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On a system of two diophantine inequalities with prime numbers

by

D. I. Tolev (Plovdiv)

1. Introduction and statement of the result. In 1952 Piatetski-Shapiro [3] considered the diophantine inequality

$$|p_1^c + \ldots + p_r^c - N| < \varepsilon, \tag{1}$$

where $c > 1$ is not an integer and $\varepsilon > 0$ is an arbitrarily small number. He showed that if $H(c)$ denotes the least r such that (1) has solutions in prime numbers $p_1, \ldots, p_r$ for arbitrarily small ε and for $N > N_0(c, \varepsilon)$, then

$$\limsup_{c\to\infty} \frac{H(c)}{c\log c} \le 4.$$

Piatetski-Shapiro also proved that $H(c) \le 5$ for $1 < c < 3/2$. In [4] the author improved this result for c close to one. More precisely, it is shown that if $1 < c < 15/14$, then the inequality

$$|p_1^c + p_2^c + p_3^c - N| < N^{-(1/c)(15/14-c)} \log^9 N$$

has solutions in prime numbers p_1, p_2, p_3 for sufficiently large N. In the present paper we shall consider the system of two inequalities with prime unknowns

$$\begin{aligned} &|p_1^c + \ldots + p_5^c - N_1| < \varepsilon_1(N_1),\\ &|p_1^d + \ldots + p_5^d - N_2| < \varepsilon_2(N_2), \end{aligned} \tag{2}$$

where c and d are different numbers greater than one but close to one and $\varepsilon_1(N_1)$, $\varepsilon_2(N_2)$ tend to zero as N_1 and N_2 tend to infinity. Of course, we have to impose a condition on the orders of N_1 and N_2 because of the inequality

$$(x_1^c + \ldots + x_5^c)^{d/c} \le x_1^d + \ldots + x_5^d \le 5^{1-d/c}(x_1^c + \ldots + x_5^c)^{d/c}$$

which holds for every positive $x_1, \ldots, x_5$ provided $1 < d < c$. We shall prove the following theorem.

THEOREM. *Suppose that c, d, α, β are real numbers satisfying the inequalities*

$$(3)\qquad 1 < d < c < 35/34,$$

$$(4)\qquad 1 < \alpha < \beta < 5^{1-d/c}.$$

Then there exist numbers $N_1^{(0)}$, $N_2^{(0)}$, depending on c, d, α, β, such that for all real numbers N_1, N_2 satisfying $N_1 > N_1^{(0)}$, $N_2 > N_2^{(0)}$ and

$$(5)\qquad \alpha \le N_2/N_1^{d/c} \le \beta,$$

the system (2) *with*

$$\varepsilon_1(N_1) = N_1^{-(1/c)(35/34-c)} \log^{12} N_1,$$
$$\varepsilon_2(N_2) = N_2^{-(1/d)(35/34-d)} \log^{12} N_2$$

has solutions in prime numbers $p_1, \ldots, p_5$.

2. Notation and an outline of the proof. Let c, d be numbers satisfying (3), and α, β numbers satisfying (4). Throughout the paper the constants in O-terms and $\ll$-symbols are absolute or depend on c, d, α, β.

$A \asymp B$ means $A \ll B \ll A$; N_1, N_2 are large numbers satisfying (5), $X = N_1^{1/c}$, $\varepsilon_1 = X^{-(35/34-c)} \log^{10} X$, $\varepsilon_2 = X^{-(35/34-d)} \log^{10} X$, $K_1 = \varepsilon_1^{-1} \log X$, $K_2 = \varepsilon_2^{-1} \log X$, η is a positive number, sufficiently small in terms of c and d, $\tau_1 = X^{3/4-c-\eta}$, $\tau_2 = X^{3/4-d-\eta}$, $e(t) = e^{2\pi i t}$, $\varphi(t) = e^{-\pi t^2}$, $\varphi_\delta(t) = \delta\varphi(\delta t)$, $\chi(t)$ is the characteristic function of the interval $[-1, 1]$, $x, y, t, t_1, t_2, \ldots$ are real numbers, k, l, m, n, q are integers, and $p, p_1, p_2, \ldots$ prime numbers.

Let λ denote a sufficiently small positive number, depending on α, β, c, d, whose value will be determined more precisely in Lemma 1. We define

$$(6)\qquad B = \sum_{\lambda X < p_1, \ldots, p_5 \le X} (\log p_1) \ldots (\log p_5) \times \chi\left(\frac{p_1^c + \ldots + p_5^c - N_1}{\varepsilon_1 \log X}\right) \chi\left(\frac{p_1^d + \ldots + p_5^d - N_2}{\varepsilon_2 \log X}\right),$$

$$(7)\qquad S(x, y) = \sum_{\lambda X < p \le X} (\log p) e(xp^c + yp^d),$$

$$(8)\qquad D = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S^5(x, y) e(-N_1 x - N_2 y) \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y)\, dx\, dy.$$

We divide the plane into three regions: Ω_1—a neighbourhood of the origin, Ω_2—an intermediate region and Ω_3—a trivial region, as follows:

$$\Omega_1 = \{(x, y) : \max(|x|/\tau_1, |y|/\tau_2) < 1\},$$
$$\Omega_2 = \{(x, y) : \max(|x|/\tau_1, |y|/\tau_2) \ge 1,\ \max(|x|/K_1, |y|/K_2) \le 1\},$$

$\Omega_3 = \{(x,y) : \max(|x|/K_1, |y|/K_2) > 1\}$.

Correspondingly, we represent the integral D as

$$D = D_1 + D_2 + D_3, \tag{9}$$

where D_i denotes the contribution to the integral D in (9) arising from the set Ω_i.

The theorem will be proved if we show that B tends to infinity as X tends to infinity. The result of Lemma 3 implies that it is sufficient to prove that D tends to infinity as X tends to infinity. The last statement is a consequence of (9) and of the inequalities

$$|D_3| \ll 1, \tag{10}$$

$$|D_2| \ll \frac{\varepsilon_1 \varepsilon_2 X^{5-c-d}}{\log X}, \tag{11}$$

$$|D_1| \gg \varepsilon_1 \varepsilon_2 X^{5-c-d}. \tag{12}$$

Inequality (10) is an easy consequence of the fact that $\varphi(t)$ tends to zero very fast as $|t|$ tends to infinity (see Lemma 4). The main difficulty is to prove (11) and (12). We estimate $|D_1|$ from below in Section 4. In Section 5 we estimate D_2. The proof of the theorem is given in Section 6.

3. Known results and some preliminary lemmas

LEMMA 1. *Let $\delta \in [\alpha, \beta]$. There exists $\lambda > 0$ depending on α, β, c, d such that for the volume V of the domain in five-dimensional space defined by*

$$t_1, \ldots, t_5 > \lambda, \quad |t_1^c + \ldots + t_5^c - 1| < \mu_1, \quad |t_1^d + \ldots + t_5^d - \delta| < \mu_2,$$

we have

$$V \gg \mu_1 \mu_2,$$

provided μ_1, μ_2 are sufficiently small.

Proof. The proof is not difficult and we omit it.

LEMMA 2. *The function $\varphi(t) = e^{-\pi t^2}$ has the properties*

$$\varphi(x) = \int_{-\infty}^{\infty} \varphi(t) e(-xt)\, dt, \tag{i}$$

$$\chi(t/\varrho) \ge \varphi(t) - e^{-\pi\varrho^2} \quad \text{for } \varrho > 0, \tag{ii}$$

$$\varphi(t) \ge e^{-\pi} \quad \text{for } |t| \le 1. \tag{iii}$$

Proof. The proof of (i) can be found for instance in [1, p. 261]; (ii) and (iii) are obvious.

LEMMA 3. *For the quantities B and D defined in* (6) *and* (8) *we have*

$$B \ge D + O(1).$$

Proof. This follows from Lemma 2.

LEMMA 4. *For the integral* D_3 *(defined in* (9)*) we have*

$$|D_3| \ll 1.$$

Proof. This follows from Lemma 2.

LEMMA 5. *If* $\mathcal{D}$ *is a region in the plane with area* $S_{\mathcal{D}}$ *whose boundary is rectifiable and has length* $L_{\mathcal{D}}$, *then for the number* $N_{\mathcal{D}}$ *of integer points in* $\mathcal{D}$ *we have the estimate*

$$|N_{\mathcal{D}} - S_{\mathcal{D}}| \ll 1 + L_{\mathcal{D}},$$

where the constant in the $\ll$*- symbol is absolute.*

Proof. See [2, p. 194].

LEMMA 6. *Let* $I = [u_1, u_2]$ *and* $J = [v_1, v_2]$ *be subintervals of the real line and let* $1 \le \Delta \le X$. *Denote by* W *the number of integers* $n_1, \ldots, n_4$ *satisfying the following conditions*:

$$\lambda X \le n_1, \ldots, n_4 \le X, \quad \Delta \le n_1 - n_2 \le 2\Delta, \quad \Delta \le n_4 - n_3 \le 2\Delta,$$
$$n_2^c + n_4^c - n_1^c - n_3^c \in I, \quad n_2^d + n_4^d - n_1^d - n_3^d \in J.$$

Then

$$W \ll X^{4-c-d}(u_2 - u_1)(v_2 - v_1) + X^{3-c}(u_2 - u_1) + X^{3-d}(v_2 - v_1) + \Delta X.$$

Proof. It is clear that

$$W \ll \sum_{\substack{\lambda X \le n_1, n_2 \le X \\ \Delta \le n_1 - n_2 \le 2\Delta}} W(n_1, n_2), \tag{13}$$

where $W(n_1, n_2)$ denotes the number of integral points in the region $\mathcal{D}$ in the (x, y)-plane, defined by

$$\lambda X \le x, y \le X, \quad \Delta \le x - y \le 2\Delta,$$
$$x^c - y^c \in n_1^c - n_2^c + I, \quad x^d - y^d \in n_1^d - n_2^d + J.$$

(As usual, if $I = [u_1, u_2]$ then $\lambda + I$ denotes the interval $[\lambda + u_1, \lambda + u_2]$.) We may assume that $\mathcal{D}$ is not empty, otherwise $W(n_1, n_2) = 0$. By Lemma 5 we have

$$W(n_1, n_2) \ll S_{\mathcal{D}} + L_{\mathcal{D}} + 1, \tag{14}$$

where $S_{\mathcal{D}}$, $L_{\mathcal{D}}$ denote the area and the length of the boundary of $\mathcal{D}$. Consider the map

$$\Phi : (x, y) \mapsto (u = x^c - y^c, v = x^d - y^d).$$

It is a bijection between the domain $\{0 < y < x\}$ in the (x,y)-plane and the domain $\{0 < v < u^{d/c}\}$ in the (u,v)-plane. We have

$$\left|\frac{D(u,v)}{D(x,y)}\right| = -cd(xy)^{d-1}(x^{c-d} - y^{c-d}).$$

In $\mathcal{D}$ we have

$$x \asymp X, \quad y \asymp X, \quad x - y \asymp \Delta, \tag{15}$$

therefore in this region $|D(u,v)/D(x,y)| \asymp \Delta X^{c+d-3}$. Hence

$$\begin{aligned} S_{\mathcal{D}} &= \iint\limits_{\Phi(\mathcal{D})} \left|\frac{D(x,y)}{D(u,v)}\right| du\,dv \ll \Delta^{-1}X^{3-c-d} \iint\limits_{\Phi(\mathcal{D})} du\,dv \\ &\ll \Delta^{-1}X^{3-c-d}(u_2 - u_1)(v_2 - v_1), \end{aligned} \tag{16}$$

because $\Phi(\mathcal{D})$ is a subset of the rectangle $\mathcal{K}$ in the (u,v)-plane, defined by

$$u \in n_1^c - n_2^c + I, \quad v \in n_1^d - n_2^d + J.$$

Let us now estimate $L_{\mathcal{D}}$. Denote by $l_{\mathcal{D}}$ the curve which is the boundary of $\mathcal{D}$. It is easy to see that it consists of finitely many parts l_0 such that $\Phi(l_0)$ is either a segment lying on the boundary of $\mathcal{K}$ or the graph of an increasing differentiable function $v = v(u)$ defined for $u' \le u \le u''$, where

$$u', u'' \in n_1^c - n_2^c + I, \quad v(u'), v(u'') \in n_1^d - n_2^d + J. \tag{17}$$

Consider the second case. It is clear that the curve l_0 in the (x,y)-plane can be parametrized in the following way:

$$x = x(u, v(u)), \quad y = y(u, v(u)), \quad u' \le u \le u''.$$

(Here $x(u,v)$ and $y(u,v)$ are the components of Φ^{-1}.) Then for the length L_0 of l_0 we have

$$\begin{aligned} L_0 &= \int_{u'}^{u''} \sqrt{\left(\frac{d}{du}x(u,v(u))\right)^2 + \left(\frac{d}{du}y(u,v(u))\right)^2}\, du \\ &\ll \int_{u'}^{u''} (|x_u(u,v(u))| + |y_u(u,v(u))| \\ &\qquad + v'(u)|x_v(u,v(u))| + v'(u)|y_v(u,v(u))|)\, du. \end{aligned} \tag{18}$$

It is easy to verify that the partial derivatives of $x(u,v)$ and $y(u,v)$ satisfy

$$x_u = \frac{1}{cx^{d-1}(x^{c-d} - y^{c-d})}, \quad x_v = \frac{-y^{c-d}}{dx^{d-1}(x^{c-d} - y^{c-d})},$$

$$y_u = \frac{1}{cy^{d-1}(x^{c-d} - y^{c-d})}, \quad y_v = \frac{-x^{c-d}}{dy^{d-1}(x^{c-d} - y^{c-d})}.$$

Therefore by (15) we conclude that in $\Phi(\mathcal{D})$ we have

$x_u \asymp \Delta^{-1}X^{2-c}, \quad -x_v \asymp \Delta^{-1}X^{2-d}, \quad y_u \asymp \Delta^{-1}X^{2-c}, \quad -y_v \asymp \Delta^{-1}X^{2-d}.$

Hence by (17) and (18) we obtain

$$\begin{aligned} L_0 &\ll \int\limits_{u'}^{u''} (\Delta^{-1}X^{2-c} + \Delta^{-1}X^{2-d}v'(u))\,du \\ &\ll \Delta^{-1}X^{2-c}(u_2-u_1) + \Delta^{-1}X^{2-d}(v_2-v_1). \end{aligned}$$

If $\Phi(l_0)$ is a segment lying on the boundary of $\mathcal{K}$, we proceed in the same way, and we obtain the same estimate for L_0. Therefore

$$L_{\mathcal{D}} \ll \Delta^{-1}X^{2-c}(u_2-u_1) + \Delta^{-1}X^{2-d}(v_2-v_1). \tag{19}$$

The assertion of the lemma follows from (13), (14), (16) and (19).

4. The integral over the neighbourhood of the origin. In this section we estimate from below the quantity $|D_1|$. Set

$$I(x,y) = \int\limits_{\lambda X}^{X} e(xt^c + yt^d)\,dt. \tag{20}$$

We shall show that in Ω_1 the sum $S(x,y)$ is "close" to the integral $I(x,y)$, which implies that D_1 is "close" to

$$H_1 = \iint\limits_{\Omega_1} I^5(x,y)e(-N_1x - N_2y)\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy. \tag{21}$$

Outside Ω_1 the integral $I(x,y)$ is "small", so H_1 is "close" to

$$H = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} I^5(x,y)e(-N_1x - N_2y)\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy. \tag{22}$$

In turn this integral is greater than the volume of a domain in five-dimensional space, which we are able to estimate from below.

Lemma 7. *If $S(x,y)$ and $I(x,y)$ are defined by* (7) *and* (20) *then for $(x,y)\in\Omega_1$ we have*

$$S(x,y) = I(x,y) + O(Xe^{-(\log X)^{1/5}}).$$

Proof. We proceed as in the proof of Lemma 14 of [4] and the result follows.

Lemma 8. *We have*

$$E = \iint\limits_{\Omega_1} |S(x,y)|^4\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy \ll \varepsilon_1\varepsilon_2X^{4-c-d}\log^8 X.$$

P r o o f. It is clear that

$$E \ll \varepsilon_1\varepsilon_2 \iint_{\Omega_1} |S(x,y)\overline{S(x,y)}|^2\,dx\,dy \tag{23}$$
$$= \varepsilon_1\varepsilon_2 \iint_{\Omega_1} \Big| \sum_{\lambda X<p\le X} \log^2 p + 2\,\mathrm{Re} \sum_{\lambda X<p_2<p_1\le X} (\log p_1)(\log p_2) \times e(x(p_1^c-p_2^c)+y(p_1^d-p_2^d))\Big|^2\,dx\,dy$$
$$\ll \varepsilon_1\varepsilon_2\tau_1\tau_2X^2\log^2 X+\varepsilon_1\varepsilon_2E_1,$$

where

$$E_1 = \iint_{\Omega_1} \Big| \sum_{\lambda X<p_2<p_1\le X} (\log p_1)(\log p_2)e(x(p_1^c-p_2^c)+y(p_1^d-p_2^d))\Big|^2\,dx\,dy.$$

We divide the sum over p_1, p_2 above into $O(\log X)$ sums in each of which the summation is over p_1, p_2 such that $\Delta\le p_1-p_2<2\Delta$, where $1\le\Delta\le X$. We then have

$$E_1 \ll E_2\log^2 X, \tag{24}$$

where

$$E_2 = \iint_{\Omega_1} \Big| \sum_{\substack{\lambda X<p_1,p_2\le X\\ \Delta\le p_1-p_2<2\Delta}} (\log p_1)(\log p_2)e(x(p_1^c-p_2^c)+y(p_1^d-p_2^d))\Big|^2\,dx\,dy$$

and Δ is chosen in such a way that E_2 is maximal. Clearly,

$$E_2 = \iint_{\Omega_1} \sum_{\substack{\lambda X<p_1,\dots,p_4\le X\\ \Delta\le p_1-p_2<2\Delta\\ \Delta\le p_4-p_3<2\Delta}} (\log p_1)\dots(\log p_4) \times e(x(p_1^c-p_2^c+p_3^c-p_4^c)+y(p_1^d-p_2^d+p_3^d-p_4^d))\,dx\,dy$$
$$= \sum_{\substack{\lambda X<p_1,\dots,p_4\le X\\ \Delta\le p_1-p_2<2\Delta\\ \Delta\le p_4-p_3<2\Delta}} (\log p_1)\dots(\log p_4) \int_{-\tau_1}^{\tau_1} e(x(p_1^c-p_2^c+p_3^c-p_4^c))\,dx \times \int_{-\tau_2}^{\tau_2} e(y(p_1^d-p_2^d+p_3^d-p_4^d))\,dy.$$

Hence

$$E_2 \ll E_3\log^4 X, \tag{25}$$

where

$$E_3 = \sum_{\substack{\lambda X < n_1, \dots, n_4 \le X \\ \Delta \le n_1 - n_2 < 2\Delta \\ \Delta \le n_4 - n_3 < 2\Delta}} \Gamma(n_1, \dots, n_4),$$

and

$$\Gamma(n_1, \dots, n_4) = \min(\tau_1, |n_1^c - n_2^c + n_3^c - n_4^c|^{-1}) \min(\tau_2, |n_1^d - n_2^d + n_3^d - n_4^d|^{-1}).$$

For any integers k, l we define the intervals I_k, J_l as follows:

$$I_k = \begin{cases} [-1/\tau_1, 1/\tau_1] & \text{for } k = 0, \\ [2^{k-1}/\tau_1, 2^k/\tau_1] & \text{for } k \ge 1, \\ [-2^{|k|}/\tau_1, -2^{|k|-1}/\tau_1] & \text{for } k \le -1; \end{cases}$$

$$J_l = \begin{cases} [-1/\tau_2, 1/\tau_2] & \text{for } l = 0, \\ [2^{l-1}/\tau_2, 2^l/\tau_2] & \text{for } l \ge 1, \\ [-2^{|l|}/\tau_2, -2^{|l|-1}/\tau_2] & \text{for } l \le -1. \end{cases}$$

It is clear that there exist $k_0, l_0 > 0$ such that

$$k_0, l_0 \ll \log X \tag{26}$$

and

$$E_3 \ll \sum_{\substack{|k| \le k_0 \\ |l| \le l_0}} E(k, l), \tag{27}$$

where

$$E(k, l) = \sum_{n_1, \dots, n_4;\, (28)} \Gamma(n_1, \dots, n_4).$$

Here $n_1, \dots, n_4$ satisfy the conditions imposed in (28):

$$\begin{gathered} \lambda X \le n_1, \dots, n_4 \le X, \\ \Delta \le n_1 - n_2 \le 2\Delta, \\ \Delta \le n_4 - n_3 \le 2\Delta, \\ n_2^c + n_4^c - n_1^c - n_3^c \in I_k, \\ n_2^d + n_4^d - n_1^d - n_3^d \in J_l. \end{gathered} \tag{28}$$

By the definition of $\Gamma(n_1, \dots, n_4)$ we get

$$E(k, l) \ll \frac{\tau_1 \tau_2}{2^{|k|+|l|}} \sum_{n_1, \dots, n_4;\, (28)} 1.$$

We estimate the last sum by Lemma 6 to obtain

$$E(k, l) \ll X^{4-c-d}.$$

The assertion of the lemma follows from the last inequality and from (23)–(27).

LEMMA 9. *For the integral $I(x,y)$ defined by* (20) *we have*

$$F=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}|I(x,y)|^4\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy\ll\varepsilon_1\varepsilon_2X^{4-c-d}\log^4X.$$

Proof. Define

$$h(t_1,t_2)=e(x(t_1^c-t_2^c)+y(t_1^d-t_2^d)).$$

We have

$$\begin{aligned}I(x,y)\overline{I(x,y)}&=\int\int\limits_{\lambda X<t_1,t_2<X}h(t_1,t_2)\,dt_1\,dt_2\\&=2\operatorname{Re}\int\int\limits_{\substack{\lambda X<t_1,t_2<X\\X^{-1}<t_1-t_2}}h(t_1,t_2)\,dt_1\,dt_2+O(1).\end{aligned}$$

Hence

$$F\ll1+\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\Big|\int\int\limits_{\substack{\lambda X<t_1,t_2<X\\X^{-1}<t_1-t_2}}h(t_1,t_2)\,dt_1\,dt_2\Big|^2\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy.$$

We represent the integral over t_1, t_2 as a sum of no more than $O(\log X)$ integrals

$$J_\Delta=\int\int\limits_{\substack{\lambda X<t_1,t_2<X\\\Delta<t_1-t_2<2\Delta}}h(t_1,t_2)\,dt_1\,dt_2,$$

where $X^{-1}\le\Delta\le X$. We then have

$$F\ll1+F_1\log^2X,\tag{29}$$

where

$$F_1=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}|J_\Delta|^2\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy$$

and Δ is chosen in such a way that the integral F_1 is maximal. We have

$$\begin{aligned}F_1=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\int\int\int\limits_{\substack{\lambda X<t_1,\dots,t_4<X\\\Delta<t_1-t_2<2\Delta\\\Delta<t_4-t_3<2\Delta}}&e(x(t_1^c-t_2^c+t_3^c-t_4^c)+y(t_1^d-t_2^d+t_3^d-t_4^d))\\&\times\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dt_1\dots dt_4\,dx\,dy\end{aligned}$$

and by Lemma 2(i),

$$F_1=\int\int\int\int\limits_{\substack{\lambda X<t_1,\dots,t_4<X\\\Delta<t_1-t_2<2\Delta\\\Delta<t_4-t_3<2\Delta}}\varphi\Big(\frac{t_1^c-t_2^c+t_3^c-t_4^c}{\varepsilon_1}\Big)\varphi\Big(\frac{t_1^d-t_2^d+t_3^d-t_4^d}{\varepsilon_2}\Big)\,dt_1\dots dt_4.$$

We change the variables as follows:

$$u_1 = t_1^c - t_2^c, \quad u_2 = t_4^c - t_3^c, \quad u_3 = t_1^d - t_2^d, \quad u_4 = t_4^d - t_3^d.$$

For the Jacobian determinant we have

$$\left| \frac{D(u_1, \ldots, u_4)}{D(t_1, \ldots, t_4)} \right| \asymp \Delta^2 X^{2c+2d-6}.$$

Hence

$$F_1 \ll \Delta^{-2} X^{6-2c-2d} I_1 I_2, \tag{30}$$

where

$$I_1 = \int\int_{u_1, u_2 \asymp \Delta X^{c-1}} \varphi\left(\frac{u_1 - u_2}{\varepsilon_1}\right) du_1\, du_2,$$

$$I_2 = \int\int_{u_3, u_4 \asymp \Delta X^{d-1}} \varphi\left(\frac{u_3 - u_4}{\varepsilon_2}\right) du_3\, du_4.$$

By Lemma 2(ii) we have

$$I_1 \ll X^{-2} + \int\int_{u_1, u_2 \asymp \Delta X^{c-1}} \chi\left(\frac{u_1 - u_2}{\varepsilon_1 \log X}\right) du_1\, du_2 \ll \varepsilon_1 \Delta X^{c-1} \log X.$$

Analogously

$$I_2 \ll \varepsilon_2 \Delta X^{d-1} \log X.$$

The estimates (29) and (30) imply

$$F \ll \varepsilon_1 \varepsilon_2 X^{4-c-d} \log^4 X.$$

The lemma is proved.

LEMMA 10. *For the integrals H_1 and H defined by* (21) *and* (22) *we have*

$$|H - H_1| \ll \frac{\varepsilon_1 \varepsilon_2 X^{5-c-d}}{\log X}.$$

Proof. Clearly

$$\begin{aligned} |H - H_1| &\ll \int\int_{\mathbb{R}^2 \setminus \Omega_1} |I(x,y)|^5 \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y)\, dx\, dy \\ &\ll \max_{\mathbb{R}^2 \setminus \Omega_1} |I(x,y)| \int\int_{\mathbb{R}^2} |I(x,y)|^4 \varphi_{\varepsilon_1}(x) \varphi_{\varepsilon_2}(y)\, dx\, dy. \end{aligned} \tag{31}$$

It is not difficult to see that

$$\max_{\mathbb{R}^2 \setminus \Omega_1} |I(x,y)| \ll X^{5/6}.$$

We estimate the integral (31) using Lemma 9 and the result follows.

LEMMA 11. *The integral H defined by* (22) *satisfies*

$$H \gg \varepsilon_1 \varepsilon_2 X^{5-c-d}.$$

Proof. This follows from (5) and Lemmas 1 and 2.

LEMMA 12. *The integral D_1 defined by* (9) *satisfies*

$$|D_1| \gg \varepsilon_1\varepsilon_2 X^{5-c-d}.$$

Proof. If H_1 is defined by (21) then

$$\begin{aligned} |D_1 - H_1| &\ll \iint_{\Omega_1} |S^5(x,y) - I^5(x,y)|\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy \\ &\ll \max_{\Omega_1} |S(x,y) - I(x,y)| \\ &\quad \times \iint_{\Omega_1} (|S(x,y)|^4 + |I(x,y)|^4)\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy. \end{aligned}$$

Hence, by Lemmas 7–9,

$$|D_1 - H_1| \ll \frac{\varepsilon_1\varepsilon_2 X^{5-c-d}}{\log X}.$$

This estimate and Lemma 10 imply

$$D_1 = H + O\biggl(\frac{\varepsilon_1\varepsilon_2 X^{5-c-d}}{\log X}\biggr).$$

Now we use Lemma 11 and the result follows.

5. The integral over the intermediate region

LEMMA 13. *For the sum $S(x,y)$ defined in* (7) *we have uniformly for $(x,y) \in \Omega_2$,*

$$|S(x,y)| \ll \varepsilon_1\varepsilon_2 \frac{X^{3-c-d}}{\log^{10} X}.$$

Proof. The proof is a standard application of Vaughan's identity (see [5]). See also Lemma 10 in [4].

LEMMA 14. *We have*

$$L = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |S(x,y)|^4 \varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy \ll X^2\log^6 X.$$

Proof. It is clear that

$$\begin{aligned} S(x,y)\overline{S(x,y)} &= \sum_{\lambda X < p \le X} \log^2 p \\ &\quad + 2\operatorname{Re} \sum_{\lambda X < p_2 < p_1 \le X} (\log p_1)(\log p_2) e(x(p_1^c - p_2^c) + y(p_1^d - p_2^d)). \end{aligned}$$

This implies

$$L \ll X^2 \log^2 X + L_1, \tag{32}$$

where

$$L_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big|\sum_{\lambda X < p_2 < p_1 \le X}(\log p_1)(\log p_2)e(x(p_1^c - p_2^c) + y(p_1^d - p_2^d))\Big|^2 \times \varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy.$$

We divide the sum over p_1, p_2 into no more than $O(\log X)$ sums in each of which the summation is over p_1, p_2 such that $\Delta \le p_1 - p_2 < 2\Delta$, where $1 \le \Delta \le X$. Then we have

$$L_1 \ll L_2 \log^2 X, \tag{33}$$

where

$$L_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big|\sum_{\substack{\lambda X < p_2 < p_1 \le X\\ \Delta \le p_1 - p_2 < 2\Delta}}(\log p_1)(\log p_2)e(x(p_1^c - p_2^c) + y(p_1^d - p_2^d))\Big|^2 \times \varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy$$

and Δ is chosen in such a way that L_2 is maximal. By Lemma 2(i), (ii) we have

$$\begin{aligned} L_2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{\substack{\lambda X < p_1,\ldots,p_4 \le X\\ \Delta \le p_1 - p_2 < 2\Delta\\ \Delta \le p_4 - p_3 < 2\Delta}}(\log p_1)(\log p_2)(\log p_3)(\log p_4)\\ &\quad\times e(x(p_1^c - p_2^c + p_3^c - p_4^c) + y(p_1^d - p_2^d + p_3^d - p_4^d))\\ &\qquad\qquad\times \varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy\\ &= \sum_{\substack{\lambda X < p_1,\ldots,p_4 \le X\\ \Delta \le p_1 - p_2 < 2\Delta\\ \Delta \le p_4 - p_3 < 2\Delta}}(\log p_1)(\log p_2)(\log p_3)(\log p_4)\\ &\quad\times \varphi\bigg(\frac{p_1^c - p_2^c + p_3^c - p_4^c}{\varepsilon_1}\bigg)\varphi\bigg(\frac{p_1^d - p_2^d + p_3^d - p_4^d}{\varepsilon_2}\bigg)\\ &\ll 1 + L_3\log^4 X, \end{aligned} \tag{34}$$

where L_3 denotes the number of integers $n_1, \ldots, n_4$ satisfying

$$\lambda X \le n_1, \ldots, n_4 \le X, \quad \Delta \le n_1 - n_2 \le 2\Delta, \quad \Delta \le n_4 - n_3 \le 2\Delta,$$
$$n_1^c - n_2^c + n_3^c - n_4^c \in I, \quad n_1^d - n_2^d + n_3^d - n_4^d \in J,$$

and where $I = [-\varepsilon_1 \log X, \varepsilon_1 \log X]$ and $J = [-\varepsilon_2 \log X, \varepsilon_2 \log X]$. By Lemma 6 we have

$$L_3 \ll \varepsilon_1\varepsilon_2 X^{4-c-d}\log^2 X + \varepsilon_1 X^{3-c}\log X + \varepsilon_2 X^{3-d}\log X + \Delta X \ll X^2$$

and the result follows from (32)–(34).

LEMMA 15. *For the integral D_2 defined by* (9) *the following estimate holds*:
$$|D_2| \ll \frac{\varepsilon_1\varepsilon_2 X^{5-c-d}}{\log X}.$$

Proof. We have
$$|D_2| \ll \max_{\Omega_2} |S(x,y)| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |S(x,y)|^4 \varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy$$
and the result follows from Lemmas 13 and 14.

6. Proof of the Theorem. Lemma 3 shows that for the sum
$$B = \sum_{\lambda X < p_1,\ldots,p_5 \le X} (\log p_1)\ldots(\log p_5) \times \chi\left(\frac{p_1^c+\ldots+p_5^c-N_1}{\varepsilon_1\log X}\right)\chi\left(\frac{p_1^d+\ldots+p_5^d-N_2}{\varepsilon_2\log X}\right)$$
we have
$$B \ge D + O(1), \tag{35}$$
where D is defined by (8). On the other hand,
$$D = D_1 + D_2 + D_3. \tag{36}$$
From Lemma 12 we have
$$|D_1| \gg \varepsilon_1\varepsilon_2 X^{5-c-d}. \tag{37}$$
Lemma 15 states that
$$|D_2| \ll \frac{\varepsilon_1\varepsilon_2 X^{5-c-d}}{\log X}, \tag{38}$$
and Lemma 4 gives us
$$|D_3| \ll 1. \tag{39}$$
Consequently, by (35)–(39) we have
$$B \gg \varepsilon_1\varepsilon_2 X^{5-c-d}.$$
The Theorem is proved.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PLOVDIV
"TSAR ASEN" 24
PLOVDIV 4000, BULGARIA

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