}^{y+u_r}(y+u_r-t)\,dt\Big)\,dy\\
&\ll \exp(-f_r(u_r))\int\limits_{D_r}^{\infty}\exp\Big(-\frac{r^{3/2}y^2}{20(\log x\log_2 x)^{1/2}}\Big)\,dy\\
&\ll \exp(-f_r(u_r))\exp\Big(-\frac{r^{3/2}D_r^2}{20(\log x\log_2 x)^{1/2}}\Big)(\log x)^{1/4}\\
&\ll \exp(-f_r(u_r))\log^{-10}x.
\end{aligned}
$$

To simplify J_1 note that, for $\frac{1}{3}L_r(x)\le y\le 3L_r(x)$ we have, as $x\to\infty$,

$$
f_r^{(3)}(y)\sim\frac{-3\log x\log\log x}{y^4},\qquad f_r^{(4)}(y)\sim\frac{12\log x\log\log x}{y^5}
$$

and

$$
\Big(\xi'\Big(\frac{\log x}{y+u_r}\Big)\Big)^{1/2}=\Big(1+O\Big(\frac{D_r}{u_r}\Big)\Big)\Big(\xi'\Big(\frac{\log x}{u_r}\Big)\Big)^{1/2}.
$$

Therefore

$$
\begin{aligned}
(4.6)\ J_1 = &\int\limits_{-D_r}^{D_r}\Big(\xi'\Big(\frac{\log x}{y+u_r}\Big)\Big)^{1/2}\exp(-f_r(y+u_r))\,dy\\
&=\Big(\xi'\Big(\frac{\log x}{u_r}\Big)\Big)^{1/2}\Big(1+O\Big(\Big(\frac{\log_2 x}{\log x}\Big)^{1/4}\Big)\Big)\exp(-f_r(u_r))\\
&\quad\times\int\limits_{-D_r}^{D_r}\exp\big\{-\tfrac12 f_r''(u_r)y^2-\tfrac16 f_r^{(3)}(u_r)y^3+O(L_r^{-3}(x)y^4)\big\}\,dy\\
&=\Big(\xi'\Big(\frac{\log x}{u_r}\Big)\Big)^{1/2}\Big(1+O\Big(\Big(\frac{\log_2 x}{\log x}\Big)^{1/4}\Big)\Big)
\end{aligned}
$$

$$\times \exp(-f_r(u_r)) \int_{-D_r}^{D_r} \exp\big(-\tfrac{1}{2} f_r''(u_r) y^2\big)\big(1 - \tfrac{1}{6} f_r^{(3)}(u_r) y^3 + O(L_r^{-3}(x) y^4 (1 + L_r^{-1}(x) y^2))\big)\, dy$$

$$= \left(\xi'\left(\frac{\log x}{u_r}\right)\right)^{1/2} \left(1 + O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/4}\right)\right) \times \exp(-f_r(u_r)) \int_{-D_r}^{D_r} \exp\big(-\tfrac{1}{2} f_r''(u_r) y^2\big)\, dy$$

$$= \left(2\pi\xi'\left(\frac{\log x}{u_r}\right)\right)^{1/2} (f_r''(u_r))^{-1/2} \times \left(1 + O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/4}\right)\right) \exp(-f_r(u_r)),$$

since for $a, c > 0$,

$$\int_{-a}^{a} e^{-cx^2}\, dx = \int_{-\infty}^{\infty} e^{-cx^2}\, dx + O\left(\frac{1}{ac} \int_{a}^{\infty} 2cxe^{-cx^2}\, dx\right) = \left(\frac{\pi}{c}\right)^{1/2} + O\left(\frac{e^{-ac^2}}{ac}\right).$$

Theorem 1 follows now from (4.3), (4.5) and (4.6).

5. The number of integers $\le x$ for which $P(n) \equiv l \pmod{k}$. In this section we shall prove the asymptotic formula (2.10) for $S_0(x)$, the number of integers $\le x$ for which $P(n) \equiv l \pmod{k}$. If n is counted by $S_0(x)$, then $n = 1$ (if $l \equiv 1 \pmod{k}$) or $n = mp$, where $P(m) \le p$, $p \equiv l \pmod{k}$ and p denotes primes. Thus, for $0 < \alpha < 1$ to be determined later, we have

$$S_0(x) = \sum_{mp \le x,\, P(m) \le p,\, p \equiv l \ (\mathrm{mod}\, k)} 1 + O(1) = \sum_{p \le x,\, p \equiv l \ (\mathrm{mod}\, k)} \Psi\left(\frac{x}{p}, p\right) + O(1) = \sum_{\substack{\exp(\log^\alpha x) < p \le x \\ p \equiv l \ (\mathrm{mod}\, k)}} \Psi\left(\frac{x}{p}, p\right) + O\big(x \exp\big(-\tfrac{1}{3} \log^{1-\alpha} x\big)\big), \tag{5.1}$$

since (3.10) yields

$$\sum_{p\le\exp(\log^\alpha x),\, p\equiv l\ (\mathrm{mod}\, k)} \Psi\Big(\frac{x}{p}, p\Big) \ll \sum_{p\le\exp(\log^\alpha x)} \frac{x}{p}\exp\Big(-\frac{\log x}{2\log p}\Big)$$
$$\ll x\exp\big(-\tfrac{1}{3}\log^{1-\alpha} x\big).$$

Now we use the prime number theorem for arithmetic progressions in essentially the strongest known form, namely

$$\sum_{p\le x,\, p\equiv l\ (\mathrm{mod}\, k)} 1 = \frac{x}{\varphi(k)}\int_2^x \frac{dt}{\log t} + \Delta(x), \tag{5.2}$$
$$\Delta(x) \ll x\exp(-\log^{3/5-\varepsilon} x)$$

for any given $\varepsilon > 0$. Therefore

$$\sum_{\exp(\log^\alpha x)<p\le x,\, p\equiv l\ (\mathrm{mod}\, k)} \Psi\Big(\frac{x}{p}, p\Big) \tag{5.3}$$
$$= \frac{1}{\varphi(k)}\int_{\exp(\log^\alpha x)}^x \Psi\Big(\frac{x}{t}, t\Big)\frac{dt}{\log t} + \int_{\exp(\log^\alpha x)}^x \Psi\Big(\frac{x}{t}, t\Big)\, d\Delta(t)$$
$$= \frac{1}{\varphi(k)}\int_{\exp(\log^\alpha x)}^x \Psi\Big(\frac{x}{t}, t\Big)\frac{dt}{\log t} + I,$$

say. Integration by parts and (3.10) give

$$I = \Psi\Big(\frac{x}{t}, t\Big)\Delta(t)\Big|_{e^{\log^\alpha x}}^x - \int_{e^{\log^\alpha x}}^x \Delta(t)\, d\Psi\Big(\frac{x}{t}, t\Big) \tag{5.4}$$
$$= O(x\exp(-\log^{3/5-\varepsilon} x)) + O\big(x\exp\big(-\tfrac{1}{2}\log^{1-\alpha} x\big)\big)$$
$$+ \int_1^{xe^{-\log^\alpha x}} \Delta\Big(\frac{x}{t}\Big)\, d\Psi\Big(t, \frac{x}{t}\Big)$$
$$\ll x\exp(-\log^{3/5-\varepsilon} x) + x\exp\big(-\tfrac{1}{2}\log^{1-\alpha} x\big)$$
$$+ \int_1^{xe^{-\log^\alpha x}} \Big|\Delta\Big(\frac{x}{t}\Big)\Big|\, d[t]$$
$$\ll x\exp(-\log^{3/5-\varepsilon} x) + x\exp\big(-\tfrac{1}{2}\log^{1-\alpha} x\big)$$
$$+ x\sum_{n\le xe^{-\log^\alpha x}} \frac{1}{n}\exp\Big(-\Big(\log\frac{x}{n}\Big)^{3/5-\varepsilon}\Big)$$

$$\ll x\exp\left(-\tfrac{1}{2}\log^{1-\alpha}x\right)+x\exp(-\log^{3\alpha/5-\varepsilon}x)$$
$$\ll x\exp(-\log^{3/8-\varepsilon}x)$$

with the choice $\alpha = 5/8$.

From (5.1), (5.3) and (5.4) we obtain

$$S_0(x)=\frac{1}{\varphi(k)}\int\limits_{\exp(\log^{5/8}x)}^{x}\Psi\left(\frac{x}{t},t\right)\frac{dt}{\log t}+O(x\exp(-\log^{3/8-\varepsilon}x)). \tag{5.5}$$

Now observe that (5.5) holds also for $k=1$, in which case we trivially have

$$S_0(x)=x+O(1).$$

Hence we obtain

$$\int\limits_{\exp(\log^{5/8}x)}^{x}\Psi\left(\frac{x}{t},t\right)\frac{dt}{\log t}=x+O(x\exp(-\log^{3/8-\varepsilon}x)), \tag{5.6}$$

and (2.10) follows from (5.5) and (5.6).

We remark that the use of the conditional result $\Delta(x)\ll x^{1/2+\varepsilon}$, which is a consequence of the Generalized Riemann Hypothesis for L-functions, would not by our method of proof lead to any substantial improvements. It would give $O(x\exp(-C\log^{1/2}x))$ $(C>0)$ for the error term in (2.10), and the same for (2.14), while (2.11) and (2.15) would remain unaffected. For (2.14) we would have to use the conditional improvement of (3.1) under the Riemann hypothesis (see p. 81 of E. Saias [11]).

6. Sum of reciprocals of $P^r(n)$. In this section we shall prove the asymptotic expansion (2.11). If $L_r(x)$ is defined by (4.1), then

$$\begin{aligned}S_r(x)&=\sum_{p\le x,\,p\equiv l\ (\mathrm{mod}\ k)}\frac{1}{p^r}\Psi\left(\frac{x}{p},p\right)+O(1)\\&=(1+O(\log^{-A}x))\sum_{\substack{\exp(L_r(x)/3)<p\le\exp(3L_r(x))\\ p\equiv l\ (\mathrm{mod}\ k)}}\frac{1}{p^r}\Psi\left(\frac{x}{p},p\right)\end{aligned} \tag{6.1}$$

for any given $A>0$. This is similar to (4.2), the contribution of $p\le\exp(L_r(x)/3)$ being estimated by (3.4) and that of $p>\exp(3L_r(x))$ trivially by using $\Psi(x,y)\le x$. Now set for brevity

$$U:=\exp\left(\tfrac{1}{3}L_r(x)\right),\qquad V:=\exp(3L_r(x)).$$

By using (5.2) we have

$$
\begin{aligned}
(6.2)\qquad &\sum_{U\le p\le V,\, p\equiv l \,(\mathrm{mod}\, k)} \frac{1}{p^r}\Psi\Big(\frac{x}{p},p\Big)\\
&= \frac{1}{\varphi(k)}\int_U^V \Psi\Big(\frac{x}{t},t\Big)\frac{dt}{t^r\log t} + \int_U^V \Psi\Big(\frac{x}{t},t\Big)\frac{d\Delta(t)}{t^r}\\
&= \frac{1}{\varphi(k)}\int_U^V \Psi\Big(\frac{x}{t},t\Big)\frac{dt}{t^r\log t}\\
&\quad + \Psi\Big(\frac{x}{t},t\Big)\frac{\Delta(t)}{t^r}\Big|_U^V - \int_U^V \Delta(t)\, d\Big(\frac{1}{t^r}\Psi\Big(\frac{x}{t},t\Big)\Big).
\end{aligned}
$$

The integrated terms are $\ll x\delta_r(x)\log^{-A} x$ for any fixed $A>0$. This follows from (3.4), (1.4) and (1.16). The last integral in (6.2) equals

$$
\begin{aligned}
(6.3)\qquad & -r\int_U^V \frac{\Delta(t)}{t^{r+1}}\Psi\Big(\frac{x}{t},t\Big)\,dt + \int_U^V \frac{1}{t^r}\Delta(t)\,d\Psi\Big(\frac{x}{t},t\Big)\\
&\ll \exp(-\log^{3/10-\varepsilon} x)\\
&\quad\times\Big(x\int_U^V \varrho\Big(\frac{\log x}{\log t}-1\Big)\frac{dt}{t^{r+1}} + x^{1-r}\int_{x/V}^{x/U} t^{r-1}\,d\Psi\Big(t,\frac{x}{t}\Big)\Big).
\end{aligned}
$$

Using (3.8) we obtain

$$
\begin{aligned}
\int_U^V \varrho\Big(\frac{\log x}{\log t}-1\Big)\frac{dt}{t^{r+1}} &\ll \int_U^V \frac{\log x}{\log t}\log_2 x\cdot \varrho\Big(\frac{\log x}{\log t}\Big)\frac{dt}{t^{r+1}}\\
&\ll \delta_r(x)(\log x\log_2 x)^{1/2},
\end{aligned}
$$

while from (3.4) and (1.4) we obtain

$$
\begin{aligned}
-x^{1-r}\int_{x/V}^{x/U} t^{r-1}d\Psi\Big(t,\frac{x}{t}\Big) &= \int_U^V t^{1-r}d\Psi\Big(\frac{x}{t},t\Big)\\
&= \Psi\Big(\frac{x}{t},t\Big)t^{1-r}\Big|_U^V + (r-1)\int_U^V \frac{1}{t^r}\Psi\Big(\frac{x}{t},t\Big)\,dt\\
&\ll x\delta_r(x)(\log x\log_2 x)^{1/2}.
\end{aligned}
$$

Thus it follows from (6.1)–(6.3) that, for any fixed $A > 0$,

$$S_r(x) = \frac{1}{\varphi(k)} \int_U^V \Psi\left(\frac{x}{t}, t\right) \frac{dt}{t^r \log t} + O(x\delta_r(x) \log^{-A} x). \tag{6.4}$$

Then applying (3.1) we find that

$$\begin{aligned} \int_U^V \Psi\left(\frac{x}{t}, t\right) \frac{dt}{t^r \log t} \\ = (1 + O(\exp(-\log^{3/10-\varepsilon} x))) \int_U^V \Lambda\left(\frac{x}{t}, t\right) \frac{dt}{t^r \log t}. \end{aligned} \tag{6.5}$$

In the last integral we may use (3.6) with x replaced by x/t and y replaced by t, since (3.5) will be satisfied for $U \le t \le V$. Therefore

$$\begin{aligned} \int_U^V \Lambda\left(\frac{x}{t}, t\right) \frac{dt}{t^r \log t} &= x \sum_{j=0}^{J} a_j \int_U^V \varrho^{(j)}\left(\frac{\log x}{\log t} - 1\right) \frac{dt}{t^{r+1} \log^{j+1} t} \\ &\quad + O\left(x \int_U^V \left|\varrho^{(J+1)}\left(\frac{\log x}{\log t} - 1\right)\right| \frac{dt}{t^{r+1} \log^{J+2} t}\right). \end{aligned} \tag{6.6}$$

It remains to evaluate the integrals

$$I_{j,k}(x,r) := \int_U^V \varrho^{(j)}\left(\frac{\log x}{\log t} - 1\right) \frac{dt}{t^{r+1} \log^k t} \qquad (r > 0) \tag{6.7}$$

when $j \ge 1$ and k are given integers, in particular when $k = j+1$. Integrating by parts, using (1.4) and (3.8) we obtain, for any fixed $A > 0$,

$$\begin{aligned} I_{j,k}(x,r) &= \frac{-1}{\log x} \int_U^V \frac{1}{t^r \log^{k-2} t}\, d\left\{\varrho^{(j-1)}\left(\frac{\log x}{\log t} - 1\right)\right\} \\ &= \frac{1}{\log x} \int_U^V \varrho^{(j-1)}\left(\frac{\log x}{\log t} - 1\right) \\ &\quad \times \left(\frac{-r}{t^{r+1} \log^{k-2} t} + \frac{2-k}{t^{r+1} \log^{k-1} t}\right) dt + O(\delta_r(x) \log^{-A} x). \end{aligned}$$

Hence we have the recursive formula

$$\begin{aligned} I_{j,k}(x,r) &= \frac{-1}{\log x}(r I_{j-1,k-2}(x,r) + (k-2) I_{j-1,k-1}(x,r)) \\ &\quad + O(\delta_r(x) \log^{-A} x), \end{aligned} \tag{6.8}$$

and in particular,

$$I_{1,2}(x,r) = \frac{-r}{\log x} I_{0,0}(x,r) + O(\delta_r(x)\log^{-A} x) \tag{6.9}$$
$$= \frac{-r}{\log x} \int_2^x \varrho\Big(\frac{\log x}{\log t} - 1\Big)\frac{dt}{t^{r+1}} + O(\delta_r(x)\log^{-A} x).$$

If we use induction on m $(1 \le m \le j)$, (6.8) and $\binom{m}{k-1} + \binom{m}{k} = \binom{m+1}{k}$, then we obtain

$$I_{j,j+1}(x,r) = \frac{(-1)^m}{\log^m x}\Big\{ r^m I_{j-m,j+1-2m}(x,r) \tag{6.10}$$
$$+ \sum_{k=1}^{m} \binom{m}{k} r^{m-k}(j-m)(j-m+1)\ldots(j-m+k-1)$$
$$\times I_{j-m,j+1-2m+k}(x,r)\Big\} + O(\delta_r(x)\log^{-A} x).$$

Hence for $m = j$ it follows from (6.10) that

$$I_{j,j+1}(x,r) = \Big(\frac{-r}{\log x}\Big)^j I_{0,1-j}(x,r) + O(\delta_r(x)\log^{-A} x). \tag{6.11}$$

Thus from (6.4)–(6.6) and (6.11) we obtain

$$S_r(x) = \frac{x}{\varphi(k)} \sum_{j=0}^{J} \frac{a_j}{\log^j x} \int_2^x \varrho\Big(\frac{\log x}{\log t} - 1\Big) P_{j-1,r}(\log t)\frac{dt}{t^{r+1}} \tag{6.12}$$
$$+ O(x\delta_r(x)\log^{-A} x)$$
$$+ O\Big(\frac{x}{\log^{J+1} x}\int_2^x \varrho\Big(\frac{\log x}{\log t} - 1\Big)(\log t)^J \frac{dt}{t^{r+1}}\Big)$$

with

$$P_{-1,r}(x) = 1/x, \qquad P_{m,r}(x) = (-r)^{m+1}x^m \qquad (m = 0,1,2,\ldots). \tag{6.13}$$

As in (6.9) we have, by using (1.4), replaced the limits of integration U and V by 2 and x, respectively. In doing this we have created an error which is certainly

$$\ll x\delta_r(x)\log^{-A} x.$$

Note that the absolute value signs in (6.6) are unimportant, since by (3.8) the function $\varrho^{(k)}(u)$ is of constant sign for $u \ge u_0(k)$. The integrals in (6.12) are further transformed by using

$$\varrho\Big(\frac{\log x}{\log t} - 1\Big)\frac{dt}{t\log t} = d\Big(\varrho\Big(\frac{\log x}{\log t}\Big)\Big),$$

which easily follows from (1.3). This gives

$$\int_2^x \varrho\Big(\frac{\log x}{\log t}-1\Big)P_{-1,r}(\log t)\frac{dt}{t^{r+1}} = (r+O(\log^{-A} x))\delta_r(x), \tag{6.14}$$

and for $j \ge 1$

$$\begin{aligned}\int_2^x \varrho\Big(\frac{\log x}{\log t}-1\Big)P_{j-1,r}(\log t)\frac{dt}{t^{r+1}} &= O(\delta_r(x)\log^{-A} x)\\ &+ r\int_2^x \varrho\Big(\frac{\log x}{\log t}\Big)\Big(P_{j-1,r}(\log t)\Big(\log t-\frac{1}{r}\Big)-P'_{j-1,r}(\log t)\frac{\log t}{r}\Big)\frac{dt}{t^{r+1}}.\end{aligned} \tag{6.15}$$

Therefore if we insert (6.14) and (6.15) in (6.12) we obtain (2.11) with $Q_{0,r}(x)=r$, and in view of (6.13) for $j=1,2,\ldots$ we have explicitly

$$\begin{aligned}Q_{j,r}(x) &= ra_j\Big(P_{j-1,r}(x)\Big(x-\frac{1}{r}\Big)-P'_{j-1,r}(x)\frac{x}{r}\Big)\\ &= (-r)^j a_j(rx^j - jx^{j-1}),\end{aligned} \tag{6.16}$$

where a_j is given by (3.7). Since $a_1=\gamma-1$ we obtain the expression for $Q_{1,r}(x)$ given by (2.12).

To obtain (2.13) from (2.11) note that we may write

$$\begin{aligned}S_r(x) = \frac{rx}{\varphi(k)}&\int_{\exp(u_r-D_r)}^{\exp(u_r+D_r)} \varrho\Big(\frac{\log x}{\log t}\Big)\\ &\times\Big(1+(1-\gamma)\frac{(r\log t-1)}{\log x}+O\Big(\frac{\log^2 t}{\log^2 x}\Big)\Big)\frac{dt}{t^{r+1}},\end{aligned} \tag{6.17}$$

where u_r is defined by (2.2) and D_r by (4.4). This follows by the analysis of Section 4, where $\delta_r(x)$ was evaluated. Hence in the integral in (6.17) we have

$$\log t = u_r + O(D_r) = u_r + O((\log x)^{1/4}(\log_2 x)^{3/4}),$$

and (2.13) follows.

Since by (6.14) the relevant range of integration in the expression for $S_r(x)$ is $[\exp(u_r-D_r),\exp(u_r+D_r)]$, it is seen that in (2.11) one can write the error term outside the integral and obtain

$$\begin{aligned}S_r(x) &= \frac{x}{\varphi(k)}\sum_{j=0}^{J}\frac{1}{\log^j x}\int_2^x Q_{j,r}(\log t)\varrho\Big(\frac{\log x}{\log t}\Big)\frac{dt}{t^{r+1}}\\ &+ O\Big(x\Big(\frac{\log_2 x}{\log x}\Big)^{(J+1)/2}\int_2^x \varrho\Big(\frac{\log x}{\log t}\Big)\frac{dt}{t^{r+1}}\Big)\end{aligned} \tag{6.18}$$

in view of (2.2). Similarly one can write (2.15) with the error term outside the integral (it will be like the one in (6.18) but with $J+2$ in place of $J+1$). However, (2.15) has the advantage that one sees immediately that the error term is of the smaller order of magnitude than the main terms, which is not obvious in (6.18).

7. The sum in which $P^2(n)\,|\,n$. We turn now to the proof of Theorem 3, which follows the method of the proof of Theorem 2. For this reason we shall be relatively brief and stress only the salient points. We start with the proof of (2.14). Since

$$T_{-1}(x) = \sum_{mp^2\le x,\, p\equiv l \pmod k,\, P(m)\le p} p + O(1),$$

analogously to (5.5) we obtain

$$T_{-1}(x) = \frac{1}{\varphi(k)} \int\limits_{\exp(\log^{5/8} x)}^{\sqrt{x}} t\Psi\left(\frac{x}{t^2}, t\right)\frac{dt}{\log t} + O(x\exp(-\log^{3/8-\varepsilon} x)). \tag{7.1}$$

However, in the case of $S_0(x)$ we knew that $S_0(x) = x + O(1)$ when $k = 1$, whereas nothing analogous seems to hold for $T_{-1}(x)$ when $k = 1$. For this reason we shall prove (2.14) directly from (7.1), noting that this method incidentally provides an alternative proof of (2.10). For $\Psi(x/t^2, t)$ in (7.1) and $t \le x^{1/3}$ we use (3.1), noting that (3.3) will hold, and for $x^{1/3} \le t \le x^{1/2}$ we use $\Lambda(x, y) = \Psi(x, y) = [x]$ $(y \ge x)$. Thus

$$T_{-1}(x) = \frac{1}{\varphi(k)} \int\limits_{\exp(\log^{5/8} x)}^{\sqrt{x}} y\Lambda\left(\frac{x}{y^2}, y\right)\frac{dy}{\log y} + O(x\exp(-\log^{3/8-\varepsilon} x)). \tag{7.2}$$

Now we use the representation (3.2) for $\Lambda(x, y)$, invert the order of integration and make the change of variable $(\log(x/t))/\log y - 2 = v$. This gives

$$\int\limits_{\exp(\log^{5/8} x)}^{\sqrt{x}} y\Lambda\left(\frac{x}{y^2}, y\right)\frac{dy}{\log y} = x \int\limits_{\exp(\log^{5/8} x)}^{\sqrt{x}} \frac{1}{y\log y}\left(\int\limits_{1-0}^{x/y^2} \varrho\left(\frac{\log x - \log t}{\log y} - 2\right) d\left(\frac{[t]}{t}\right)\right) dy$$

$$= x \int_{1-0}^{X} \left(\int_{\exp(\log^{5/8} x)}^{(x/t)^{1/2}} \varrho\left(\frac{\log x - \log t}{\log y} - 2\right) \frac{dy}{y \log y}\right) d\left(\frac{[t]}{t}\right)$$

$$= x \int_{1-0}^{X} \left(\int_{0}^{W} \frac{\varrho(v)}{v+2}\, dv\right) d\left(\frac{[t]}{t}\right)$$

($X = xe^{-2\log^{5/8} x}$, $W = (\log x - \log t)/\log^{5/8} x - 2$). From (1.4) we have

$$g(x) := \int_{x}^{\infty} \frac{\varrho(v)}{v+2}\, dv \ll e^{-x}, \quad g'(x) = -\frac{\varrho(x)}{x+2} \ll e^{-x}. \tag{7.3}$$

Integrating by parts, using (7.3) and $d([t]/t) = -d(\{t\}/t)$, we then obtain

$$\int_{\exp(\log^{5/8} x)}^{\sqrt{x}} y\Lambda\left(\frac{x}{y^2}, y\right) \frac{dy}{\log y} \tag{7.4}$$

$$= x \int_{1-0}^{X} \left(\int_{0}^{\infty} \frac{\varrho(v)}{v+2}\, dv\right) d\left(\frac{[t]}{t}\right) + x \int_{1-0}^{X} g(W)\, d\left(\frac{\{t\}}{t}\right)$$

$$= Cx + O(e^{2\log^{5/8} x}) + xg(W)\frac{\{t\}}{t}\bigg|_{t=1-0}^{t=X}$$

$$+ x \int_{1-0}^{X} \frac{\{t\}}{t} g'(W) \frac{dt}{t \log^{5/8} x}$$

$$= Cx + O(x \exp(-\log^{3/8-\varepsilon} x)).$$

Thus (2.14) follows from (7.2) and (7.4). Note that

$$C = \int_{0}^{\infty} \frac{\varrho(v)}{v+2}\, dv < \int_{0}^{\infty} \frac{\varrho(v)}{v+1}\, dv = -\int_{0}^{\infty} \varrho'(v+1)\, dv = \varrho(1) = 1,$$

hence $T_{-1}(x)$ is proportional to $S_0(x)$.

To prove (2.15) note first that

$$T_r(x) = \sum_{p \le \sqrt{x},\, p \equiv l \pmod{k}} \frac{1}{p^r} \Psi\left(\frac{x}{p^2}, p\right) + O(1) \tag{7.5}$$

$$= \frac{1}{\varphi(k)} \int_{Y}^{Z} \Psi\left(\frac{x}{t^2}, t\right) \frac{dt}{t^r \log t} + O(x\delta_{r+1}(x) \log^{-A} x)$$

for any fixed $A > 0$. This follows analogously to (6.4), where we set

$$Y := \exp\left(\tfrac{1}{3} L_{r+1}(x)\right), \quad Z := \exp(3L_{r+1}(x)),$$

and $L_r(x)$ is given by (4.1). Applying (3.1) we obtain

$$\int_Y^Z \Psi\Big(\frac{x}{t^2}, t\Big)\frac{dt}{t^r\log t} = (1 + O(\exp(-\log^{3/10-\varepsilon} x)))\int_Y^Z \Lambda\Big(\frac{x}{t^2}, t\Big)\frac{dt}{t^r\log t}. \tag{7.6}$$

In the last integral we may use (3.6) to obtain, similarly to (6.6),

$$\begin{aligned}\int_Y^Z \Lambda\Big(\frac{x}{t^2}, t\Big)\frac{dt}{t^r\log t} &= x\sum_{j=0}^{J} a_j \int_Y^Z \varrho^{(j)}\Big(\frac{\log x}{\log t} - 2\Big)\frac{dt}{t^{r+2}\log^{j+1} t}\\ &\quad + O\Big(x\int_Y^Z \Big|\varrho^{(J+1)}\Big(\frac{\log x}{\log t} - 2\Big)\Big|\frac{dt}{t^{r+2}\log^{J+2} t}\Big).\end{aligned} \tag{7.7}$$

For the integrals

$$J_{j,k}(x,r) := \int_Y^Z \varrho^{(j)}\Big(\frac{\log x}{\log t} - 2\Big)\frac{dt}{t^{r+2}\log^k t} \qquad (r > -1),$$

where $j \ge 1$ and k are integers, a recursive formula analogous to (6.8) holds:

$$\begin{aligned}J_{j,k}(x,r) = \frac{-1}{\log x}((r+1)J_{j-1,k-2}(x,r)\\ + (k-2)J_{j-1,k-1}(x,r)) + O(\delta_{r+1}(x)\log^{-A} x)\end{aligned} \tag{7.8}$$

for any fixed $A > 0$. If we set $k = j+1$ in (7.8) and iterate j times, then from (7.5)–(7.7) we shall obtain

$$\begin{aligned}T_r(x) &= \frac{x}{\varphi(k)}\sum_{j=0}^{J}\frac{a_j}{\log^j x}\int_Y^Z \varrho\Big(\frac{\log x}{\log t} - 2\Big)q_{j-1,r}(\log t)\frac{dt}{t^{r+2}}\\ &\quad + O(x\delta_{r+1}(x)\log^{-A} x)\\ &\quad + O\Big(\frac{x}{\log^{J+1} x}\int_Y^Z \varrho\Big(\frac{\log x}{\log t} - 2\Big)\log^J t\frac{dt}{t^{r+2}}\Big),\end{aligned} \tag{7.9}$$

where, analogously to (6.13), we have

$$q_{-1,r}(x) = 1/x, \quad q_{m,r}(x) = (-r-1)^{m+1}x^m \quad (m = 0, 1, 2, \ldots).$$

To transform the integrals in (7.9) we use

$$d\varrho\left(\frac{\log x}{\log t}-1\right)=\varrho\left(\frac{\log x}{\log t}-2\right)\frac{dt}{(1-\log t/\log x)t\log t},$$

$$d\varrho\left(\frac{\log x}{\log t}\right)=\varrho\left(\frac{\log x}{\log t}-1\right)\frac{dt}{t\log t}.$$

We obtain

$$\begin{aligned}(7.10)\qquad \int_Y^Z \varrho\left(\frac{\log x}{\log t}-2\right)q_{-1,r}(\log t)\frac{dt}{t^{r+2}} &= \int_Y^Z \varrho\left(\frac{\log x}{\log t}\right)\left(p_{1,r}(\log t)+\frac{p_{2,r}(\log t)}{\log x}\right)\frac{dt}{t^{r+2}} \\ &\quad + O(\delta_{r+1}(x)\log^{-A} x),\end{aligned}$$

where $p_{1,r}(x)=(r+1)^2x-r-1$ and $p_{2,r}(x)$ is a quadratic function in x. In the general case, if $P_{j,r}(x)$ denotes a generic polynomial of degree j (≥ 1) in x whose coefficients depend on r, we have

$$\begin{aligned}(7.11)\qquad \int_Y^Z \varrho\left(\frac{\log x}{\log t}-2\right)P_{j-1,r}(\log t)\frac{dt}{t^{r+2}} &= \int_Y^Z \left(P_{j,r}(\log t)+\frac{P_{j+1,r}(\log t)}{\log x}\right)\frac{1}{t^{r+1}}\,d\varrho\left(\frac{\log x}{\log t}-1\right) \\ &\quad + O(\delta_{r+1}(x)\log^{-A} x) \\ &= \int_Y^Z \varrho\left(\frac{\log x}{\log t}-1\right)\left(P_{j,r}(\log t)+\frac{P_{j+1,r}(\log t)}{\log x}\right)\frac{dt}{t^{r+2}} \\ &\quad + O(\delta_{r+1}(x)\log^{-A} x) \\ &= \int_Y^Z \left(P_{j+1,r}(\log t)+\frac{P_{j+2,r}(\log t)}{\log x}\right)\frac{1}{t^{r+1}}\,d\varrho\left(\frac{\log x}{\log t}\right) \\ &\quad + O(\delta_{r+1}(x)\log^{-A} x) \\ &= \int_Y^Z \varrho\left(\frac{\log x}{\log t}\right)\left(P_{j+1,r}(\log t)+\frac{P_{j+2,r}(\log t)}{\log x}\right)\frac{dt}{t^{r+2}} \\ &\quad + O(\delta_{r+1}(x)\log^{-A} x).\end{aligned}$$

If we insert (7.10) and (7.11) in (7.9) and replace the limits of integration Y and Z by 2 and x, respectively, we obtain (2.15) with $R_{1,r}(x)$ given by (2.16).

To obtain (2.17) note that, analogously to (6.14), we have

$$T_r(x) = \frac{(r+1)x}{\varphi(k)} \int\limits_{\exp(u_{r+1}-D_{r+1})}^{\exp(u_{r+1}+D_{r+1})} \varrho\left(\frac{\log x}{\log t}\right) \times \left((r+1)\log t - 1 + O\left(\frac{\log^2 t}{\log x}\right)\right)\frac{dt}{t^{r+2}}, \tag{7.12}$$

where u_r is defined by (2.2) and D_r by (4.4). Thus for t in (7.12) we have

$$\log t = u_{r+1} + O(D_{r+1}) = u_{r+1} + O((\log x)^{1/4}(\log_2 x)^{3/4}),$$

which gives

$$T_r(x) = ((r+1)u_{r+1} + O((\log x)^{1/4}(\log_2 x)^{3/4}))\frac{(r+1)x}{\varphi(k)}\delta_{r+1}(x). \tag{7.13}$$

But from (2.13) we have

$$S_{r+1}(x) = \left(1 + \frac{(r+1)(1-\gamma)}{\log x}u_{r+1} + O\left(\left(\frac{\log_2 x}{\log x}\right)^{3/4}\right)\right) \times \frac{(r+1)x}{\varphi(k)}\delta_{r+1}(x), \tag{7.14}$$

hence (2.17) follows from (7.13), (7.14) and (2.3).

In concluding let it be mentioned that the foregoing methods may be used to yield an asymptotic expansion of the general sum

$$\sum_{n\le x,\, P(n)\equiv l \pmod k,\, P^m(n)|n} \frac{1}{P^r(n)}$$

when $m \ge 1$, $1 \le l \le k$, $(k,l) = 1$ are fixed integers and r is a fixed real satisfying $r \ge 1 - m$. Also in Theorems 2 and 3 one can suppose that $1 \le k \le \log^N x$ for any fixed $N > 0$. Namely in (5.2) we shall have then $\Delta(x) \ll x\exp(-C(N)\log^{1/2} x)$ for some $C(N) > 0$. The asymptotic formulas (2.11) and (2.15) will not be affected, but in (2.10) and (2.14) the exponent $3/8$ of $\log x$ will be replaced by the slightly weaker $3/10$.

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