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Arithmetic of half integral weight theta-series

by

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0. Introduction and notations. One of powerful methods of studying representations of quadratic forms by forms is via theta-series. Many authors did a great deal of work in this direction. Most of them, however, worked in the case when the representing quadratic form has an even number of variables. One reason for this is that quadratic forms with odd number of variables are associated with half integral weight theta-series whose transformation formulas involve branch problems.

In this article, we study the behavior of half integral weight theta-series under Hecke operators. We give an explicit formula of a given theta-series of half integral weight acted on by a Hecke operator as a linear combination of theta-series. As an application, we prove that generic theta-series of half integral weight are simultaneous eigenfunctions with respect to certain Hecke operators. For integral weight theta-series, analogous results were given by A. N. Andrianov [A2] in 1979.

For $g \in M_m(\mathbb{C})$, $h \in M_{m,n}(\mathbb{C})$, let $g[h] = {}^t hgh$, where ${}^t h$ is the transpose of h. For $g \in M_{2n}(\mathbb{R})$, let A_g, B_g, C_g, and D_g denote the $n \times n$ block matrices in the upper left, upper right, lower left, and lower right corners of g, respectively. Let $\mathcal{N}_m$ be the set of all semi-positive definite (eigenvalues ≥ 0), semi-integral (diagonal entries and twice nondiagonal entries are integers), symmetric $m \times m$ matrices, and $\mathcal{N}_m^+$ be its subset consisting of positive definite (eigenvalues > 0) matrices.

Let $G_n = GSp_n^+(\mathbb{R}) = \{g \in M_{2n}(\mathbb{R}) : J_n[g] = rJ_n,\ r > 0\}$ where $J_n = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ and $r = r(g)$ is a real number determined by g. Let $\Gamma^n = Sp_n(\mathbb{Z}) = \{M \in M_{2n}(\mathbb{Z}) : J_n[M] = J_n\}$. Let $\mathcal{H}_n = \{Z \in M_n(\mathbb{C}) : {}^t Z = Z,\ \operatorname{Im} Z \text{ is positive definite}\}$. For $g \in G_n$ and $Z \in \mathcal{H}_n$, we set

$$g\langle Z\rangle = (A_g Z + B_g)(C_g Z + D_g)^{-1} \in \mathcal{H}_n.$$

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For $Z \in M_n(\mathbb{C})$, let $e(Z) = \exp(2\pi i\sigma(Z))$ where $\sigma(Z)$ is the trace of Z. Finally, let $< n >= n(n+1)/2$ for $n \in \mathbb{Z}$.

For other standard terminologies and basic facts, we refer the readers to [A1], [M], [O].

1. Hecke rings. Let G be a multiplicative group and let Γ be its subgroup. Let L be a semigroup of G contained in the commensurator of Γ in G, i.e., $\Gamma^g = g^{-1}\Gamma g \cap \Gamma$ is of finite index in both $g^{-1}\Gamma g$ and Γ for any $g \in L$. Let (Γ, L) be a Hecke pair, i.e., $\Gamma L = L\Gamma = L$. Let $V = V(\Gamma, L)$ be the vector space over $\mathbb{C}$ spanned by left cosets (Γg), $g \in L$. Let $\mathcal{L} = \mathcal{L}(\Gamma, L)$ be the subspace of V consisting of $X = \sum a_i(\Gamma g_i)$, $a_i \in \mathbb{C}$, such that $XM = X$, for all $M \in \Gamma$, where $XM = \sum a_i(\Gamma g_i M)$. If we write $(\Gamma g \Gamma) = \sum_{i=1}^{\mu}(\Gamma g_i)$, $g,\ g_i \in L$, when $\Gamma g \Gamma$ is the disjoint union of $\Gamma g_i,\ i = 1, \ldots, \mu$, then the double cosets $(\Gamma g \Gamma)$, $g \in L$, form a basis for the subspace $\mathcal{L}$. $\mathcal{L}$ is in fact a ring, which is called the *Hecke ring* of the pair (Γ, L), with the multiplication defined by $X_1X_2 = \sum a_i b_j(\Gamma g_i h_j)$ for any $X_1 = \sum a_i(\Gamma g_i)$, $X_2 = \sum b_j(\Gamma h_j) \in \mathcal{L}$.

Let (Γ_1, L_1), (Γ_2, L_2) be two Hecke pairs such that

$$(1.1) \qquad \Gamma_2 \subset \Gamma_1, \quad \Gamma_1 L_2 = L_1, \quad \text{and} \quad \Gamma_1 \cap L_2 L_2^{-1} \subset \Gamma_2\,.$$

Then the map $\epsilon = \epsilon(\mathcal{L}_1, \mathcal{L}_2) : \mathcal{L}_1 = \mathcal{L}(\Gamma_1, L_1) \to \mathcal{L}_2 = \mathcal{L}(\Gamma_2, L_2)$ defined by $\epsilon(X) = \sum a_i(\Gamma_2 g_i) \in \mathcal{L}_2$ for any $X \in \mathcal{L}_1$, where X may be written in the form $X = \sum a_i(\Gamma_1 g_i)$ with $g_i \in L_2$ because of the second condition of (1.1), is an injective ring homomorphism. Moreover, ϵ is an isomorphism if $[\Gamma_1 : \Gamma_1^g] = [\Gamma_2 : \Gamma_2^g]$ for every $g \in L_2$.

Let $\widehat{G}$ be another multiplicative group and $\gamma : \widehat{G} \to G$ and $j : \Gamma \to \widehat{G}$ be surjective and injective homomorphisms, respectively, such that $\gamma \circ j = 1$ on Γ and $\operatorname{Ker}\gamma \subset C(\widehat{G})$, the center of $\widehat{G}$. For each $g \in L$, we define a homomorphism $\varrho = \varrho_g : \Gamma^g \to \widehat{G}$ by

$$(1.2) \qquad j(gMg^{-1}) = \zeta j(M)\zeta^{-1}\varrho(M) \quad \text{for every } M \in \Gamma^g$$

where $\zeta \in \widehat{G}$ such that $\gamma(\zeta) = g$. $\varrho_g(M)$ is independent of the choice of ζ because $\operatorname{Ker}\gamma \subset C(\widehat{G})$. We call ϱ_g the *lifting homomorphism* of g. It is known [Zh1] that if (Γ, L) is a Hecke pair and $[\Gamma : \operatorname{Ker}\varrho_g]$ is finite for any $g \in L$, then $(\widehat{\Gamma},\ \widehat{L})$ is also a Hecke pair where $\widehat{\Gamma} = j(\Gamma)$ and $\widehat{L} = \gamma^{-1}(L)$, and that if ϱ_g is trivial, then $(\widehat{\Gamma}\zeta\widehat{\Gamma}) = \sum_{i=1}^{\mu}(\widehat{\Gamma}\zeta_i)$ if and only if $(\Gamma g \Gamma) = \sum_{i=1}^{\mu}(\Gamma g_i)$, where $\zeta, \zeta_i \in \widehat{L}$ and $g, g_i \in L$ such that $\gamma(\zeta) = g$ and $\gamma(\zeta_i) = g_i$.

Let n, q be positive integers and p be a prime relatively prime to q. Let $L^n = L^n_p = \{g \in M_{2n}(\mathbb{Z}[p^{-1}]) : J_n[g] = p^{\delta}J_n,\ \delta \in \mathbb{Z}\}$ where $\delta = \delta(g)$ is an integer determined by g. Let $\Gamma^n_0(q) = \{M \in \Gamma^n : C_M \equiv 0 \pmod q\}$ and $L^n_0(q) = L^n_{0,p}(q) = \{g \in L^n : C_g \equiv 0 \pmod q\}$. Let $\Gamma^n_0 = \{M \in \Gamma^n : C_M = 0\}$ and $L^n_0 = L^n_{0,p} = \{g \in L^n : C_g = 0\}$. Finally, let $\Lambda^n = SL_n(\mathbb{Z})$

and $V^n = V_p^n = \{D \in M_n(\mathbb{Z}[p^{-1}]) : \det D = p^\delta,\ \delta \in \mathbb{Z}\}$. Then (Γ^n, L^n), $(\Gamma_0^n(q), L_0^n(q))$, (Γ_0^n, L_0^n), and (Λ^n, V^n) are Hecke pairs. We denote their corresponding Hecke rings by $\mathcal{L}^n = \mathcal{L}_p^n$, $\mathcal{L}_0^n(q) = \mathcal{L}_{0,p}^n(q)$, $\mathcal{L}_0^n = \mathcal{L}_{0,p}^n$, and $\mathcal{D}^n = \mathcal{D}_p^n$, respectively. We let $E^n = E_p^n = \{g \in L^n : \delta(g) \in 2\mathbb{Z}\}$, $E_0^n(q) = E_{0,p}^n(q) = E^n \cap L_0^n(q)$, and $E_0^n = E_{0,p}^n = E^n \cap L_0^n$. Then (Γ^n, E^n), $(\Gamma_0^n(q), E_0^n(q))$, and (Γ_0^n, E_0^n) are also Hecke pairs whose corresponding Hecke rings are denoted by $\mathcal{E}^n = \mathcal{E}_p^n$, $\mathcal{E}_0^n(q) = \mathcal{E}_{0,p}^n(q)$, and $\mathcal{E}_0^n = \mathcal{E}_{0,p}^n$, respectively. These are called the *even subrings* of $\mathcal{L}^n$, $\mathcal{L}_0^n(q)$, and $\mathcal{L}_0^n$, respectively.

Since Hecke pairs $(\Gamma_0^n(q),\ L_0^n(q))$ and (Γ_0^n, L_0^n) satisfy the conditions (1.1), we have a monomorphism $\beta^n = \epsilon(\mathcal{L}_0^n(q), \mathcal{L}_0^n) : \mathcal{L}_0^n(q) \to \mathcal{L}_0^n$,

$$\beta^n\Big(\sum a_i(\Gamma_0^n(q)g_i)\Big) = \sum a_i(\Gamma_0^n g_i) \tag{1.3}$$

where g_i are chosen to be in L_0^n. Similarly, we have an injective homomorphism $\alpha^n = \epsilon(\mathcal{L}, \mathcal{L}_0^n(q)) : \mathcal{L}^n \to \mathcal{L}_0^n(q)$, which is in fact an isomorphism because $[\Gamma^n : (\Gamma^n)^g] = [\Gamma_0^n(q) : (\Gamma_0^n(q))^g]$ for any $g \in L_0^n(q)$.

We introduce a homomorphism $\psi_n : \mathcal{L}_0^n \to \mathbb{C}_n[\boldsymbol{x}]$, where $\mathbb{C}_n[\boldsymbol{x}] = \mathbb{C}[x_0^{\pm1}, \ldots, x_n^{\pm1}]$. Let $X \in \mathcal{L}_0^n$. Then X can be written in the form $X = \sum a_i(\Gamma_0^n g_i)$, where $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix} \in L_0^n$, with $\delta_i = \delta(g_i) \in \mathbb{Z}$, $B_i \in M_n(\mathbb{Z}[p^{-1}])$, $D_i \in V^n$ and $D_i^* = ({}^tD)^{-1}$. We define $\omega_n : \mathcal{L}_0^n \to \mathcal{D}^n[t^{\pm1}]$ by

$$\omega_n(X) = \sum a_i t^{\delta_i}(\Lambda^n D_i)\,.$$

Then ω_n is a surjective ring homomorphism. Let $W = \sum a_i t^{\delta_i}(\Lambda^n D_i) \in \mathcal{D}^n[t^{\pm1}]$. We may assume that each D_i is an upper triangular matrix with diagonal entries $p^{d_{i1}}, \ldots, p^{d_{in}}$. We define $\phi_n : \mathcal{D}^n[t^{\pm1}] \to \mathbb{C}_n[\boldsymbol{x}]$ by

$$\phi_n(W) = \sum a_i x_0^{\delta_i}\Big(\prod_{1\le j\le n}(x_j p^{-j})^{d_{in}}\Big)\,.$$

Then ϕ_n is an injective ring homomorphism. Finally, we set

$$\psi_n = \phi_n \circ \omega_n : \mathcal{L}_0^n \to \mathbb{C}_n[\boldsymbol{x}]\,. \tag{1.4}$$

The Hecke rings we introduced above are local Hecke rings at p. We will not use global Hecke rings in this context except $\mathcal{D}_{\mathbb{Q}}^n$, the Hecke ring of the Hecke pair $(\Lambda^n, GL_n^+(\mathbb{Q}))$ where $GL_n^+(\mathbb{Q}) = \{D \in GL_n(\mathbb{Q}) : \det D > 0\}$, and its subring

$$\mathcal{D}_{\mathbb{Z}}^n = \Big\{\sum a_i(\Lambda^n D_i) \in \mathcal{D}_{\mathbb{Q}}^n : D_i \in M_n(\mathbb{Z}),\ \det D_i > 0\Big\}\,. \tag{1.5}$$

It is well known that $\mathcal{D}_{\mathbb{Q}}^n = \bigoplus_p \mathcal{D}_p^n$ where p runs over all rational primes.

2. The lifted Hecke rings. Let $\widehat{G}_n = \{(g, \alpha(Z)) : g \in G_n,\ \alpha(Z)$ is holomorphic on $\mathcal{H}_n,\ \alpha(Z)^2 = t(\det g)^{-1/2}\det(C_g Z + D_g)$ for some $t \in \mathbb{C}$,

$|t| = 1\}$. Then $\widehat{G}_n$ is a multiplicative group under the multiplication defined by $(g,\alpha(Z))(h,\beta(Z)) = (gh, \alpha(h\langle Z\rangle)\beta(Z))$ and is called the *universal covering group* of G_n.

Let $\gamma : \widehat{G}_n \to G$ be the projection $\gamma(g,\alpha(z)) = g$. We define an action of $\widehat{G}_n$ on $\mathcal{H}_n$ by $\zeta\langle Z\rangle = \gamma(\zeta)\langle Z\rangle$ for $\zeta \in \widehat{G}_n$, $Z \in \mathcal{H}_n$. Note that $\operatorname{Ker}\gamma \subset C(\widehat{G}_n)$.

For a moment, we assume $4 \mid q$. Let

$$\theta^n(Z) = \sum_{M\in M_{1,n}(\mathbb{Z})} e({}^tMMZ) = \sum_{N\in M_{n,1}(\mathbb{Z})} e(Z[N]), \quad Z \in \mathcal{H}_n. \tag{2.1}$$

$\theta^n(Z)$ is called the *standard theta-function.* For $M \in \Gamma_0^n(q)$, we define

$$j(M,Z) = \frac{\theta^n(M\langle Z\rangle)}{\theta^n(Z)}, \quad Z \in \mathcal{H}_n. \tag{2.2}$$

It is well known [S1] that $(M, j(M,Z)) \in \widehat{G}_n$. So the map $j : \Gamma_0^n(q) \to \widehat{G}_n$ defined by $j(M) = (M, j(M,Z))$ is a well defined injective homomorphism such that $\gamma\circ j = 1$ on $\Gamma_0^n(q)$. Hence we can define the lifting homomorphism ϱ_g for each $g \in L_0^n(q)$ and conclude that $(\widehat{\Gamma}_0^n(q), \widehat{L}_0^n(q))$ is a Hecke pair where $\widehat{\Gamma}_0^n(q) = j(\Gamma_0^n(q))$ and $\widehat{L}_0^n(q) = \gamma^{-1}(L_0^n(q))$ because $[\Gamma_0^n(q) : \operatorname{Ker}\varrho_g]$ is finite for each $g \in L_0^n(q)$ (see [Zh1]). Similarly $(\widehat{\Gamma}_0^n, \widehat{L}_0^n)$ is a Hecke pair where $\widehat{\Gamma}_0^n = j(\Gamma_0^n)$ and $\widehat{L}_0^n = \gamma^{-1}(L_0^n)$. We denote their corresponding Hecke rings by $\widehat{\mathcal{L}}_0^n(q) = \widehat{\mathcal{L}}_{0,p}^n(q)$ and $\widehat{\mathcal{L}}_0^n = \widehat{\mathcal{L}}_{0,p}^n$, respectively. Also $(\widehat{\Gamma}_0^n(q), \widehat{E}_0^n(q))$ and $(\widehat{\Gamma}_0^n, \widehat{E}_0^n)$ are Hecke pairs, where $\widehat{E}_0^n(q) = \gamma^{-1}(E_0^n(q))$ and $\widehat{E}_0^n = \gamma^{-1}(E_0^n)$, and we denote their corresponding Hecke rings by $\widehat{\mathcal{E}}_0^n(q) = \widehat{\mathcal{E}}_{0,p}^n(q)$ and $\widehat{\mathcal{E}}_0^n = \widehat{\mathcal{E}}_{0,p}^n$, which are the even subrings of $\widehat{\mathcal{L}}_0^n(q)$ and $\widehat{\mathcal{L}}_0^n$, respectively.

Hecke pairs $(\widehat{\Gamma}_0^n(q), \widehat{L}_0^n(q))$ and $(\widehat{\Gamma}_0^n, \widehat{L}_0^n)$ also satisfy (1.1). So we have an injective homomorphism $\widehat{\beta}^n = \epsilon(\widehat{\mathcal{L}}_0^n(q), \widehat{\mathcal{L}}_0^n) : \widehat{\mathcal{L}}_0^n(q) \to \widehat{\mathcal{L}}_0^n$,

$$\widehat{\beta}^n\Big(\sum a_i(\widehat{\Gamma}_0^n(q)\zeta_i)\Big) = \sum a_i(\widehat{\Gamma}_0^n\zeta_i) \tag{2.3}$$

where ζ_i are chosen to be in $\widehat{L}_0^n$.

For each $g \in L_0^n$, the lifting homomorphism $\varrho_g : (\Gamma_0^n)^g \to \widehat{G}_n$ is trivial [Zh1]. From this we obtain a surjective ring homomorphism

$$\pi_k^n : \widehat{\mathcal{L}}_0^n \to \mathcal{L}_0^n, \quad \pi_k^n(\widehat{\Gamma}_0^n\zeta\widehat{\Gamma}_0^n) = \tau(\zeta)^{-2k}(\Gamma_0^n g\Gamma_0^n) \tag{2.4}$$

where k is a positive half integer, i.e., $k = m/2$ for some odd integer $m \geq 1$, $\zeta = (g,\alpha(Z)) \in \widehat{L}_0^n$, and $\tau(\zeta) = \alpha(Z)/|\alpha(Z)|$.

Let $g_s^n = \operatorname{diag}(I_{n-s}, pI_s, p^2I_{n-s}, pI_s) \in E_0^n$ for $s = 0,1,\ldots,n$. Let $T_s^n = (\Gamma_0^n(q)g_s^n\Gamma_0^n(q)) \in \mathcal{E}_0^n(q)$ and let $\mathcal{L}_0^n(T) = \mathcal{L}_{0,p}^n(T)$ be the subring $\mathbb{C}[T_0^n,\ldots, T_{n-1}^n, (T_n^n)^{\pm 1}]$ of $\mathcal{E}_0^n(q)$. Similarly, let $\widehat{T}_s^n = (\widehat{\Gamma}_0^n(q)\widehat{g}_s^n\widehat{\Gamma}_0^n(q)) \in \widehat{\mathcal{E}}_0^n(q)$, where $\widehat{g}_s^n = (g_s^n, p^{(n-s)/2}) \in \widehat{E}_0^n$ for $s = 0,1,\ldots,n$, and let $\widehat{\mathcal{L}}_0^n(T) = \widehat{\mathcal{L}}_{0,p}^n(T)$ be the subring $\mathbb{C}[\widehat{T}_0^n, \ldots, \widehat{T}_{n-1}^n, (\widehat{T}_n^n)^{\pm 1}]$ of $\widehat{\mathcal{E}}_0^n(q)$. We define

$$\mathbb{L}_0^n(T) = \mathbb{L}_{0,p}^n(T) = (\pi_k^n \circ \widehat{\beta}^n)(\widehat{\mathcal{L}}_0^n(T)) \subset \mathcal{E}_0^n\,. \tag{2.5}$$

Let S_n be the permutation group on $\{x_1, x_2, \ldots, x_n\}$. Let W_n be the group of automorphisms of $\mathbb{C}_n[\boldsymbol{x}]$ generated by S_n and σ_i, $i = 0, \ldots, n$, where σ_i are automorphisms of $\mathbb{C}_n[\boldsymbol{x}]$ defined by

$$\sigma_0 : x_0 \mapsto -x_0\,;\ x_j \mapsto x_j, \quad \forall j \neq 0\,,$$
$$\sigma_i : x_0 \mapsto x_0 x_i\,;\ x_i \mapsto x_i^{-1}\,;\ x_j \mapsto x_j, \quad \forall j \neq 0, i, \text{ for } i = 1, \ldots, n\,.$$

Let $W_n[\boldsymbol{x}]$ be the subring of $\mathbb{C}_n[\boldsymbol{x}]$ consisting of all W_n-invariant elements. Then

$$\psi_n : \mathbb{L}_0^n(T) \to W_n[\boldsymbol{x}] \tag{2.6}$$

is an isomorphism [Zh2]. Note that this implies that $\mathbb{L}_0^n(T)$ is a commutative ring.

Let $\Delta^n(\boldsymbol{x}) = (x_0^2 x_1 \ldots x_n)$, and $R_i^n(\boldsymbol{x}) = s_i(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1})$ for $i = 0, \ldots, 2n$, where $s_i(\cdot)$ denotes the elementary symmetric polynomial of degree i in the corresponding variables. It is known [A2] that $W_n[\boldsymbol{x}]$ is generated by $\Delta^n(\boldsymbol{x})^{\pm 1}$ and $R_i^n(\boldsymbol{x})$, $i = 1, \ldots, n$.

3. Hecke polynomials. Let

$$r^n(y) = \prod_{1 \le j \le n} (1 - x_j^{-1} y)(1 - x_j y) = \sum_{i=0}^{2n} (-1)^i R_i^n(\boldsymbol{x}) y^i\,. \tag{3.1}$$

W_n only permutes the factors of $r^n(y)$ and hence the coefficients $R_i^n(\boldsymbol{x})$ are W_n-invariant. By (2.6), there exist $R_i^n \in \mathbb{L}_0^n(T)$ such that $\psi_n(R_i^n) = R_i^n(\boldsymbol{x})$ for all $i = 0, \ldots, 2n$. Let $\Delta^n = (\pi_k^n \circ \widehat{\beta}^n)(\widehat{T}_n^n) = p(\Gamma_0^n I_{2n} \Gamma_0^n)$. Then $\psi_n(\Delta^n) = p^{-<n>} \Delta^n(\boldsymbol{x})$. Therefore, we obtain

$$\mathbb{L}_0^n(T) = \mathbb{C}[R_1^n, \ldots, R_n^n, (\Delta^n)^{\pm 1}]\,. \tag{3.2}$$

Let $R^n(y)$ be a polynomial over $\mathbb{L}_0^n(T)$ defined by

$$R^n(y) = \sum_{i=0}^{2n} (-1)^i R_i^n y^i \in \mathbb{L}_0^n(T)[y]\,. \tag{3.3}$$

Such a polynomial over a Hecke ring is called a *Hecke polynomial.*

Let $\Pi_s^n = (\Gamma_0^n h_s^n \Gamma_0^n) \in \mathcal{L}_0^n$, where $h_s^n = \mathrm{diag}(pI_{n-s}, I_s, I_{n-s}, pI_s) \in L_0^n$, $s = 0, 1, \ldots, n$. Let $A = {}^t A \in M_i(\mathbb{Z})$ and $r(A) = r_p(A)$ be the rank of A modulo p, where p is a prime. If $r = r(A) \ge 0$, then there exist $U \in M_i(\mathbb{Z})$ and $A' \in M_r(\mathbb{Z})$ such that p is relatively prime to $(\det U\ \det A')$

and $A_1 \equiv A[U] \pmod p$, where $A_1 = \mathrm{diag}(A', 0_{i-r})$. We define

$$\kappa(A) = \kappa_p(A) = \begin{cases} \varepsilon_p^{-r}\left(\dfrac{(-1)^r \det A'}{p}\right) & \text{if } r > 0\,, \\ 1 & \text{if } r = 0\,, \end{cases}$$

where $(-)$ is the Legendre symbol and ε_p is a complex number defined by $\varepsilon_p = 1$ for $p \equiv 1 \pmod 4$ and $\varepsilon_p = \sqrt{-1}$ for $p \equiv 3 \pmod 4$. Then $\kappa(A)$ is independent of the choice of U and A'. Let $\{A\} = \{A\}_p$ be the set of all $A_1 = {}^tA_1 \in M_i(\mathbb{Z})$ such that $A_1 \equiv A[U] \pmod p$ for some $U \in GL_i(\mathbb{Z})$; call it the *p-class* of A. Note that $\kappa(A)$, $r(A)$ are invariants of the p-class of A.

Let $D^n_{ij} = \mathrm{diag}(I_{n-i-j}, pI_i, p^2 I_j)$ for $0 \le i, j, i+j \le n$, and let $B^n_{ij}(A) = \mathrm{diag}(0_{n-i-j}, A, 0_j)$ for $A = {}^tA = M_i(\mathbb{Z})$. Then

$$g^n_{ij}(A) = \begin{pmatrix} p^2(D^n_{ij})^* & B^n_{ij}(A) \\ 0 & D^n_{ij} \end{pmatrix} \in E^n_0 \quad \text{and} \quad \Pi^n_{ij}(A) = (\Gamma^n_0 g^n_{ij}(A) \Gamma^n_0) \in \mathcal{E}^n_0\,.$$

Moreover, $\Pi^n_{ij}(A) = \Pi^n_{ij}(A_1)$ if $A_1 \in \{A\}$. For $0 \le r \le i$ and a half integer k, we set

$$\Pi^{n,r}_{ij}(\kappa) = \sum_{\{A\}, r(A)=r} \kappa(A)^{-2k} \Pi^n_{ij}(A)\,.$$

Let

$$\varphi_p(l) = \prod_{a=1}^{l} (p^a - 1) \quad \text{for } l \ge 1 \quad (\varphi_p(0) = 1)\,,$$

$$\varphi^+_p(l) = \prod_{\substack{2 \le a \le l \\ a \text{ even}}} (p^a - 1) \quad \text{for } l \ge 2 \quad (\varphi^+_p(0) = \varphi^+_p(1) = 1)\,,$$

and let

$$\sigma^n_{ij} = \frac{\varphi_p(n-i+j)(-p)^{j/2}}{\varphi_p(n-i)\varphi^+_p(j)} \quad \text{or } 0$$

for j even or odd, respectively, where $0 \le i, j, i+j \le n$. Let

$$\begin{gathered} X^n_-(y) = \sum_{i=0}^{n} (-1)^i X^n_{-i} y^i, \quad X^n_+(y) = \sum_{i=0}^{n} (-1)^i X^n_{+i} y^i\,, \\ B^n(\kappa, y) = \sum_{i=0}^{n} (-1)^i B^n_i(\kappa) y^i \end{gathered} \tag{3.4}$$

where $X^n_{-i} = \Delta^{-1} \Pi^n_0 \Pi^n_{n-i}$, $X^n_{+i} = \Delta^{-1} \Pi^n_i \Pi^n_n$, and

$$B^n_i(\kappa) = p^{<n-i>} \Delta^{-1} \sum_{j=0}^{i} \sigma^n_{ij} \Pi^{n,i-j}_{n0}(\kappa)$$

for $i = 0, 1, \ldots, n$. Here $\Delta = p^{<n>} \Delta^n$.

The following is an analogue of Andrianov's result on the factorization of Hecke polynomials concerning integral weight Siegel modular forms [A2].

PROPOSITION 3.1. $R^n(y) = X^n_-(y)B^n(\kappa, y)X^n_+(y)$.

Proof. See [Zh2].

Let $\mathcal{C}^n_- = \mathcal{C}^n_{-p} = \{X \in \mathcal{L}^n_0 : X\Pi^n_0 = \Pi^n_0 X\}$ and $\mathcal{C}^n_+ = \mathcal{C}^n_{+p} = \{X \in \mathcal{L}^n_0 : X\Pi^n_n = \Pi^n_n X\}$. It is well known [A2] that $\mathcal{C}^n_-$ and $\mathcal{C}^n_+$ are commutative subrings of $\mathcal{L}^n_0$ with no zero divisors. Let $\mathcal{C}^n_-[[y]]$ and $\mathcal{C}^n_+[[y]]$ be the formal power series rings in y over $\mathcal{C}^n_-$ and $\mathcal{C}^n_+$, respectively. Then $X^n_-(y)$ and $X^n_+(y)$ are invertible in $\mathcal{C}^n_-[[y]]$ and $\mathcal{C}^n_+[[y]]$, respectively, because their constant term $(\Gamma^n_0 I_{2n}\Gamma^n_0)$ is the unity of $\mathcal{L}^n_0$, and we denote their inverses by $X^-_n(y)$ and $X^+_n(y)$, respectively. If we write

$$X^-_n(y) = \sum_{i=0}^{\infty} X^{-i}_n y^i \in \mathcal{C}^n_-[[y]] \quad \text{and} \quad X^+_n(y) = \sum_{i=0}^{\infty} X^{+i}_n y^i \in \mathcal{C}^n_+[[y]],$$

then

$$\begin{aligned} X^{-i}_n &= p^{-in} \sum_{\substack{D\in\Lambda^n\backslash M_n(\mathbb{Z})/\Lambda^n \\ \det D = p^i}} \left(\Gamma^n_0 \begin{pmatrix} D & 0 \\ 0 & D^* \end{pmatrix} \Gamma^n_0\right), \\ X^{+i}_n &= p^{-in} \sum_{\substack{D\in\Lambda^n\backslash M_n(\mathbb{Z})/\Lambda^n \\ \det D = p^i}} \left(\Gamma^n_0 \begin{pmatrix} D^* & 0 \\ 0 & D \end{pmatrix} \Gamma^n_0\right). \end{aligned} \tag{3.5}$$

Observe that $X^{-i}_n, X^{+i}_n \in \mathcal{E}^n_0$.

4. Siegel modular forms of half integral weight. Let n, q be a positive integers with $4 \mid q$. Let χ be a Dirichlet character modulo q. Let p be a prime relatively prime to q. Let k be a positive half integer. For a complex-valued function F on $\mathcal{H}_n$ and $\zeta = (g, \alpha(Z)) \in \widehat{G}_n$, we set

$$(F|_k\zeta)(Z) = r(g)^{nk/2-<n>}\alpha(Z)^{-2k}F(g\langle Z\rangle), \qquad Z \in \mathcal{H}_n. \tag{4.1}$$

Since the map $Z \to g\langle Z\rangle$ is an analytic automorphism of $\mathcal{H}_n$ and $\alpha(Z) \neq 0$ on $\mathcal{H}_n$, $F|_k\zeta$ is holomorphic on $\mathcal{H}_n$ if F is. Also from the definition it follows that $F|_k\zeta_1|_k\zeta_2 = F|_k\zeta_1\zeta_2$ for $\zeta_1, \zeta_2 \in \widehat{G}_n$.

A function $F : \mathcal{H}_n \to \mathbb{C}$ is called a *Siegel modular form* of degree n, weight k, level q, with character χ if the following conditions hold: (i) F is holomorphic on $\mathcal{H}_n$, (ii) $F|_k\widehat{M} = \chi(\det D_M)F$ for every $\widehat{M} = (M, j(M, Z)) \in \widehat{\Gamma}^n_0(q)$, and (iii) $F|_k(M, \alpha(z))$ is bounded as $\operatorname{Im} z \to \infty$, $z \in \mathcal{H}_1$, for every $(M, \alpha(z)) \in \widehat{G}_1$ with $M \in SL_2(\mathbb{Z})$ when $n = 1$. It is known [Koe] that the boundedness condition (iii) follows from (i) and (ii) for $n \geq 2$. We denote the set of all such Siegel modular forms by $\mathcal{M}^n_k(q, \chi)$. This is a finite-dimensional vector space over $\mathbb{C}$ [Si2].

A function $F : \mathcal{H}_n \to \mathbb{C}$ is called an *even* or *odd modular form* of degree n if F satisfies (i), (ii)$'$ $(\det D_M)^s F(M\langle Z\rangle) = F(Z)$, $Z \in \mathcal{H}_n$ for every $M \in \Gamma_0^n$, where $s = 0$ for even and $s = 1$ for odd modular forms, and (iii)$'$ $F(z)$ is bounded as $\operatorname{Im} z \to \infty$, $z \in \mathcal{H}_1$ when $n = 1$. We denote the sets of all even modular forms by $\mathcal{M}_0^n$ and of odd modular forms by $\mathcal{M}_1^n$. They are also vector spaces over $\mathbb{C}$.

Let $F \in \mathcal{M}_k^n(q,\chi)$ and $\chi(-1) = (-1)^s$ for $s = 0$ or 1. For $M \in \Gamma_0^n$, we have $\widehat{M} = (M, j(M,Z)) = (M,1)$ and $\det D_M = \pm 1$. So, F satisfies (ii)$'$, (iii)$'$ and hence

$$\mathcal{M}_k^n(q,\chi) \subset \mathcal{M}_s^n \quad \text{if } \chi(-1) = (-1)^s\,. \tag{4.2}$$

For $F \in \mathcal{M}_k^n(q,\chi)$ and $\widehat{X} = \sum a_i(\widehat{\Gamma}_0^n(q)\zeta_i) \in \widehat{\mathcal{E}}_0^n(q)$, we set

$$F|_{k,\chi}\widehat{X} = \sum a_i \chi(\det A_i) F|_k \zeta_i\,, \tag{4.3}$$

where $A_i = A_{\gamma(\zeta_i)}$. There is a good reason for using the even subring $\widehat{\mathcal{E}}_0^n(q)$ instead of $\widehat{\mathcal{L}}_0^n(q)$: the action of double cosets in $\widehat{\mathcal{L}}_0^n(q) - \widehat{\mathcal{E}}_0^n(q)$ on $\mathcal{M}_k^n(q,\chi)$ is trivial [Zh1], i.e., for $F \in \mathcal{M}_k^n(q,\chi)$ and $\widehat{X} = (\widehat{\Gamma}_0^n(q)\zeta\widehat{\Gamma}_0^n(q)) \in \widehat{\mathcal{L}}_0^n(q) - \widehat{\mathcal{E}}_0^n(q)$, we have $F|_{k,\chi}\widehat{X} = 0$.

As for $F \in \mathcal{M}_s^n$ and $X = \sum a_i(\Gamma_0^n g_i) \in \mathcal{L}_0^n$, we set

$$F|_{k,\chi}X = \sum a_i \chi(\det A_i) F|_k \widetilde{g}_i \tag{4.4}$$

where

$$\widetilde{g}_i = (g_i, (\det g_i)^{-1/4} |\det D_i|^{1/2}) \in \widehat{L}_0^n\,, \tag{4.5}$$

$A_i = A_{g_i}$, and $\chi(-1) = (-1)^s$.

$\widehat{X}$ and X acting on modular spaces as above are called *Hecke operators*. It follows from the definitions that $F|_{k,\chi}\widehat{X}_1 \in \mathcal{M}_k^n(q,\chi)$ if $F \in \mathcal{M}_k^n(q,\chi)$ and $F|_{k,\chi}\widehat{X}_1|_{k,\chi}\widehat{X}_2 = F|_{k,\chi}\widehat{X}_1\widehat{X}_2$ for any $\widehat{X}_1, \widehat{X}_2 \in \widehat{\mathcal{E}}_0^n(q)$. Similarly, for $F \in \mathcal{M}_s^n$ and $X_1, X_2 \in \mathcal{L}_0^n$, we have $F|_{k,\chi}X_1 \in \mathcal{M}_s^n$ and $F|_{k,\chi}X_1|_{k,\chi}X_2 = F|_{k,\chi}X_1X_2$, where $\chi(-1) = (-1)^s$.

Let $\chi(-1) = (-1)^s$, with $s = 0$ or 1, $F \in \mathcal{M}_k^n(q,\chi) \subset \mathcal{M}_s^n$, and $\widehat{X} = \sum a_i(\widehat{\Gamma}_0^n(q)\zeta_i) \in \widehat{\mathcal{E}}_0^n(q)$, where $\zeta_i = (g_i, \alpha_i(Z)) \in \widehat{E}_0^n$ with $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix}$ and $\alpha_i(Z) = t_i p^{-n\delta_i/4}(\det D_i)^{1/2}$ for some $t_i \in \mathbb{C}$, $|t_i| = 1$, $\delta_i \in 2\mathbb{Z}$. We choose the usual branch for $(\det D_i)^{1/2}$ when $\det D_i < 0$. Since $j(M,Z) = 1$ for any $M \in \Gamma_0^n$, from (2.3) and (2.4) it follows that

$$(\pi_k^n \circ \widehat{\beta}^n)(\widehat{X}) = \sum a_i (t_i\varepsilon_i)^{-2k}(\Gamma_0^n g_i) \in \mathcal{E}_0^n$$

where $\varepsilon_i = 1$ or $\sqrt{-1}$ according as $\det D_i > 0$ or $\det D_i < 0$. So (4.1) and

(4.3)–(4.5) yield

$$\begin{aligned}
F|_{k,\chi}(\pi_k^n\circ\widehat{\beta}^n)(\widehat{X})
&=\sum a_i(t_i\varepsilon_i)^{-2k}\chi(\det p^{\delta_i}D_i^*)F|_k\widetilde{g}_i\\
&=\sum a_i(t_i\varepsilon_i)^{-2k}\chi(\det p^{\delta_i}D_i^*)(p^{\delta_i})^{nk/2-<n>}\\
&\quad\times(p^{-n\delta_i/4}|\det D_i|^{1/2})^{-2k}F(g_i\langle Z\rangle)\\
&=\sum a_i\chi(\det p^{\delta_i}D_i^*)(p^{\delta_i})^{nk-<n>}(t_i(\det D_i)^{1/2})^{-2k}F(g_i\langle Z\rangle)
\end{aligned}$$

so that

$$F|_{k,\chi}\widehat{X}=F|_{k,\chi}(\pi_k^n\circ\widehat{\beta}^n)(\widehat{X})\,. \tag{4.6}$$

Let $\mathcal{M}_s^n[[y]]$ and $\mathcal{L}_0^n[[y]]$ be the rings of formal power series in y over $\mathcal{M}_s^n$ and $\mathcal{L}_0^n$, respectively. For $F(y)=\sum_{i=0}^\infty F_iy^i\in\mathcal{M}_s^n[[y]]$ and $X(y)=\sum_{j=0}^\infty X_jy^j\in\mathcal{L}_0^n[[y]]$, we generalize (4.4) formally as follows:

$$F(y)|_{k,\chi}X(y)=\sum_{l=0}^\infty\Big(\sum_{i+j=l}F_i|_{k,\chi}X_j\Big)y^l\in\mathcal{M}_s^n[[y]] \tag{4.7}$$

for a half integer k and a character χ satisfying $\chi(-1)=(-1)^s$. Observe that

$$F(y)|_{k,\chi}X_1(y)X_2(y)=F(y)|_{k,\chi}X_1(y)|_{k,\chi}X_2(y)$$

for $F(y)\in\mathcal{M}_s^n[[y]]$, $X_1(y),X_2(y)\in\mathcal{L}_0^n[[y]]$. We say that $F(y)\in\mathcal{M}_s^n[[y]]$ is *defined at* $\tau\in\mathbb{C}$ if $F(\tau)$ converges absolutely and uniformly on every subset $\mathcal{H}_n(c)$ of $\mathcal{H}_n$ where $\mathcal{H}_n(c)=\{Z\in\mathcal{H}_n:\operatorname{Im}Z\ge c\}$ for $c>0$.

We now introduce an action of $\mathcal{D}_{\mathbb{Q}}^n$ on $\mathcal{M}_s^n$, $s=0$ or 1. Let $F\in\mathcal{M}_s^n$ and $W=\sum a_i(\Lambda^nD_i)\in\mathcal{D}_{\mathbb{Q}}^n$. We define

$$(F|W)(Z)=\sum a_iF(Z[{}^tD_i]),\qquad Z\in\mathcal{H}_n\,. \tag{4.8}$$

For $D\in V^n\cap M_n(\mathbb{Z})$, we set

$$g_D=\begin{pmatrix}D&0\\0&D^*\end{pmatrix}\in E_0^n\qquad\text{and}\qquad T_D=(\Gamma_0^ng_DG_0^n)\in\mathcal{E}_0^n\,.$$

Then $T_D=\sum_{D_i\in\Lambda^n\backslash\Lambda^nD\Lambda^n}(\Gamma_0^ng_{D_i})$. So if $\chi(-1)=(-1)^s$, then (4.4), (4.5) and (4.8) imply that

$$\begin{aligned}
F|_{k,\chi}T_D&=\sum_{D_i\in\Lambda^n\backslash\Lambda^nD\Lambda^n}\chi(\det D_i)F|_k\widetilde{g}_{D_i}\\
&=\chi(\det D)(\det D)^kF|(\Lambda^nD\Lambda^n).
\end{aligned} \tag{4.9}$$

5. Action of $B^n(\kappa,y)$ on $\mathcal{M}_s^n$. Let n, q, χ, p, and k be as above. For $F\in\mathcal{M}_s^n$ ($s=0$ or 1), we set

$$F|_{k,\chi}B^n(\kappa,y)=\sum_{i=0}^{n}(-1)^i(F|_{k,\chi}B_i^n(\kappa))y^i\,. \tag{5.1}$$

For $0\le r\le n$, $N\in\mathcal{N}_n$, we let

$$l^n(\kappa,r,N)\ =\sum_{\substack{A={}^tA\in M_n(\mathbb{F}_p)\\ r(A)=r}}\kappa(A)^{-2k}e\bigg(\frac{NA}{p}\bigg)\,.$$

Zhuravlev [Zh2] showed

$$F|_{k,\chi}B^n(\kappa,y)=\sum_{N\in\mathcal{N}_n}B^n(\kappa,y,N)f(N)e(NZ),\qquad Z\in\mathcal{H}_n\,, \tag{5.2}$$

where

$$B^n(\kappa,y,N)=\sum_{i=0}^{n}(-1)^ip^{<n-i>-<n>}\Big(\sum_{j=0}^{i}\alpha_{ij}^n l^n(\kappa,i-j,N)\Big)y^i \tag{5.3}$$

and $F(Z)=\sum_{N\in\mathcal{N}_n}f(N)e(NZ)$ (see (6.2)).

For semi-integral $n\times n$ matrices N_1, N_2, we write $N_1\equiv N_2 \pmod p$ if $(N_1-N_2)/p$ is again semi-integral, and write $N_1{\sim}N_2 \pmod p$ if there exists $U{\in}M_n(\mathbb{Z})$ such that $N_1{\equiv}N_2[U] \pmod p$ and p is relatively prime to $\det 2U$. The following properties of $B^n(\kappa,y,N)$ are also due to Zhuravlev [Zh2]:

$$B^n(\kappa,y,N_1)=B^n(\kappa,y,N_2)\qquad\text{if } N_1\sim N_2 \pmod p \tag{5.4}$$

for $N_1, N_2\in\mathcal{N}_n$, and

$$B^n(\kappa,y,N)=B^{n-1}(\kappa,y,N')\qquad\text{if } N\sim\begin{pmatrix}N'&0\\0&0\end{pmatrix}\pmod p \tag{5.5}$$

for $N\in\mathcal{N}_n$ and $N'\in\mathcal{N}_{n-1}$. Finally, if N is non-degenerate modulo p, i.e., p is relatively prime to $\det 2N$, then

$$\begin{aligned}&B^n(\kappa,y,N)\\ &=\begin{cases}\displaystyle\prod_{0\le i\le n/2-1}\Big(1-\frac{y^2}{p^{2i+1}}\Big) & \text{for } n \text{ even}\,,\\ \displaystyle\Big(1-\chi_{k,N}^n(p)\frac{y}{p^{n/2}}\Big)\prod_{0\le i\le (n-3)/2}\Big(1-\frac{y^2}{p^{2i+1}}\Big) & \text{for } n \text{ odd}\,,\end{cases}\end{aligned} \tag{5.6}$$

where

$$\chi_{k,N}^n(p)=\bigg(\frac{(-1)^{(2k-n)/2}2\det 2N}{p}\bigg) \tag{5.7}$$

for n odd and $(-)$ is the Legendre symbol.

6. Zharkovskaya's commutation relation. Let n, q, χ, p and k be as above. Let $F \in \mathcal{M}_s^n$. We define $\Phi : \mathcal{M}_s^n \to \mathcal{M}_s^{n-1}$ by

$$(6.1) \qquad (\Phi F)(Z') = \lim_{\lambda \to +\infty} F\left(\begin{pmatrix} Z' & 0 \\ 0 & i\lambda \end{pmatrix}\right), \qquad Z' \in \mathcal{H}_{n-1} \text{ and } \lambda > 0\,.$$

Φ is well defined and is called the *Siegel operator* ($\mathcal{M}_s^0 = \mathbb{C}$, $\mathcal{H}_0 = \{0\}$). Every $F \in \mathcal{M}_s^n$, hence every $F \in \mathcal{M}_k^n(q,\chi)$ if $\chi(-1) = (-1)^s$, has a Fourier expansion of the form

$$(6.2) \qquad F(Z) = \sum_{N \in \mathcal{N}_n} f(N) e(NZ), \qquad Z \in \mathcal{H}_n\,.$$

Then from (6.1) and (6.2) it follows that

$$(6.3) \qquad (\Phi F)(Z') = \sum_{N' \in \mathcal{N}_{n-1}} f\left(\begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix}\right) e(N'Z'), \qquad Z' \in \mathcal{H}_{n-1}$$

($\mathcal{N}_0 = \{0\}$) and that $\Phi F \in \mathcal{M}_k^{n-1}(q,\chi)$ if $F \in \mathcal{M}_k^n(q,\chi)$.

Let $X = \sum a_i(\Gamma_0^n g_i) \in \mathcal{L}_0^n$ where $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix} \in L_0^n$. By multiplying g_i on the left by $\begin{pmatrix} U_i^* & 0 \\ 0 & U_i \end{pmatrix} \in \Gamma_0^n$ for a suitable $U_i \in GL_n(\mathbb{Z})$, we may assume that all the D_i are of the form $D_i = \begin{pmatrix} D_i' & * \\ 0 & p^{d_i} \end{pmatrix}$, $d_i \in \mathbb{Z}$, where $D_i' \in V^{n-1}$ is upper triangular. We set

$$(6.4) \qquad \Psi(X,u) = \sum a_i u^{-\delta_i} (up^{-n})^{d_i} (\Gamma_0^{n-1} g_i') \in \mathcal{L}_0^{n-1}[u^{\pm 1}]$$

where $g_i' = \begin{pmatrix} p^{\delta_i} (D_i')^* & B_i' \\ 0 & D_i' \end{pmatrix} \in L_0^{n-1}$ and $\mathcal{L}_0^{n-1}[u^{\pm 1}]$ is the polynomial ring in u, u^{-1} over $\mathcal{L}_0^{n-1}$. Here B_i' and D_i' denote the blocks of size $(n-1)\times(n-1)$ in the upper left corners of B_i and D_i, respectively. If $n = 1$, we set $\Psi(X,u) = \sum a_i u^{-\delta_i}(up^{-1})^{d_i}$. Note that δ_i, d_i are uniquely determined by the left coset $(\Gamma_0^n g_i)$ for each i. $\Psi(-,u) : \mathcal{L}_0^n \to \mathcal{L}_0^{n-1}[u^{\pm 1}]$ is a well defined ring homomorphism (see [Z]).

We define a ring homomorphism $\eta(-,u) : \mathbb{C}_n[\boldsymbol{x}] \to \mathbb{C}_{n-1}[\boldsymbol{x}, u^{\pm 1}]$ by

$$(6.5) \qquad \begin{cases} x_0 \mapsto x_0 u^{-1}\,;\ x_n \mapsto u\,;\ x_i \mapsto x_i,\ i \neq 0, n & \text{when } n > 1\,, \\ x_0 \mapsto u^{-1}\,;\ x_1 \mapsto u \quad \text{when } n = 1\ (\mathbb{C}_0[\boldsymbol{x}] = \mathbb{C})\,. \end{cases}$$

It is known [A2] that the following diagram commutes :

$$(6.6) \qquad \begin{array}{ccc} \mathcal{L}_0^n & \xrightarrow{\ \psi_n\ } & \mathbb{C}_n[\boldsymbol{x}] \\ {\scriptstyle \Psi(-,u)}\downarrow & & \downarrow{\scriptstyle \eta(-,u)} \\ \mathcal{L}_0^{n-1}[u^{\pm 1}] & \xrightarrow{\ \psi_{n-1}\times 1_u\ } & \mathbb{C}_{n-1}[\boldsymbol{x}][u^{\pm 1}] \end{array}$$

where $\psi_{n-1} \times 1_u$ is the ring homomorphism that coincides with ψ_{n-1} on $\mathcal{L}_0^{n-1}$ and fixes u.

We state the following theorem concerning a commutation relation, called Zharkovskaya's relation, between Hecke operators and the Siegel operator acting on Siegel modular forms of half integral weight.

THEOREM 6.1. *Let $F \in \mathcal{M}_k^n(q,\chi)$ and $\widehat{X} \in \widehat{\mathcal{E}}_0^n(q)$, where k is a half integer. Then*

$$\Phi(F|_{k,\chi}\widehat{X}) = (\Phi F)|_{k,\chi}\Psi(Y, p^{n-k}\chi(p)^{-1})$$

where $Y = (\pi_k^n \circ \widehat{\beta}^n)(\widehat{X}) \in \mathcal{E}_0^n$. (If $n = 1$, then the action on the right hand side is nothing but multiplication of complex numbers.)

Proof. See [KKO].

The analogue of this formula for the integral weight Siegel modular forms was given by Andrianov [A2]. The following result is also given by Andrianov.

THEOREM 6.2. *$\Psi(-,u) : \mathbb{L}^n(T) \to \mathbb{L}^{n-1}(T)$ is a surjective ring homomorphism for any $u \in \mathbb{C}$, $u \neq 0$.*

Proof. See [A2].

For later use, we introduce a decomposition of $F \in \mathcal{M}_s^n$. Let

$$F(Z) = \sum_{N \in \mathcal{N}_n} f(N)e(NZ), \qquad Z \in \mathcal{H}_n\,.$$

We define the *r-component* $F_r(Z)$ of $F(Z)$ for $0 \le r \le n$ by

$$F_r(Z) = \sum_{\substack{N \in \mathcal{N}_n \\ \mathrm{rank}(N) = r}} f(N)e(NZ), \qquad Z \in \mathcal{H}_n\,, \tag{6.7}$$

so that $F(Z) = \sum_{r=0}^n F_r(Z)$. One can easily show that

$$(F|_{k,\chi}X)_r = F_r|_{k,\chi}X, \qquad X \in \mathcal{L}_0^n\,. \tag{6.8}$$

7. Theta-series of half integral weight. Let $Q \in \mathcal{N}_m^+$. The *level* q of Q is defined to be the smallest positive integer such that $q(2Q)^{-1}$ is integral with even diagonal entries. It is well known [Og] that q is divisible by 4 when m is odd. We define the *theta-series* of degree n associated with Q by

$$\theta^n(Z,Q) = \sum_{X \in M_{m,n}(\mathbb{Z})} e(Q[X]Z) = \sum_{N \in \mathcal{N}_n} r(N,Q)e(NZ), \qquad Z \in \mathcal{H}_n\,, \tag{7.1}$$

where $r(N,Q) = |\{X \in M_{m,n}(\mathbb{Z}) : Q[X] = N\}| < \infty$.

When m is even, the following is known [A-M]:

$$\theta^n(M\langle Z\rangle,Q) = \chi_Q(\det D_M)\det(C_MZ+D_M)^{m/2}\theta^n(Z,Q), \quad Z\in\mathcal{H}_n\,, \tag{7.2}$$

for $M\in\Gamma_0^n(q)$ where χ_Q is the Dirichlet character defined by

$$\chi_Q(d)=(d/|d|)^{m/2}\left(\frac{(-1)^{m/2}\det 2Q}{|d|}\right)_{\mathrm{Jac}} \quad \text{if } q>1 \tag{7.3}$$

and $\chi_Q(d)=1$ if $q=1$ for integers d relatively prime to q.

From (2.1), (7.1) and (7.3) it follows that

$$\theta^n(Z)^2=\theta^n(Z,I_2) \quad \text{and} \quad \chi_{I_2}(d)=\operatorname{sign}(d)\left(\frac{-4}{|d|}\right)_{\mathrm{Jac}}=\pm1\,.$$

So (2.2), (7.2) and (7.3) show that for any $M\in\Gamma_0^n(q)$

$$j(M,Z)^2=\chi_{I_2}(\det D_M)\det(C_MZ+D_M)\,. \tag{7.4}$$

We fix an odd m in what follows. Let $Q^\star=\operatorname{diag}(Q,I_3)\in\mathcal{N}_{m+3}^+$. Then the level $q^\star$ of $Q^\star$ is the same as the level q of Q. Since $m+3$ is even and

$$\theta^n(Z,Q^\star)=\frac{\theta^n(Z,Q)}{\theta^n(Z)}\theta^n(Z)^4\,,$$

by applying (7.2)–(7.4), we obtain

$$\theta^n(M\langle Z\rangle)=\chi^\star(\det D_M)\det(C_MZ+D_M)^{(m-1)/2}j(M,Z)\theta^n(Z,Q) \tag{7.5}$$

for any $M\in\Gamma_0^n(q)$ where $\chi^\star$ is the character of $Q^\star$ (see (7.3)). From (4.1) and (7.5) it follows that

$$\theta^n(Z,Q)|_k\widehat{M}=\chi_Q(\det D_M)\theta^n(Z,Q), \quad Z\in\mathcal{H}_n\,, \tag{7.6}$$

for any $\widehat{M}=(M,j(M,Z))\in\widehat{\Gamma}_0^n(q)$ where $k=m/2$ is a half integer and

$$\chi_Q(d)=\chi^\star(d)\chi_{I_2}(d)^{(1-m)/2}=\left(\frac{2\det 2Q}{|d|}\right)_{\mathrm{Jac}}. \tag{7.7}$$

So we have the following theorem:

Theorem 7.1. *Let $Q\in\mathcal{N}_m^+$, m odd. Then*

$$\theta^n(Z,Q)\in\mathcal{M}_k^n(q,\chi)\subset\mathcal{M}_0^n$$

where $k=m/2$ is a half integer, q is the level of Q, and $\chi=\chi_Q$ is the Dirichlet character (7.7).

Proof. Clear from the above and (4.2).

See [C-J],[A1] and [St] for the explicit formulas for $\det(C_MZ+D_M)^{-m/2}\times\theta^n(M\langle Z\rangle,Q)/\theta^n(Z,Q)$ and $j(M,Z)\det(C_MZ+D_M)^{-1/2}$, respectively, for

$M \in \Gamma_0^n(q)$, where m is odd and $\det(C_M Z + D_M)^{1/2}$ is under the usual branch.

8. Theta operators. Let m, n be positive integers. Let Θ_m^n be the vector space over $\mathbb{C}$ spanned by $\theta^n(Z,Q)$, $Q \in \mathcal{N}_m^+$, and let $\Theta_m^n(q,d)$ be its subspace spanned by $\theta^n(Z,Q)$, $Q \in \mathcal{N}_m^+$, with $d = \det 2Q$ and $q =$ the level of Q for given positive integers d and q. If m is odd, then Theorem 7.1 shows that

$$\Theta_m^n \subset \mathcal{M}_0^n \quad \text{and} \quad \Theta_m^n(q,d) \subset \mathcal{M}_k^n(q,\chi)$$

where

$$\chi(\det D_M) = \left(\frac{2d}{|\det D_M|}\right)_{\mathrm{Jac}} \quad \text{for any } M \in \Gamma_0^n(q)\,.$$

Observe that $\det D_M$ is relatively prime to q and hence to d because q and d have exactly the same prime factors [Og].

Let $Q \in \mathcal{N}_m^+$. We denote the genus of Q by $[Q]$, i.e., $[Q]$ is the set of all matrices in $\mathcal{N}_m^+$ that are locally equivalent to Q everywhere. In global notation, we may define $[Q]$ by the set of all $Q_1 \in \mathcal{N}_m^+$ such that $\det 2Q_1 = \det 2Q$ and $2Q_1 \equiv 2Q[U] \pmod{8(\det 2Q)^3}$ for some $U \in M_m(\mathbb{Z})$ (see [Si2]).

Let (Q) be the class of Q, i.e., the set of $Q_1 \in \mathcal{N}_m^+$ such that $2Q_1 = 2Q[U]$ for some $U \in GL_m(\mathbb{Z})$. Obviously $(Q) \subset [Q]$. It is well known that $[Q]$ contains a finite number of classes (see, for instance, [O]). Note that $\theta^n(Z,Q_1) = \theta^n(Z,Q)$ for any $Q_1 \in (Q)$. Also note that $\det 2Q$ and the level of Q are invariants of $[Q]$ and hence

$$\Theta_m^n[Q] \subset \Theta_m^n(q,d) \subset \Theta_m^n$$

if $q =$ the level of Q and $d = \det 2Q$, where $\Theta_m^n[Q]$ is the subspace of Θ_m^n spanned by $\theta^n(Z,Q_i)$, $Q_i \in [Q]$.

It is well known [Si1] that

$$\Phi(\theta^n(Z,Q)) = \theta^{n-1}(Z',Q)$$

where Φ is the Siegel operator (6.1) and $Z = \begin{pmatrix} Z' & * \\ * & * \end{pmatrix} \in \mathcal{H}_n$, $Z' \in \mathcal{H}_{n-1}$. In particular, $\Phi : \Theta_m^n[Q] \to \Theta_m^{n-1}[Q]$, $\Phi : \Theta_m^n(q,d) \to \Theta_m^{n-1}(q,d)$ are epimorphisms for all $n \geq 1$ and isomorphisms [F] if $n > m$.

We now introduce theta operators. Let $m, n \geq 1$ and let p be a prime relatively prime to q. Let $\alpha : L_0^m \to \mathbb{C}^\times$ be a character such that $\alpha(\Gamma_0^m) = 1$. For $X = (\Gamma_0^m g_0 \Gamma_0^m) \in \mathcal{L}_0^m$ with $g_0 = \begin{pmatrix} p^\delta D_0^* & B_0 \\ 0 & D_0 \end{pmatrix} \in L_0^m$ and $\theta^n(Z,Q) \in \Theta_m^n$ with $Q \in \mathcal{N}_m^+$, we set

$$\theta^n(Z,Q) \circ_\alpha X = \alpha(g_0) \sum_{\substack{D \in \Lambda D_0 \Lambda/\Lambda \\ p^\delta Q[D^*] \in \mathcal{N}_m^+}} l_X(Q,D)\theta^n(Z, p^\delta Q[D^*]) \tag{8.1}$$

where $\Lambda = \Lambda^m$ and

$$l_X(Q, D) = \sum_{B \in B_X(D)/ \bmod D} e(QBD^{-1}) \,. \tag{8.2}$$

Here

$$B_X(D) = \left\{ B \in M_m(\mathbb{Z}[p^{-1}]) : \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \in \Gamma_0^m g_0 \Gamma_0^m \right\}$$

and $B_1, B_2 \in B_X(D)$ are said to be *congruent modulo D on the right* if $(B_1 - B_2)D^{-1} \in M_m(\mathbb{Z})$. This congruence is obviously an equivalence relation and the summation in (8.2) is over equivalence classes in $B_X(D)$ modulo D on the right. We extend (8.1) by linearity to the whole space Θ_m^n and the whole ring $\mathcal{L}_0^m$. Elements of $\mathcal{L}_0^m$ in this action are called *theta operators*.

We set

$$\mathcal{L}_{00}^m = \left\{ \sum a_i (\Gamma_0^m g_i \Gamma_0^m) \in \mathcal{L}_0^m : \delta_i m - 2b_i = 0, b_i = \log_p |\det D_i| \right\}$$

where $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix} \in L_0^m$ and let $\mathcal{E}_{00}^m = \mathcal{E}_0^m \cap \mathcal{L}_{00}^m$.

We prove the following theorem:

THEOREM 8.1. (1) *The action* (8.1) *is a well-defined action of* $\mathcal{L}_0^m$ *on* Θ_m^n.

(2) $\Theta_m^n(q, d)$ *is invariant under the theta operators of* $\mathcal{L}_{00}^m$ *if* p *is relatively prime to* q.

(3) $\Theta_m^n[Q]$ *is invariant under the theta operators of* $\mathcal{E}_{00}^m$ *if* p *is relatively prime to* $2q$, *where* q *is the level of* Q.

Proof. This theorem is proved for the case of m even by Andrianov [A2]. So, we restrict ourselves to the case of m odd. Let

$$\varepsilon(Z, Q) = \sum_{U \in \Omega} e(Q[U]Z), \qquad Z \in \mathcal{H}_m \,, \tag{8.3}$$

where $\Omega = GL_m(\mathbb{Z})$. $\varepsilon(Z, Q)$ is called the *epsilon-series* of Q. For every $M = \begin{pmatrix} D^* & B \\ 0 & D \end{pmatrix} \in \Gamma_0^m$ with $D \in \Omega$, we have

$$\varepsilon(M\langle Z\rangle, Q) = \sum_{U \in \Omega} e(Q[UD^*]Z) e(Q[U]BD^{-1}) = \varepsilon(Z, Q) \,. \tag{8.4}$$

Note that $e(Q[U]BD^{-1}) = 1$ because $Q[U] \in \mathcal{N}_m^+$ and BD^{-1} is integral symmetric [M]. From (8.4) and the definition of even modular forms it follows that $\varepsilon(Z, Q) \in \mathcal{M}_0^m$. Let

$$\mathcal{A}_m = \left\{ \sum a_i \varepsilon(Z, Q_i) : Q_i \in \mathcal{N}_m^+ \right\} \subset \mathcal{M}_0^m \,.$$

Let $k = m/2$ and χ be a character satisfying $\chi(-1) = 1$. Let $X = (\Gamma_0^m g_0 \Gamma_0^m) \in \mathcal{L}_0^m$ with $g_0 = \begin{pmatrix} p^\delta D_0^* & B_0 \\ 0 & D_0 \end{pmatrix} \in L_0^m$. Then

$$X = \sum_{\substack{D\in\Omega\backslash\Omega D_0\Omega \\ B\in B_X(D)/\bmod D}} (\Gamma_0^m g)$$

where $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix}$. By (4.1) and (4.5)

$$\varepsilon(Z,Q) = \sum_{U\in\Omega} e(Q[U^*]Z) = \sum_{U\in\Omega} e(QZ)|_k \widetilde{M}_U \tag{8.5}$$

where $M_U = \begin{pmatrix} U^* & 0 \\ 0 & U \end{pmatrix} \in \Gamma_0^m$ and $\widetilde{M}_U = (M_U, 1)$. Hence

$$\varepsilon(Z,Q)|_{k,\chi}X = \sum_{\substack{D\in\Omega\backslash\Omega D_0\Omega \\ B\in B_X(D)/\bmod D}} \sum_{U\in\Omega} \chi(\det p^\delta D^*) e(QZ)|_k \widetilde{M}_U|_k \widetilde{g}$$

where $\widetilde{g} = (g, p^{-\delta m/4}|\det D|^{1/2})$ (see (4.5)). Since

$$M_U g = \begin{pmatrix} p^\delta (UD)^* & U^*B \\ 0 & UD \end{pmatrix} \text{ and } U^*B_X(D)/\bmod D = B_X(UD)/\bmod UD$$

for any $U \in \Omega$, we have

$$\varepsilon(Z,Q)|_{k,\chi}X = \sum_{\substack{D\in\Omega D_0\Omega \\ B\in B_X(D)/\bmod D}} \chi(\det p^\delta D^*) e(QZ)\,|_k\, \widetilde{g}\,.$$

We may rewrite this as

$$\varepsilon(Z,Q)|_{k,\chi}X = \sum_{U\in\Omega} \sum_{\substack{D\in\Omega D_0\Omega/\Omega \\ B\in B_X(D)/\bmod D}} \chi(\det p^\delta D^*) e(QZ)|_k \widetilde{g}|_k \widetilde{M}_U\,. \tag{8.6}$$

Now let

$$\iota(Z,Q) = \sum_{B\in B_X(D)/\bmod D} \chi(\det p^\delta D^*) e(QZ)|_k \widetilde{g}\,. \tag{8.7}$$

Then from (4.1) and (4.5) it follows that

$$\iota(Z,Q) = \alpha_{k,\chi}(g_0) e(p^\delta Q[D^*]Z) \sum_{B\in B_X(D)/\bmod D} e(QBD^{-1}) \tag{8.8}$$

where $\alpha_{k,\chi} : L_0^m \to \mathbb{C}^\times$ is the character defined by

$$\alpha_{k,\chi}(g) = \chi(p^{\delta m - b}) p^{\delta(mk - <m>) - bk} \tag{8.9}$$

for any $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \in L_0^m$, where $b = \log_p |\det D|$. If we take $B + AD$ instead of B as a representative of $B_X(D)/\bmod D$ where ${}^tA = A \in M_m(\mathbb{Z})$, then

$$e(Q(B + AD)D^{-1}) = e(QBD^{-1})e(QA) = e(QBD^{-1})\,.$$

So (8.8) is independent of the choice of representatives B of $B_X(D)/\bmod D$. Let $K_S = \begin{pmatrix} I_m & S \\ 0 & I_m \end{pmatrix} \in \Gamma_0^m$ with ${}^tS = S \in M_m(\mathbb{Z})$. Then $\widetilde{K}_S = (K_S, 1)$ and $gK_S = \begin{pmatrix} p^\delta D^* & p^\delta D^* S + B \\ 0 & D \end{pmatrix}$ so that $\{B + p^\delta D^* S\}$ is a complete set of representatives of $B_X(D)/\bmod D$ if $\{B\}$ is. Therefore, $\iota(Z,Q)|_k \widetilde{K}_S = \iota(Z,Q)$ by (8.7). Applying $|_k \widetilde{K}_S$ on the right hand side of (8.8), we obtain

$$\iota(Z,Q) = \alpha_{k,\chi}(g_0) e(p^\delta Q[D^*]Z) e(p^\delta Q[D^*]S) l_X(Q,D)\,. \tag{8.10}$$

So, if $l_X(Q,D) \neq 0$, then $e(p^\delta Q[D^*]S) = 1$ for any ${}^tS = S \in M_m(\mathbb{Z})$. This clearly implies that $p^\delta Q[D^*] \in \mathcal{N}_m^+$. In other words, if $p^\delta Q[D^*] \notin \mathcal{N}_m^+$, then $l_X(Q,D) = 0$. From this and (8.5), (8.6), (8.10) it follows that

$$\varepsilon(Z,Q)|_{k,\chi} X = \alpha_{k,\chi}(g_0) \sum_{\substack{D \in \Omega D_0 \Omega/\Omega \\ p^\delta Q[D^*] \in \mathcal{N}_m^+}} l_X(Q,D) \varepsilon(Z, p^\delta Q[D^*]) \in \mathcal{A}_m\,.$$

Choosing a complete set of representatives $\{D_i\}$ of $\Omega D_0 \Omega/\Omega$ such that $\det D_i = \det D_0$, we may rewrite the above as follows:

$$\varepsilon(Z,Q)|_{k,\chi} X = \alpha_{k,\chi}(g_0) \sum_{\substack{D \in \Lambda D_0 \Lambda/\Lambda \\ p^\delta Q[D^*] \in \mathcal{N}_m^+}} l_X(Q,D) \varepsilon(Z, p^\delta Q[D^*])\,. \tag{8.11}$$

We now define a linear map $\vartheta_m^n : \mathcal{A}_m \to \Theta_m^n$ by

$$\vartheta_m^n(\varepsilon(Z,Q)) = \theta^n(Z,Q), \qquad Q \in \mathcal{N}_m^+\,.$$

Obviously ϑ_m^n is a well-defined epimorphism. (8.1) and (8.11) yield

$$\vartheta_m^n(\varepsilon(Z,Q)\,|_{k,\chi} X) = \theta^n(Z,Q) \circ_\alpha X, \qquad X \in \mathcal{L}_0^m\,, \tag{8.12}$$

where $\alpha = \alpha_{k,\chi}$ is the character (8.9). Observe that

$$\varepsilon(Z,Q)|_{k,\chi} X_1\,|_{k,\chi} X_2 = \varepsilon(Z,Q)|_{k,\chi} X_1 X_2$$

implies

$$\theta^n(Z,Q) \circ_\alpha X_1 \circ_\alpha X_2 = \theta^n(Z,Q) \circ_\alpha X_1 X_2\,.$$

From the surjectivity of ϑ_m^n, (8.11) and the above, (1) follows.

Let p be relatively prime to q. To prove (2), it is enough to show that $\det 2Q_1 = d$ and the level of $Q_1 = q$ if $Q_1 = p^\delta Q[D^*] \in \mathcal{N}_m^+$, where $d = \det 2Q$ and $q =$ the level of Q. Clearly $\det 2Q_1 = d$. Let q_1 be the level of

Q_1. Then $q(2Q_1)^{-1}p^{\delta_1} = qp^{\delta_1-\delta}(2Q)^{-1}[D]$ is integral for some $\delta_1 \geq 0$. So $q_1 \mid qp^{\delta_1}$, which implies $q_1 \mid q$. Similarly $q \mid q_1$. This proves (2).

For (3), let δ be even. Since we restricted ourselves to the case of m odd, the level q of Q is divisible by 4. So we may replace $2q$ by q in this case. Let $D_1 = p^{\delta/2}D^*$ so that $\det D_1 = \pm 1$. Since q and $d = \det 2Q$ have the same prime factors, p is relatively prime to d and hence one can find $U \in M_m(\mathbb{Z})$ such that $U \equiv D_1 \pmod{8d^3}$. Since $2Q_1 = 2Q[D_1]$, we have $2Q_1 \equiv 2Q[U] \pmod{8d^3}$. Therefore $Q_1 \in [Q]$ if $Q_1 = p^\delta Q[D^*] \in \mathcal{N}_m^+$ and this proves (3).

9. Action of $\widehat{\mathcal{L}}_0^n(T)$ on $\Theta_m^n[Q]$. Let $Q \in \mathcal{N}_m^+$ with m odd. We set $\Psi = \Psi_Q : \widehat{\mathcal{L}}_0^n(T) \to \widehat{\mathcal{L}}_0^{n-1}(T)$ by requiring the following diagram to commute:

$$\begin{array}{ccc} \widehat{\mathcal{L}}_0^n(T) & \xrightarrow[\pi_k^n \circ \hat{\beta}^n]{\sim} & \mathbb{L}_0^n(T) \\ {\scriptstyle \Psi=\Psi_Q}\big\downarrow & & \big\downarrow {\scriptstyle \Psi(-,p^{n-k}\chi_Q^{-1}(p))} \\ \widehat{\mathcal{L}}_0^{n-1}(T) & \xrightarrow[\pi_k^{n-1} \circ \hat{\beta}^{n-1}]{\sim} & \mathbb{L}_0^{n-1}(T) \end{array} \tag{9.1}$$

where $k = m/2$ and χ_Q is the character (7.7). Since the right vertical arrow is surjective by Theorem 6.2, Ψ is also surjective. We let Ψ^r be the rth iteration of Ψ for $r > 0$ and $\Psi^0 =$ the identity map. For $\widehat{X} \in \widehat{\mathcal{L}}_0^{n-r}(T)$, $0 \leq r \leq n$, let $\Psi^{-r}(\widehat{X})$ denote any element in $\widehat{\mathcal{L}}_0^n(T)$ whose image under Ψ^r is $\widehat{X}$.

Let $X = (\Gamma_0^m g \Gamma_0^m) \in \mathcal{L}_0^m$ for $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \in L_0^m$. We define the *signature* $s(X)$ of X by $s(X) = 2b - m\delta$ where $b = \log_p |\det D|$. A linear combination of double cosets with the same signature $s \in \mathbb{Z}$ in $\mathcal{L}_0^m$ is said to be *s-homogeneous of signature s*. For general $X = \sum_i a_i(\Gamma_0^m g_i) \in \mathcal{L}_0^m$ with $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix}$ and $b_i = \log_p |\det D_i|$, we denote the s-homogeneous part of signature s in X by $X_{(s)}$, i.e.,

$$X_{(s)} = \sum_{i,2b_i - m\delta_i = s} a_i(\Gamma_0^m g_i)\,.$$

Let $\widehat{X} \in \widehat{\mathcal{L}}_0^m(T)$ and $Y = (\pi_k^m \circ \widehat{\beta}^m)(\widehat{X}) \in \mathbb{L}_0^m(T)$. We define a homomorphism $\xi^m = \xi_Q^m : \widehat{\mathcal{L}}_0^m(T) \to \mathcal{L}_0^m$ by

$$\xi^m(\widehat{X}) = \sum_{s \geq 0} (\chi_Q(p)p^{m/2})^s Y_{(-2s)} X_m^{+s} \tag{9.2}$$

(see (3.5) for X_m^{+s}). Observe that $\xi^m(\widehat{X}) \in \mathcal{E}_{00}^m$ for any $\widehat{X} \in \widehat{\mathcal{L}}_0^m(T)$.

We now prove the following theorem. For m even, it is also due to Andrianov [A2].

THEOREM 9.1. *Let $m, n \geq 1$ be integers, m odd, $m \geq n$. Let $Q \in \mathcal{N}_m^+$ with level q, $4 \mid q$. Let p be a prime relatively prime to q. Then for $\widehat{X} \in \widehat{\mathcal{L}}_0^n(T)$, we have*

$$\theta^n(Z,Q)|_{k,\chi}\widehat{X} = \theta^n(Z,Q) \circ_\alpha \xi^m(\Psi^{n-m}(\widehat{X})) \tag{9.3}$$

where $k = m/2$, $\chi = \chi_Q$, and $\alpha = \alpha_{k,\chi}$ (see (7.7) and (8.9)).

P r o o f. Assume for a moment that (9.3) holds when $n = m$, i.e.,

$$\theta^m(Z,Q)|_{k,\chi}\widehat{X} = \theta^m(Z,Q) \circ_\alpha \xi^m(\widehat{X}) \,. \tag{9.4}$$

When $r = m - n > 0$, we apply Φ^r (the rth iteration of the Siegel operator Φ) to (9.4). Then from Theorem 6.1, (8.1) and (9.1) it follows that

$$\Phi^r(\theta^m(Z,Q)|_{k,\chi}\widehat{X}) = \Phi^r(\theta^m(Z,Q))\,|_{k,\chi}\Psi^r(\widehat{X}) = \theta^n(Z,Q)\,|_{k,\chi}\Psi^r(\widehat{X})$$

and

$$\Phi^r(\theta^m(Z,Q) \circ_\alpha \xi^m(\widehat{X})) = \theta^n(Z,Q) \circ_\alpha \xi^m(\widehat{X}) \,.$$

Therefore, it suffices to show (9.4). We now let $G(Z)$ be the m-component of $\theta^m(Z,Q)$ (see (6.7)). From the definition of $\varepsilon(Z,Q)$ and the left coset decomposition of

$$\{D \in M_m(\mathbb{Z}) : \det D \neq 0\} = \bigcup_{D \in \Lambda \backslash M_m^+(\mathbb{Z})} \Omega D \,,$$

it follows that

$$\begin{aligned} G(Z) &= \sum_{\substack{D \in M_m(\mathbb{Z}) \\ \det D \neq 0}} e(Q[D]Z) = \sum_{D \in \Lambda \backslash M_m^+(\mathbb{Z})} \sum_{U \in \Omega} e(Q[UD]Z) \\ &= \sum_{D \in \Lambda \backslash M_m^+(\mathbb{Z})} \varepsilon(Z[{}^tD], Q) \end{aligned}$$

where $\Lambda = \Lambda^m$, $\Omega = \Omega^m$, and $M_m^+(\mathbb{Z}) = \{D \in M_m(\mathbb{Z}) : \det D > 0\}$.

Let

$$W(a) = \sum_{\substack{D \in \Lambda \backslash M_m^+(\mathbb{Z})/\Lambda \\ \det D = a}} (\Lambda D \Lambda) = \sum_{\substack{D \in \Lambda \backslash M_m^+(\mathbb{Z}) \\ \det D = a}} (\Lambda D) \in \mathcal{D}_{\mathbb{Z}}^m$$

where a is a positive integer (see (1.5) for $\mathcal{D}_{\mathbb{Z}}^m$). Then from (4.8), we have

$$G(Z) = \sum_{a=1}^{\infty} \varepsilon(Z,Q) \mid W(a) = \sum_{d=0}^{\infty} \sum_{\substack{a>0 \\ (a,p)=1}} \varepsilon(Z,Q)|W(p^d)|W(a) \tag{9.5}$$

for any fixed prime p. The second equality follows from the commutativity $W(a)W(b) = W(b)W(a)$ for $(a,b) = 1$ (see [Zh2]). We let p be the given prime and let $Y = (\pi_k^m \circ \widehat{\beta}^m)(\widehat{X}) \in \mathbb{L}_0^m(T)$, $\widehat{X} \in \widehat{\mathcal{L}}_0^m(T)$. Then

$$G(Z)|_{k,\chi}\widehat{X} = G(Z)|_{k,\chi}Y = \sum_{d=0}^{\infty} \sum_{\substack{a>0 \\ (a,p)=1}} \varepsilon(Z,Q)|\, W(p^d)|W(a)|_{k,\chi}Y$$
$$= \sum_{d=0}^{\infty} \sum_{\substack{a>0 \\ (a,p)=1}} \varepsilon(Z,Q)|W(p^d)|_{k,\chi}Y|W(a)$$

(see [A2] for the last equality). From (4.9) and (3.5) it follows that

$$\varepsilon(Z,Q)|W(p^d) = \sum_{\substack{D\in\Lambda\backslash M_m^+(\mathbb{Z})/\Lambda \\ \det D = p^d}} \varepsilon(Z,Q)|(\Lambda D\Lambda) = \tau^d \varepsilon(Z,Q)|_{k,\chi}X_m^{-d}\,, \tag{9.6}$$

where $\tau = \chi(p)p^k$. Therefore

$$G(Z)|_{k,\chi}\widehat{X} = \sum_{\substack{a>0 \\ (a,p)=1}} \Big(\sum_{d=0}^{\infty} \tau^d \varepsilon(Z,Q)|_{k,\chi}X_m^{-d}Y\Big)\Big|\, W(a)\,. \tag{9.7}$$

We now consider $F(y) = \sum_{i=0}^{\infty} F_i y^i \in \mathcal{M}_0^m[[y]]$. If $F(y)$ is defined at $\tau \in \mathbb{C}$, then clearly $F(\tau) = \sum_{i=0}^{\infty} F_i \tau^i \in \mathcal{M}_0^m$. Let

$$Y(y) = X_m^-(y)YX_-^m(y) \in \mathcal{L}_0^m[[y]]\,.$$

Then by Proposition 3.1,

$$Y(y) = B^m(\kappa,y)X_+^m(y)R^m(y)^{-1}YR^m(y)X_m^+(y)B^m(\kappa,y)^{-1}\,.$$

Since $Y \in \mathbb{L}_0^m(T)$, $R^m(y) \in \mathbb{L}_0^m(T)[y]$, and $\mathbb{L}_0^m(T)$ is commutative, we have $R^m(y)^{-1}YR^m(y) = Y$ and hence

$$Y(y) = B^m(\kappa,y)X_+^m(y)YX_m^+(y)B^m(\kappa,y)^{-1}\,. \tag{9.8}$$

We now assume for a moment the following holds:

$$\varepsilon(Z,Q)|_{k,\chi}Y(y) = \Big(\sum_{i=0}^{\infty} \varepsilon(Z,Q)|_{k,\chi}Y_{(-2i)}X_m^{+i}y^i\Big) + \varepsilon_1(y) \tag{9.9}$$

where $\varepsilon_1(y) \in \mathcal{A}_m[y] \subset \mathcal{M}_0^m[y]$, which vanishes at $y = \tau = \chi(p)p^k$. Since

$$X_m^-(y)Y = Y(y)X_m^-(y)\,,$$

from (9.9) it follows that

$$\begin{aligned} &\varepsilon(Z,Q)|_{k,\chi}X_m^-(y)Y \\ &\quad = \Big(\Big(\sum_{i=0}^{\infty} \varepsilon(Z,Q)|_{k,\chi}Y_{(-2i)}X_m^{+i}y^i\Big) + \varepsilon_1(y)\Big)\Big|_{k,\chi}X_m^-(y)\,. \end{aligned} \tag{9.10}$$

Evaluate both sides of (9.10) at $y = \tau = \chi(p)p^k$. Then (9.2) gives

$$\sum_{d=0}^{\infty} \tau^d \varepsilon(Z,Q)|_{k,\chi} X_m^{-d} Y = \sum_{d=0}^{\infty} \tau^d \varepsilon(Z,Q)|_{k,\chi} \xi^m(\widehat{X})|_{k,\chi} X_m^{-d}\,.$$

So, by (9.5)–(9.7),

$$G(Z)|_{k,\chi}\widehat{X} = \sum_{a=1}^{\infty} (\varepsilon(Z,Q)|_{k,\chi}\xi^m(\widehat{X}))|W(a)$$

and hence the m-component of $\theta^m(Z,Q)|_{k,\chi}\widehat{X}$ and that of $\theta^m(Z,Q) \circ_\alpha \xi^m(\widehat{X})$ coincide. Therefore,

$$\theta^m(Z,Q)|_{k,\chi}\widehat{X} - \theta^m(Z,Q) \circ_\alpha \xi^m(\widehat{X}) \in \mathcal{M}_k^m(q,\chi)$$

such that its m-component is 0. Such a form is called a *singular form* and it is well known [F] that there are no non-zero singular forms if $2k \geq m$. So the theorem follows.

It only remains to prove (9.9). Let $B^m(\kappa, y, N)$ be the polynomial in (5.3). From (5.2), (5.4), and the definition of $\varepsilon(Z,Q)$ it follows that $\varepsilon(Z,Q)|_{k,\chi}B^m(\kappa,y) = B^m(\kappa,y,Q)\varepsilon(Z,Q)$. So,

$$\text{(9.11)} \qquad \varepsilon(Z,Q)|_{k,\chi}B^m(\kappa,y)^{-1} = B^m(\kappa,y,Q)^{-1}\varepsilon(Z,Q)\,.$$

Note that $\varepsilon(Z,Q)|_{k,\chi}X = 0$ if the signature $s(X)$ of X is positive for $X \in \mathcal{E}_0^m$ (see (8.11)). But $s(X_{+i}^m) = 2i > 0$ if $i > 0$ and hence

$$\text{(9.12)} \qquad \varepsilon(Z,Q)|_{k,\chi}X_+^m(y) = \varepsilon(Z,Q)|_{k,\chi}X_{+0}^m = \varepsilon(Z,Q)\,.$$

We may write $YX_m^+(y) = \sum_{d=0}^{\infty}\sum_i Y_{(i)}X_m^{+d}y^d$. Since $s(Y_{(i)}X_m^{+d}) = i+2d$, we set

$$\varepsilon(Z,Q)|_{k,\chi}YX_m^+(y) = \sum_{i+2d\leq 0} \varepsilon(Z,Q)|_{k,\chi}Y_{(i)}X_m^{+d}y^d\,.$$

So, (9.8) and (9.12) imply

$$\begin{aligned} &\varepsilon(Z,Q)|_{k,\chi}Y(y) \\ &\qquad = B^m(\kappa,y,Q)\Big(\sum_{i+2d\leq 0}\varepsilon(Z,Q)|_{k,\chi}Y_{(i)}X_+(d)y^d\Big)\Big|_{k,\chi}B^m(\kappa,y)^{-1}\,. \end{aligned}$$

The expression in parenthesis on the right hand side can be written as

$$\sum_{\substack{i,d,j\\ i+2d\leq 0}} a_{i,d,j}\varepsilon(Z,Q_{i,d,j})y^d$$

where $Q_{i,d,j} \in \mathcal{N}_m^+$ and $a_{i,d,j} \in \mathbb{C}$ (see (8.11)). So, from (9.11) it follows that

$$\varepsilon(Z,Q)|_{k,\chi}Y(y) = B^m(\kappa,y,Q)\sum_{\substack{i,d,j\\ i+2d\leq 0}} a_{i,d,j}B^m(\kappa,y,Q_{i,d,j})^{-1}\varepsilon(Z,Q_{i,d,j})y^d\,.$$

By (5.4), $B^m(\kappa, y, Q) = B^m(\kappa, y, Q_{i,d,j})$ if $i + 2d = 0$. Hence,

$$\varepsilon(Z, Q)|_{k,\chi} Y(y) = \sum_{\substack{i,d,j \\ i+2d=0}} a_{i,d,j} \varepsilon(Z, Q_{i,d,j}) y^d + \varepsilon_1(y)$$

where

$$\varepsilon_1(y) = B^m(\kappa, y, Q) \sum_{\substack{i,d,j \\ i+2d<0}} a_{i,d,j} B^m(\kappa, y, Q_{i,d,j})^{-1} \varepsilon(Z, Q_{i,d,j}) y^d .$$

One can easily check that $Q_{i,d,j}$ is degenerate modulo p if $i + 2d < 0$. So, from (5.5)–(5.7) it follows that $(1 - \chi^m_{k,Q}(p) p^{-m/2} y)$ divides $\varepsilon_1(y)$ and that $\varepsilon_1(y)$ vanishes at $y = \tau = p^{m/2} \chi^m_{k,Q}(p)$, where

$$\chi^m_{k,Q}(p) = \left(\frac{(-1)^{(2k-m)/2} 2 \det(2Q)}{p} \right),$$

which coincides with $\chi(p) = \chi_Q(p)$ of (7.7) because $k = m/2$. This proves (9.9) and hence the proof is complete.

Theorems 8.1 and 9.1 say that $\theta^n(Z, Q)$, $Q \in \mathcal{N}_m^+$, acted on by a Hecke operator $\widehat{X} \in \widehat{\mathcal{L}}_0^n(T)$, can be written as a linear combination of $\theta^n(Z, Q_i)$, $Q_i \in [Q]$.

10. Generic theta-series. Let $Q \in \mathcal{N}_m^+$. Let $Q_1, \ldots, Q_h$ be the full set of representatives of the classes in the genus $[Q]$ of Q. We define the generic theta-series of degree n associated with $[Q]$ by

$$\theta^n(Z, [Q]) = \left(\sum_{i=1}^h \frac{\theta^n(Z, Q_i)}{e_i} \right) \left(\sum_{i=1}^h \frac{1}{e_i} \right)^{-1}, \quad Z \in \mathcal{H}_n \tag{10.1}$$

where e_i is the order of the orthogonal group $O(Q_i)$.

THEOREM 10.1. *Let $m \geq n \geq 1$ be integers with m odd. Let $Q \in \mathcal{N}_m^+$. Let q and $\chi = \chi_Q$ be the level and the character of Q, respectively. Let p be a prime relatively prime to q. Then for any $\widehat{X} \in \widehat{\mathcal{L}}_0^n(T)$,*

$$\theta^n(Z, [Q])|_{k,\chi} \widehat{X} = \lambda(\widehat{X}, \chi) \theta^n(Z, [Q]) \tag{10.2}$$

where $k = m/2$ and the eigenvalue $\lambda(\widehat{X}, \chi)$ is determined as follows: Let $f(x_0, x_1, \ldots, x_n) = (\psi_n \circ \pi_k^n \circ \widehat{\beta}^n)(\widehat{X}) \in W_n[\boldsymbol{x}]$. Then

$$\lambda(\widehat{X}, \chi) = f(p^{nk-<n>} \chi(p)^n, p^{1-k} \chi(p)^{-1}, \ldots, p^{n-k} \chi(p)^{-1}) . \tag{10.3}$$

Proof. According to Theorem 9.1, it suffices to show that $\theta^n(Z, [Q])$ is an eigenform of any theta operator $X \in \mathcal{E}_{00}^m$. Then by (8.12), this is equivalent to showing that $\varepsilon(Z,[Q])$ is an eigenform of any Hecke operator $X \in \mathcal{E}_{00}^m$, where

$$\varepsilon(Z,[Q]) = \Big(\sum_{i=1}^{h} \frac{\varepsilon(Z,Q_i)}{e_i}\Big)\Big(\sum_{i=1}^{h} \frac{1}{e_i}\Big)^{-1}. \tag{10.4}$$

By the definition of $\varepsilon(Z,Q)$,

$$\varepsilon(Z,[Q]) = \mu^{-1} \sum_{N\in[Q]} e(NZ) \tag{10.5}$$

where $\mu = \sum_{i=1}^h 1/e_i$, the mass of $[Q]$. Let

$$X = (\Gamma_0^m g_0 \Gamma_0^m) \in \mathcal{E}_{00}^m, \qquad g_0 = \begin{pmatrix} p^\delta D_0^* & B_0 \\ 0 & D_0 \end{pmatrix} \in E_0^m.$$

Then

$$X = \sum_{\substack{D\in\Omega\backslash\Omega D_0\Omega \\ B\in B_X(D)/\bmod D}} \left(\Gamma_0^m \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix}\right)$$

and hence (4.4) and (4.5) imply

$$\varepsilon(Z,[Q])|_{k,\chi} X = \sum_{D,B} \chi(\det p^\delta D^*) \varepsilon(Z,[Q])|_k \widetilde{g} \tag{10.6}$$

where $\widetilde{g} = (g, p^{-\delta m/4}|\det D|^{1/2})$, $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix}$, and the summation is over $D \in \Omega\backslash\Omega D_0 \Omega$, $B \in B_X(D)/\bmod D$. So by (4.1), (8.3), (8.4), (8.9), and (8.11),

$$\begin{aligned}
&\varepsilon(Z,[Q])|_{k,\chi}X \\
&\quad = \sum_{D,B} \chi(\det p^\delta D^*)(p^\delta)^{mk/2-<m>}(p^{-\delta m/4}|\det D|^{1/2})^{-2k} \varepsilon(g\langle Z\rangle,[Q]) \\
&\quad = \mu^{-1}\chi(p^{\delta k})p^{-2\delta<k>} \sum_{\substack{Q_0\in[Q] \\ D,B}} e(Q_0(p^\delta Z[D^{-1}] + BD^{-1})) \\
&\quad = \mu^{-1}\chi(p^{\delta k})p^{-2\delta<k>} \sum_{\substack{Q_0,D \\ p^\delta Q_0[D^*]\in\mathcal{N}_m^+}} l_X(Q_0,D) e(p^\delta Q_0[D^*]Z).
\end{aligned}$$

According to Theorem 8.1, $p^\delta Q_0[D^*] \in [Q]$. So we have

$$\varepsilon(Z,[Q])|_{k,\chi}X = \mu^{-1}\chi(p^{\delta k})p^{-2\delta<k>} \sum_{Q_1\in[Q]} \Big(\sum_{D} l_X(p^\delta Q_1[{}^tD],D)\Big) e(Q_1 Z).$$

But it is easy to check that $\sum_D l_X(p^\delta Q_1[{}^tD],D)$ is independent of $Q_1 \in [Q]$. This proves that $\theta^n(Z,[Q])$ is an eigenform of any Hecke operator

$\widehat{X} \in \widehat{\mathcal{L}}_0^n(T)$. To prove (10.3), we apply Φ^n to (10.2) so that

$$\Phi^n(\theta^n(Z,[Q]))|_{k,\chi}\Psi^n(\widehat{X}) = \lambda(\widehat{X},\chi)\Phi^n(\theta^n(Z,[Q]))\,.$$

But $\Phi^n(\theta^n(Z,[Q])) = 1$ since Q is positive definite. Therefore, we have $\lambda(\widehat{X},\chi) = \Psi^n(\widehat{X})$ and (10.3) follows immediately from the diagram (9.1).

Schulze-Pillot [Sc] proved that

$$\theta^1(z,[Q])|_{k,\chi}T(p^2) = \lambda_p(Q)\theta^1(z,[Q]), \quad z \in \mathcal{H}_1\,, \tag{10.7}$$

where

$$\lambda_p(Q) = \begin{cases} p^{2k-2} + \chi(p)p^{k-1} + 1 & \text{if } k \text{ is an integer}\,, \\ p^{2k-2} + 1 & \text{if } k \text{ is a half integer}\,. \end{cases} \tag{10.8}$$

His $T(p^2)$ is equal to $T_0^1 + T_1^1$ if k is an integer and $\widehat{T}_0^1$ if k is an half integer. It is easy to check that

$$(\psi_1 \circ \beta^1)(T_0^1 + T_1^1) = x_0^2(1 + x_1 + x_1^2) \quad \text{and} \quad (\psi_1 \circ \pi_k^1 \circ \widehat{\beta}^1)(\widehat{T}_0^1) = x_0^2(1 + x_1^2)\,.$$

Evaluating these polynomials at $x_0 = p^{k-1}\chi(p)$, $x_1 = p^{1-k}\chi(p)^{-1}$, we obtain (10.8).

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