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An improved estimate concerning $3n+1$ predecessor sets

by

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Introduction. Consider the following operator on the set of integers:

$$T(n) := \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even,} \\ \frac{1}{2}(3n+1) & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

Now choose a starting number $x \in \mathbb{N}$, and look at its $3n+1$ *trajectory* $\{T^k(x) : k \geq 0\}$, where $T^k = T \circ \ldots \circ T$ denotes the k-fold iterate of T for $k \geq 1$, and $T^0(x) = x$. The famous and unsolved $3n+1$ *conjecture* says that any $3n+1$ trajectory eventually hits 1, for any starting number $x \in \mathbb{N}$.

There is an extensive literature on associated problems and generalizations of this conjecture (see [3] and [4]).

This paper proves an estimate on the functions

$$\vartheta_a(x) := |\{n \in \mathcal{P}(a) : n \leq ax\}| \tag{2}$$

where $\mathcal{P}(a)$ denotes the $3n+1$ predecessor set of $a \in \mathbb{Z}$, that is,

$$\mathcal{P}(a) := \{n \in \mathbb{Z} : T^k(n) = a \text{ for some } k \geq 0\}.$$

The investigation of the set $\mathcal{P}(1)$ began with Crandall [1] who succeeded in proving

$$\vartheta_1(x) \geq x^\beta \qquad \text{for some } \beta > 0 \text{ and large } x, \tag{3}$$

where the exponent has been computed to be $\beta \approx 0.057$. In 1987, Sander [5] improved Crandall's technique to show $\beta = \frac{1}{4}$ in (3). In 1989, Krasikov [2] introduced another technique to prove $\beta = \frac{3}{7}$. Here we extend Krasikov's method to obtain the estimate

$$\vartheta_a(x) \geq x^{0.48} \qquad \text{for large } x, \text{ if } a \text{ is not divisible by } 3. \tag{4}$$

Starting out from the set of Krasikov's inequalities given here in (7) it might be possible to get a further improvement of this exponent.

The improvement of Krasikov's estimate. For a given positive integer v and a given positive real number x, consider the set

$$G(v,x) := \left\{ n \in \mathbb{N} : \begin{array}{l} T^k(n) = v \text{ for some } k \ge 0 \\ T^i(n) \le x \text{ for } 0 \le i \le k \end{array} \right\}.$$

In his paper [2], Krasikov defines a function f by

$$f(v,x) = |G(v,x)|\,. \tag{5}$$

Then he puts

$$\Phi_n^m(y) := \inf\{f(v, 2^y v) : v \text{ is noncyclic and } v \equiv m \bmod 3^n\}$$

(an integer v is called *noncyclic* if $T^k(v) \ne v$ for each $k \ge 1$), which gives immediately the equation

$$\Phi_{n-1}^m(y) = \min\{\Phi_n^m(y), \Phi_n^{m+3^{n-1}}(y), \Phi_n^{m+2\cdot 3^{n-1}}(y)\}\,, \tag{6}$$

and he proves the following set of inequalities:

$$\begin{cases} \Phi_n^m(y) \ge \Phi_n^{4m}(y-2) + \Phi_{n-1}^{(4m-2)/3}(y+\alpha-2) & \text{if } m \equiv 2 \bmod 9, \\ \Phi_n^m(y) \ge \Phi_n^{4m}(y-2) & \text{if } m \equiv 5 \bmod 9, \\ \Phi_n^m(y) \ge \Phi_n^{4m}(y-2) + \Phi_{n-1}^{(2m-1)/3}(y+\alpha-1) & \text{if } m \equiv 8 \bmod 9 \end{cases} \tag{7}$$

with the constant $\alpha = \log_2 3 = 1.5849^+$. Note that (5) implies $\Phi_n^m(y) = 0$ for $y < 0$, and that $\Phi_n^m(y)$ is a nondecreasing function of y. In addition, we have $\Phi_n^m(0) \ge 1$ by the fact that $v \in G(v,v)$ gives $f(v, 2^0 v) \ge 1$ for each integer $v > 0$.

Since $G(a, ax) \subset \{n \in \mathcal{P}(a) : n \le x\}$, there is an obvious inequality between the functions ϑ_a defined in (2) and the Φ_n^m, provided a is noncyclic:

$$\vartheta_a(x) \ge \Phi_n^m(\log_2 x) \qquad \text{if } a \equiv m \bmod 3^n\,. \tag{8}$$

Krasikov uses the set (7) of inequalities for $n = 2$ to prove $\beta = \frac{3}{7}$ in the estimate (3), but he does not deal with $n \ge 3$. The following lemma provides the key to extract information out of (7) for the case $n = 3$.

Lemma 1.

$$\Phi_2^2(y) \ge \sum_{k=0}^{\infty} \Phi_2^8(y - 2 + k(\alpha - 4))\,.$$

Proof. An immediate consequence of (7) is

$$\Phi_2^2(y) \ge \Phi_2^8(y-2) + \Phi_1^2(y+\alpha-2)\,. \tag{9}$$

Moreover, we have, like Krasikov in his proof of Theorem 1 in [2],

$$\Phi_1^2(y) = \min\{\Phi_2^2(y), \Phi_2^5(y), \Phi_2^8(y)\} \ge \Phi_2^2(y-2) \tag{10}$$

since $\Phi_2^5(y) \ge \Phi_2^2(y-2)$ by (7), and $\Phi_2^8(y) \ge 1 + \Phi_1^2(y+\alpha-1) > \Phi_1^2(y)$, if $y \ge 2$. If $y < 2$ then (10) is obvious. (9) and (10) combine to give inductively

$$\Phi_2^2(y) \ge \sum_{k=0}^{n} \Phi_2^8(y-2+k(\alpha-4)) + \Phi_2^2((y-2+n(\alpha-4))+\alpha-2)\,. \ \blacksquare$$

In what follows, the transcendental function

$$g(\lambda) := \lambda^{-12} + \lambda^{\alpha-7} + \lambda^{\alpha-6} + \frac{\lambda^{\alpha-16}+\lambda^{\alpha-5}}{1-\lambda^{\alpha-4}} \tag{11}$$

will play an essential rôle. $g(\lambda)$ is a decreasing function of λ on the positive real axis, so there is a unique $\lambda_1 > 1$ with $g(\lambda_1) = 1$. This number λ_1 will be responsible for the exponent $\beta = 0.48 < \log_2 \lambda_1$ in the estimate (4).

PROPOSITION 2. *Let the real number $\lambda_0 > 1$ be given such that $g(\lambda_0) > 1$. Then $\Phi_2^8(y) \ge \lambda_0^y$ if y is sufficiently large.*

Proof. If we fix arbitrary numbers $\lambda > 1$ and $\widetilde{y} > 0$, the facts that Φ_2^8 is nondecreasing and $\Phi_2^8(0) \ge 1$ imply that there is a constant $c = c(\lambda, \widetilde{y}) > 0$ such that

$$\Phi_2^8(y) \ge c\lambda^y \qquad \text{for } 0 \le y \le \widetilde{y}\,. \tag{12}$$

Now the idea is to show—using Krasikov's inequalities (7)—that the condition $g(\lambda) > 1$ suffices to prolong the inequality (12) to all $y \ge 0$. Having done this prolongation, the claim follows by decreasing λ slightly to get rid of the constant c, while restricting the range to all sufficiently large y.

The system (7) reads for $n = 3$:

$$\begin{cases} \Phi_3^2(y) \ge \Phi_3^8(y-2) + \Phi_2^2(y+\alpha-2)\,, \\ \Phi_3^5(y) \ge \Phi_3^{20}(y-2)\,, \\ \Phi_3^8(y) \ge \Phi_3^5(y-2) + \Phi_2^5(y+\alpha-1)\,, \\ \Phi_3^{11}(y) \ge \Phi_3^{17}(y-2) + \Phi_2^5(y+\alpha-2)\,, \\ \Phi_3^{14}(y) \ge \Phi_3^2(y-2)\,, \\ \Phi_3^{17}(y) \ge \Phi_3^{14}(y-2) + \Phi_2^2(y+\alpha-1)\,, \\ \Phi_3^{20}(y) \ge \Phi_3^{26}(y-2) + \Phi_2^8(y+\alpha-2)\,, \\ \Phi_3^{23}(y) \ge \Phi_3^{11}(y-2)\,, \\ \Phi_3^{26}(y) \ge \Phi_3^{23}(y-2) + \Phi_2^8(y+\alpha-1)\,. \end{cases} \tag{13}$$

Since the functions Φ_n^m are nondecreasing, and because $\alpha > 1$ and $\Phi_2^8(0) \ge 1$, the last line of (13) implies $\Phi_3^{26}(y) \ge 1 + \Phi_2^8(y+\alpha-1) > \Phi_2^8(y)$, provided $y \ge 2$. Hence we conclude by (6)

$$\Phi_2^8(y) = \min\{\Phi_3^8(y), \Phi_3^{17}(y)\} \qquad \text{for } y \ge 2\,. \tag{14}$$

Starting with the third line of system (13) and running through this system, one arrives at the inequality

$$\begin{aligned}\Phi_3^8(y) \geq \Phi_3^{17}(y-12) + \Phi_2^5(y+\alpha-1) + \Phi_2^8(y+\alpha-6)\\ + \Phi_2^8(y+\alpha-7) + \Phi_2^5(y+\alpha-12)\,.\end{aligned}$$

By (7) and Lemma 1, one infers $\Phi_2^5(y) \geq \Phi_2^2(y-2) \geq \sum_{k=0}^{n} \Phi_2^8(y-4+k(\alpha-4))$ for any given integer $n \geq 0$. If we put

$$\begin{aligned}G_n(y) := \Phi_2^8(y-12) + \Phi_2^8(y+\alpha-6) + \Phi_2^8(y+\alpha-7)\\ + \sum_{k=0}^{n} (\Phi_2^8(y+\alpha-16+k(\alpha-4))\\ + \Phi_2^8(y+\alpha-5+k(\alpha-4)))\,,\end{aligned} \tag{15}$$

we come—using (14)—to the inequality

$$\Phi_3^8(y) \geq G_n(y) \qquad \text{for any } n \in \mathbb{N}\,. \tag{16}$$

An inspection of (15) shows that $G_n(y)$ needs the values of $\Phi_2^8(x)$ only at points in the range

$$y - 12 - (n+1)(\alpha-4) \leq x \leq y - (5-\alpha)\,.$$

Fixing an arbitrary $n \geq 0$ and a sufficiently large $\widetilde{y}$, and calculating a constant $c(\lambda, \widetilde{y})$ according to (12), we have

$$G_n(y) \geq c(\lambda, \widetilde{y})\, \lambda^y g_n(\lambda) \qquad \text{if } 12 + (n+1)(4-\alpha) \leq y \leq \widetilde{y} + (5-\alpha)\,, \tag{17}$$

where

$$g_n(\lambda) := \lambda^{-12} + \lambda^{\alpha-7} + \lambda^{\alpha-6} + \sum_{k=0}^{n} (\lambda^{\alpha-16+k(\alpha-4)} + \lambda^{\alpha-5+k(\alpha-4)})\,.$$

Analogously, chasing through the system (13) starting at the sixth line and using (14) gives

$$\Phi_3^{17}(y) \geq \Phi_2^8(y-6) + \Phi_2^2(y+\alpha-6) + \Phi_2^2(y+\alpha-1)\,.$$

As before, put

$$\begin{aligned}H_n(y) := \Phi_2^8(y-6)\\ + \sum_{k=0}^{n} (\Phi_2^8(y+\alpha-8+k(\alpha-4)) + \Phi_2^8(y+\alpha-3+k(\alpha-4)))\,,\end{aligned}$$

to get the inequality

$$\Phi_3^{17}(y) \geq H_n(y) \qquad \text{for any } n \in \mathbb{N}\,. \tag{18}$$

Again we see that $H_n(y)$ needs the values of $\Phi_2^8(x)$ only at points in the range

$$y - 4 - (n+1)(\alpha-4) \leq x \leq y - (3-\alpha)\,,$$

and we have

$$(19)\quad H_n(y) \geq c(\lambda, \widetilde{y})\,\lambda^y h_n(\lambda) \qquad \text{if } 4+(n+1)(4-\alpha) \leq y \leq \widetilde{y}+(3-\alpha)\,,$$

with the abbreviation

$$h_n(\lambda) := \lambda^{-6} + \sum_{k=0}^{n} (\lambda^{\alpha-8+k(\alpha-4)} + \lambda^{\alpha-3+k(\alpha-4)})\,.$$

Now the limiting functions

$$(20)\quad g(\lambda) = \lim_{n\to\infty} g_n(\lambda) \quad \text{and} \quad h(\lambda) := \lim_{n\to\infty} h_n(\lambda) = \lambda^{-6} + \frac{\lambda^{\alpha-8}+\lambda^{\alpha-3}}{1-\lambda^{\alpha-4}}$$

are clearly decreasing in the range $\lambda > 1$. Hence, there are unique numbers $\lambda_1, \lambda_2 > 1$ with $g(\lambda_1) = h(\lambda_2) = 1$. A simple numerical calculation shows that $\lambda_2 > \lambda_1$.

Given a number $\lambda_0 > 1$ satisfying $g(\lambda_0) > 1$ as in the assumption of Proposition 2, we know that $\lambda_0 < \lambda_1$. Choose λ' with $\lambda_0 < \lambda' < \lambda_1$ and n' with the property

$$(21)\qquad g_n(\lambda') \geq 1 \quad \text{and} \quad h_n(\lambda') \geq 1 \qquad \text{for } n \geq n'\,,$$

which is possible by (20). Moreover, put

$$y_0 := 12 + (n'+1)(4-\alpha)\,.$$

By the definition of $c(\lambda', y_0)$ above (12), we have

$$(22)\qquad \varPhi_2^8(y) \geq c(\lambda', y_0)(\lambda')^y \qquad \text{for } 0 \leq y \leq y_0\,.$$

Combine (14), (16), and (18) to get

$$\varPhi_2^8(y) = \min\{\varPhi_3^8(y), \varPhi_3^{17}(y)\} \geq \min\{G_{n'}(y), H_{n'}(y)\}\,.$$

This gives using (17) and (19)

$$\begin{aligned}\varPhi_2^8(y) &\geq c(\lambda', y_0)(\lambda')^y \min\{g_{n'}(\lambda'), h_{n'}(\lambda')\} \qquad \text{for } y_0 \leq y \leq y_0 + (3-\alpha)\\ &\geq c(\lambda', y_0)(\lambda')^y\end{aligned}$$

where the last inequality is due to (21). Using in addition inequality (22), the claim $\varPhi_2^8(y) \geq c(\lambda', y_0)(\lambda')^y$ can be proved inductively on the intervals $0 \leq y \leq y_0 + k(3-\alpha)$, which completes the proof of Proposition 2. ■

THEOREM 3. *For any integer $a > 0$ which is not divisible by* 3, *we have*

$$\vartheta_a(x) \geq x^{0.48} \qquad \textit{if } x \textit{ is sufficiently large}\,.$$

Proof. If $a \equiv 8 \bmod 3^2$, the result follows from (8) and Proposition 2:

$$\vartheta_a(x) \geq \varPhi_2^8(\log_2 x) \geq x^{\log_2 \lambda_0} \qquad \text{if } x \text{ is sufficiently large}\,,$$

where λ_0 satisfies $g(\lambda_0) > 1$. The number λ_1 with $g(\lambda_1) = 1$ and its $\log_2$ are approximately (with an error $< 10^{-3}$) given by $\lambda_1 \approx 1.397$ and $\log_2 \lambda_1 \approx 0.482$, whence the result.

If, more generally, we have only $a \not\equiv 0 \bmod 3$, it is easy to see that there is a noncyclic predecessor $b \in \mathcal{P}(a)$ satisfying $b \equiv 8 \bmod 3^2$. But this means $T^k(b) = a$ for some k, whence

$$\vartheta_a(x) \geq \vartheta_b\left(\frac{ax}{b}\right) \geq \left(\frac{a}{b}\right)^{\beta} x^{\beta} \quad \text{if } x \text{ is sufficiently large}.$$

Applying the remarks following (12) to this inequality completes the proof. ∎

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