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A note on perfect powers of the form $x^{m-1}+\ldots+x+1$

by

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1. Introduction. Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. Let $x,m,n\in\mathbb{N}$ be such that $x>1$ and $n>1$, and let $u_m(x)=(x^m-1)/(x-1)$. In [10], Shorey proved that if $m>1$, $m\equiv 1 \pmod n$ and $u_m(x)$ is an nth power, then $\max(x,m,n)<C$, where C is an effectively computable absolute constant. In [11], he further proved that if both $u_{m_1}(x)$ and $u_{m_2}(x)$ are nth powers with $m_1<m_2$ and $m_1\equiv m_2 \pmod n$, then $\max(x,m_2,n)<C$. Recently, the author [7] showed that if both $u_{m_1}(x)$ and $u_{m_2}(x)$ are nth powers with $m_1<m_2$ and $m_1\equiv m_2 \pmod n$, then $m_1=1$. For $m_1=1$, the problem is still open. In this note we prove a general result as follows.

THEOREM. *The equation*

$$\frac{x^m-1}{x-1}=y^n,\qquad x,y,m,n\in\mathbb{N},\ x>1,\ y>1,\ m>2,\ n>1, \tag{1}$$

has no solution (x,y,m,n) *satisfying* $\gcd(x\varphi(x),n)=1$, *where* $\varphi(x)$ *is Euler's function of* x.

By the above theorem, we can obtain the following result.

COROLLARY. *If* $m>1$, $m\equiv 1 \pmod n$ *and* $u_m(x)$ *is an* n*-th power, then* $(x,m,n)=(3,5,2)$.

Thus it can be seen that the above theorem contributes to solving many problems concerning the equation (1).

2. Preliminaries. Let p be an odd prime, and let $a\in\mathbb{N}$ be such that $a>1$, $p\nmid a$ and $\theta=a^{1/p}\notin\mathbb{Q}$. Then $K=\mathbb{Q}(\theta)$ is an algebraic number field of degree p. Further let $a=p_1^{k_1}\ldots p_s^{k_s}$, where $k_1,\ldots,k_s\in\mathbb{N}$, $p_1,\ldots,p_s$ are distinct primes, and let $S=\{\pm p^{r_0}p_1^{r_1}\ldots p_s^{r_s}\mid r_0,r_1,\ldots,r_s$ are nonnegative integers$\}$. Then K has an integral base of the form $\{\theta^i/I_i\mid i=0,1,\ldots,p-1\}$, where $I_i\in S$ for $i=0,1,\ldots,p-1$.

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Let O_K be the algebraic integer ring of K. Then we have $\mathbb{Z}[\theta] \subseteq O_K$. For $\alpha_1, \ldots, \alpha_r \in O_K$, let $[\alpha_1, \ldots, \alpha_r]$ be the ideal of K generated by $\alpha_1, \ldots, \alpha_r$, and let $\langle [\alpha_1, \ldots, \alpha_r] \rangle$ denote the residue class degree of $[\alpha_1, \ldots, \alpha_r]$ if $[\alpha_1, \ldots, \alpha_r]$ is a prime ideal.

LEMMA 1. *Let q be a prime. If $q \nmid ap$, $q \not\equiv 1 \pmod p$ and the congruence*

$$z^p \equiv a \pmod q \tag{2}$$

is solvable, then (2) has exactly one solution $z \equiv z_0 \pmod q$. Moreover,

$$[q] = \mathfrak{p}_1 \mathfrak{p}_2^{e_2} \ldots \mathfrak{p}_g^{e_g}, \quad e_2, \ldots, e_g \in \mathbb{N},$$

where $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_g$ are distinct prime ideals of K such that $\mathfrak{p}_1 = [q, \theta - z_0]$, $\langle \mathfrak{p}_1 \rangle = 1$ and $\langle \mathfrak{p}_j \rangle > 1$ for $j = 2, \ldots, g$.

P r o o f. By [5, Theorem 3.7.2], if $q \nmid a$ and (2) is solvable, then the number of solutions of (2) is $\gcd(p, q-1)$. Hence, if $q \not\equiv 1 \pmod p$, then (2) has exactly one solution, say $z \equiv z_0 \pmod q$. Furthermore, since $q \nmid p$, the solution is simple. This implies that

$$z^p - a \equiv (z - z_0)(h_2(z))^{e_2} \ldots (h_g(z))^{e_g} \pmod q, \quad e_2, \ldots, e_g \in \mathbb{N}, \tag{3}$$

where $h_2(z), \ldots, h_g(z) \in \mathbb{Z}[z]$ are distinct monic irreducible polynomials $\bmod\, q$ of degrees greater than one. Notice that the discriminant $\Delta(1, \theta, \ldots, \theta^{p-1}) = (-1)^{(p-1)/2} a^{p-1} p^p$. Since $q \nmid ap$, by [6, Chapter 1], we deduce from (3) that

$$[q] = [q, \theta - z_0][q, h_2(\theta)]^{e_2} \ldots [q, h_g(\theta)]^{e_g},$$

where $[q, \theta - z_0], [q, h_2(\theta)], \ldots, [q, h_g(\theta)]$ are distinct prime ideals which satisfy $\langle [q, \theta - z_0] \rangle = 1$ and $\langle [q, h_j(\theta)] \rangle > 1$ for $j = 2, \ldots, g$. The lemma is proved.

Let $\zeta = e^{2\pi\sqrt{-1}/p}$. Then $L = K(\zeta) = \mathbb{Q}(\theta, \zeta)$ is the normal extension of $K/\mathbb{Q}$. Notice that $\{\theta^i \mid i = 0, 1, \ldots, p-1\}$ and $\{\zeta^j \mid j = 0, 1, \ldots, p-2\}$ are bases of K and $\mathbb{Q}(\zeta)$ respectively. We have

LEMMA 2 ([3]). *$\{\theta^i \zeta^j \mid i = 0, 1, \ldots, p-1,\ j = 0, 1, \ldots, p-2\}$ is a base of L.*

Let U_L, W_L be the groups of units and cyclotomic units of L respectively. Then $W_L = \{\pm \zeta^l \mid l = 0, 1, \ldots, p-1\}$.

LEMMA 3. *If $\varepsilon \in U_L$, then $\varepsilon = \zeta^l \eta$, where $l \in \mathbb{Z}$ with $0 \le l \le p-1$, and η is a real unit of L.*

P r o o f. Let $\tau_i : L \to L$ be the field homomorphism defined by $\tau_i(\zeta) = \zeta$ and $\tau_i(\theta) = \theta\zeta^i$ for $i = 0, \ldots, p-1$ and $\sigma_j : L \to L$ the field homomorphism induced by $\sigma_j(\zeta) = \zeta^j$ and $\sigma_j(\theta) = \theta$ for $j = 1, \ldots, p-1$. Further, for any $\alpha \in L$, let $\tau_i\sigma_j : \tau_i\sigma_j(\alpha) = \tau_i(\sigma_j(\alpha))$. Then $\tau_i\sigma_j$ $(i = 0, \ldots, p-1,\ j = 1, \ldots, p-1)$ are distinct $p(p-1)$ distinct embeddings of L into $\mathbb{C}$, where

$\mathbb{C}$ is the set of complex numbers. Since L is a normal extension of $K/\mathbb{Q}$, $\mathrm{Gal}(L/\mathbb{Q})=\{\tau_i\sigma_j \mid i=0,1,\ldots,p-1,\ j=1,\ldots,p-1\}$ is the Galois group of $L/\mathbb{Q}$.

Let $\varrho'=\tau_0\sigma_{p-1}$. Then $\varrho'(\alpha)=\overline{\alpha}$ for any $\alpha\in L$. Hence, $\varrho(\overline{\alpha})=\varrho(\varrho'(\alpha))=\varrho'(\varrho(\alpha))=\overline{\varrho(\alpha)}$ for any $\alpha\in L$ and any $\varrho\in\mathrm{Gal}(L/\mathbb{Q})$. If $\varepsilon\in U_L$, then $\overline{\varepsilon}=\varrho'(\varepsilon)\in U_L$ and

$$\left|\varrho\left(\frac{\varepsilon}{\overline{\varepsilon}}\right)\right|=\left|\frac{\varrho(\varepsilon)}{\varrho(\overline{\varepsilon})}\right|=\left|\frac{\varrho(\varepsilon)}{\overline{\varrho(\varepsilon)}}\right|=1,\qquad \varrho\in\mathrm{Gal}(L/\mathbb{Q}).$$

This implies that $\varepsilon/\overline{\varepsilon}\in W_L$. Since $W_L=\{\pm\zeta^{2l}\mid l=0,1,\ldots,p-1\}$, we get $\varepsilon=\pm\zeta^{2l}\overline{\varepsilon}$, where $l\in\mathbb{Z}$. Let $\eta=\zeta^{-l}\varepsilon$. If $\varepsilon=-\zeta^{2l}\overline{\varepsilon}$, then

$$\eta=\zeta^{-l}\varepsilon=-\zeta^{l}\overline{\varepsilon}=-\overline{\zeta^{-l}\varepsilon}=-\overline{\eta}. \tag{4}$$

Since $\zeta\equiv\zeta^{-1}\equiv 1 \pmod{1-\zeta}$, by Lemma 2, $\alpha\equiv\overline{\alpha} \pmod{1-\zeta}$ for any $\alpha\in L$. From (4), we get $2\eta\equiv 0 \pmod{1-\zeta}$. Notice that $\eta\in U_L$, $p\mid N_{L/\mathbb{Q}}(1-\zeta)$ and p is an odd prime. That is impossible. Thus, $\varepsilon=\zeta^{2l}\overline{\varepsilon}$, $\varepsilon=\zeta^{l}\eta$ and $\eta=\zeta^{-l}\varepsilon=\zeta^{l}\overline{\varepsilon}=\overline{\zeta^{-l}\varepsilon}=\overline{\eta}$ is a real unit of L. The lemma is proved.

3. Proof of Theorem. Let (x,y,m,n) be a solution of (1)

$$\gcd(x\varphi(x),n)=1. \tag{5}$$

By [8], (1) with n even has no solutions other than $(x,y,m,n)=(3,11,5,2)$ or $(7,20,4,2)$. It suffices to consider the case $2\nmid n$. Since $n>1$, n has an odd prime factor p. Then $(x,y^{n/p},m,p)$ is a solution of (1) satisfying (5). We can therefore assume that n is an odd prime.

If $x-1$ is an nth power, then $x-1=y_1^n$ and

$$x^m-(y_1y)^n=1,\qquad x,y_1y,m,n\in\mathbb{N},\ x>1,\ y_1y>1,\ m>2,\ n>2. \tag{6}$$

By [4], we see from (6) that $n\mid x$, which contradicts (5). Therefore, $\theta:=(x-1)^{1/n}\notin\mathbb{Q}$ and $K=\mathbb{Q}(\theta)$ is an algebraic number field of degree n.

Let $x=q_1^{r_1}\ldots q_s^{r_s}$, where $r_1,\ldots,r_s\in\mathbb{N}$, and $q_1,\ldots,q_s$ are distinct primes. Then, by (5), we have $q_i\nmid x-1$, $q_i\nmid n$ and $q_i\not\equiv 1 \pmod n$ for $i=1,\ldots,s$. Notice that the congruences

$$z^n\equiv x-1 \pmod{q_i},\qquad i=1,\ldots,s,$$

have solutions $z\equiv -1 \pmod{q_i}$ $(i=1,\ldots,s)$ respectively. By Lemma 1, we get

$$[q_i]=[q_i,1+\theta]\prod_{j=2}^{g_i}\mathfrak{p}_{ij}^{e_{ij}},\qquad i=1,\ldots,s, \tag{7}$$

where $[q_i,1+\theta]$ and $\mathfrak{p}_{ij}$ are distinct prime ideals of K which satisfy $\langle[q_i,1+\theta]\rangle=1$ and $\langle\mathfrak{p}_{ij}\rangle>1$ for $i=1,\ldots,s$ and $j=2,\ldots,g_i$. Since

$N_{K/\mathbb{Q}}(1+\theta)=x$, we infer from (7) that

$$[x]=\Big(\prod_{i=1}^{s}[q_i,1+\theta]^{r_i}\Big)\Big(\prod_{i=1}^{s}\Big(\prod_{j=2}^{g_i}\mathfrak{p}_{ij}^{e_{ij}}\Big)^{r_i}\Big) = [1+\theta]\Big(\prod_{i=1}^{s}\Big(\prod_{j=2}^{g_i}\mathfrak{p}_{ij}^{e_{ij}}\Big)^{r_i}\Big). \tag{8}$$

From (1) and (8),

$$[1+y\theta]\left[\frac{1+y^n\theta^n}{1+y\theta}\right]=[1+y^n\theta^n]=[x]^m = [1+\theta]^m\Big(\prod_{i=1}^{s}\Big(\prod_{j=2}^{g_i}\mathfrak{p}_{ij}^{e_{ij}}\Big)^{r_i}\Big)^m. \tag{9}$$

Since $\gcd(x,n)=1$, the ideals $[1+y\theta]$ and $[(1+y^n\theta^n)/(1+y\theta)]$ are coprime. If $\mathfrak{p}_{ij}\mid[1+y\theta]$ for some $i,j\in\mathbb{N}$ with $1\le i\le s$ and $2\le j\le g_i$, then from (9) we get $\mathfrak{p}_{ij}^{e_{ij}r_im}\mid[1+y\theta]$. For any ideal $\mathfrak{a}$ in K, let $N\mathfrak{a}$ denote the norm of $\mathfrak{a}$. Recall that $\langle\mathfrak{p}_{ij}\rangle>1$. So we have $q^{2r_im}\mid N\mathfrak{p}^{e_{ij}r_im}$. Further, since $\mathfrak{p}_{ij}^{e_{ij}r_im}\mid[1+y\theta]$ and $N[1+y\theta]=N_{K/\mathbb{Q}}(1+y\theta)=x^m$, we get $q_i^{2r_im}\mid x^m$, a contradiction. Therefore, $\mathfrak{p}_{ij}\nmid[1+y\theta]$, and by (9),

$$[1+y\theta]=[1+\theta]^m. \tag{10}$$

Let U_K be the unit group of K. We see from (10) that

$$1+y\theta=(1+\theta)^m\varepsilon,\quad \varepsilon\in U_K,\ N_{K/\mathbb{Q}}(\varepsilon)=1. \tag{11}$$

Since $K=\mathbb{Q}[\theta]$, we have

$$\varepsilon=\varepsilon(\theta)=a_0+a_1\theta+\ldots+a_{n-1}\theta^{n-1},\quad a_0,a_1,\ldots,a_{n-1}\in\mathbb{Q}. \tag{12}$$

Let $\zeta=e^{2\pi\sqrt{-1}/n}$. Since $\theta\zeta,\ldots,\theta\zeta^{n-1}$ are conjugate numbers of θ, we get

$$1+y\theta\zeta=(1+\theta\zeta)^m\varepsilon(\theta\zeta),\quad 1+y\theta\zeta^{-1}=(1+\theta\zeta^{-1})^m\varepsilon(\theta\zeta^{-1}), \tag{13}$$

by (11). Let $L=K(\zeta)=\mathbb{Q}(\theta,\zeta)$, and let U_L, W_L be the groups of units and cyclotomic units of L respectively. Since L is a normal extension of $K/\mathbb{Q}$, we have $\varepsilon(\theta\zeta)\in U_L$, and by Lemma 3, $\varepsilon(\theta\zeta)=\zeta^l\eta$, where $l\in\mathbb{Z}$ with $0\le l\le n-1$, and η is a real unit of L. Notice that $\varepsilon(\theta\zeta^{-1})=\overline{\varepsilon(\theta\zeta)}=\zeta^{-l}\eta$. We see from (13) that

$$1+y\theta\zeta=(1+\theta\zeta)^m\zeta^l\eta,\quad 1+y\theta\zeta^{-1}=(1+\theta\zeta^{-1})^m\zeta^{-l}\eta,$$

whence we obtain

$$(1+y\theta\zeta)(1+\theta\zeta^{-1})^m-(1+y\theta\zeta^{-1})(1+\theta\zeta)^m\zeta^{2l}=0, \tag{14}$$

since $\eta\ne0$. Clearly, (14) can be written as

$$T_0(\zeta)+\theta T_1(\zeta)+\ldots+\theta^{n-1}T_{n-1}(\zeta)=0, \tag{15}$$

where

(16) $\quad T_i(\zeta)=b_{i,0}+b_{i,1}\zeta+\ldots+b_{i,n-2}\zeta^{n-2}\in\mathbb{Z}[\zeta],\qquad i=0,1,\ldots,n-1.$

By Lemma 3, we find from (14)–(16) that

(17) $$b_{i,j}=0,\qquad i=0,1,\ldots,n-1,\ j=0,1,\ldots,n-2.$$

Since $m>2$, $\theta^n=x-1$ and $\zeta^n=1$, we have

$$(1+\theta\zeta)^m=c_0+c_1\theta\zeta+\ldots+c_{n-1}(\theta\zeta)^{n-1}\in\mathbb{Z}[\theta\zeta],\qquad c_0\ge 1,\ c_1\ge 1.$$

From (14) and (15), we get

(18) $$T_0(\zeta)=c_0+c_{n-1}(x-1)y\zeta^2-c_0\zeta^{2l}-c_{n-1}(x-1)y\zeta^{2l-2},$$

(19) $$T_1(\zeta)=c_1\zeta^{n-1}+c_0y\zeta-c_1\zeta^{2l+1}-c_0y\zeta^{2l-1}.$$

If 1, ζ^2, ζ^{2l} and ζ^{2l-2} are distinct, we see from (16)–(18) that $c_0=0$, a contradiction. Therefore, there exist at least two elements of $\{1,\zeta^2,\zeta^{2l},\zeta^{2l-2}\}$ which are equal. Since $1\ne\zeta^2$ and $\zeta^{2l}\ne\zeta^{2l-2}$, it suffices to consider the following three cases.

Case 1: $1=\zeta^{2l}$. Then $l=0$, $\eta=\varepsilon(\theta\zeta)$ and

(20) $$\begin{aligned}\eta&=a_0+a_1\theta\zeta+\ldots+a_{n-1}(\theta\zeta)^{n-1}\\&=a_0+a_1\theta\zeta^{-1}+\ldots+a_{n-1}(\theta\zeta^{-1})^{n-1}=\overline{\eta}\end{aligned}$$

by (12), since η is a real unit of L. Notice that $\zeta^i\ne\zeta^{-i}$ for $i=1,\ldots,n-1$. By Lemma 2, we see from (20) that $a_1=\ldots=a_{n-1}=0$ and $\varepsilon=\varepsilon(\theta)=\varepsilon(\theta\zeta)=a_0$. Since $N_{K/\mathbb{Q}}(\varepsilon)=1$ by (11), we get $a_0=\varepsilon=1$ and

(21) $$1+y\theta=(1+\theta)^m$$

by (11). For $m>1$, (21) is impossible.

Case 2: $1=\zeta^{2l-2}$ or $\zeta^2=\zeta^{2l}$. Then $l=1$ and $T_1(\zeta)=c_1\zeta^{n-1}-c_1\zeta^3$ by (20). Since $\zeta^{n-1}\ne\zeta^3$ and $c_1\ge 1$, (16) is false.

Case 3: $\zeta^2=\zeta^{2l-2}$. Then $l=2$ and $T_0(\zeta)=c_0-c_0\zeta^4$ by (19). Since $1\ne\zeta^4$ and $c_0\ge 1$, (16) is false.

All cases are considered and the Theorem is proved.

4. Proof of Corollary

Lemma 4 ([2]). *Let $n\in\mathbb{N}$ with $n\ge 3$, and let $\mu_n=\prod_{p\mid n}p^{1/(p-1)}$. Let $a,b\in\mathbb{N}$ such that $7a/8\le b<a$ and $a\equiv b\equiv 0 \pmod{n}$. If $\lambda=4b(a-b)^{-2}\mu_n^{-1}>1$, then*

$$\left|\left(\frac{a}{b}\right)^{1/n}-\frac{X}{Y}\right|>\frac{c}{Y^\delta}$$

for any $X \in \mathbb{Z}$ and any $Y \in \mathbb{N}$, where

$$\delta = 1 + \frac{\log 2\mu_n(a+b)}{\log \lambda}, \qquad c = \frac{1}{2^{\delta+2}(a+b)}.$$

LEMMA 5 ([9]). *Let $a, b \in \mathbb{N}$ with $a > 1$ and $b > 1$. Then the equation*

$$aX^3 - bY^3 = 1, \qquad X, Y \in \mathbb{N},$$

has at most one solution (X, Y).

LEMMA 6 ([12]). *Let $a, b, c, n \in \mathbb{N}$ with $n \geq 3$. If $(ab)^{n/2-1} \geq 4c^{2n-2}(n\mu_n)^n$, where μ_n was defined in Lemma 4, then the inequality*

$$|aX^n - bY^n| \leq c, \qquad X, Y \in \mathbb{N},\ \gcd(X, Y) = 1,$$

has at most one solution (X, Y).

Proof of Corollary. Let $u_m(x)$ be an nth power which satisfies $m > 1$ and $m \equiv 1 \pmod{n}$. Then (1) has a corresponding solution (x, y, m, n). We may assume that n is a prime. By [8], if $(x, m, n) \neq (3, 5, 2)$, then n is an odd prime. Further, by Theorem, we have $n \mid x\varphi(x)$. If $n \mid x$, then we find from (1) that $y^n \equiv 1 \pmod{n}$. This implies that $y^n \equiv 1 \pmod{n^2}$ and $n^2 \mid x$. If $n \nmid x$, then $n \mid \varphi(x)$ and x has a prime factor q such that $q \equiv 1 \pmod{n}$. So we have

$$x \equiv 0 \pmod{n^2} \quad \text{or} \quad x \text{ has a prime factor } q \text{ with } q \equiv 1 \pmod{n}. \tag{22}$$

On the other hand, since $m \equiv 1 \pmod{n}$, $m = nt + 1$ and $(X, Y) = (x^t, y)$ is a solution of the equation

$$xX^n - (x-1)Y^n = 1, \qquad X, Y \in \mathbb{N}, \tag{23}$$

where $t \in \mathbb{N}$. Notice that (23) has another solution $(X, Y) = (1, 1)$. By Lemmas 5 and 6, we get $n \geq 5$ and

$$(x(x-1))^{n/2-1} < 4n^{n^2/(n-1)}. \tag{24}$$

On combining (24) with (22), we obtain

$$\begin{gathered} n = 5 \text{ and } x = 11, 22, 25, 31, 33, 41 \text{ or } 44, \\ n = 7 \text{ and } x = 29, \qquad n = 11 \text{ and } x = 23. \end{gathered} \tag{25}$$

If $2 \nmid t$, then $2 \mid m$ and

$$\frac{x^{m/2} - 1}{x - 1} = y_1^n, \qquad x^{m/2} + 1 = y_2^n, \qquad y_1, y_2 \in \mathbb{N},\ y_1 y_2 = y. \tag{26}$$

By [1], (26) is impossible for $n \leq 11$. Therefore, by (25), we get $2 \mid t$. For the pairs (x, n) in (25), by computation, $u_{2n+1}(x)$ is not an nth power. So we have $t \geq 4$.

Let $a = xn$ and $b = (x-1)n$. If $x > 1$ and (X, Y) is a solution of (24), then

$$\left|\left(\frac{a}{b}\right)^{1/n} - \frac{Y}{X}\right| < \frac{1}{n(x-1)^{1/n}X^n}. \tag{27}$$

On the other hand, by Lemma 4, if $x \ge 8$ and $4(x-1) > n^{n/(n-1)}$, then

$$\left|\left(\frac{a}{b}\right)^{1/n} - \frac{Y}{X}\right| > \frac{c}{X^\delta}, \tag{28}$$

where

$$\delta < 2 + \frac{1}{\log 2}\left(\frac{n}{n-1}\log n + \log x\right), \qquad c = \frac{1}{2^{\delta+2}n(2x-1)}.$$

Take $(X, Y) = (x^t, y)$. The combination of (27) and (28) yields

$$\log x < \frac{10\log 2 + 2n\log n/(n-1)}{(n-2)t - 1 - n\log n/((n-1)\log 2)} < 2.32 < \log 11$$

and $x \le 10$ for $n \ge 5$ and $t \ge 4$, which contradicts (25). Thus, the Corollary is proved.

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