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Common summands in partitions

by

J. H. Loxton* and Hang-Fai Yeung (Sydney, N.S.W.)

1. Introduction. Erdős and Turán have made a number of forays in statistical group theory, investigating in particular the arithmetical structure of the symmetric group S_n of order n. (See [2] and [3].) They posed many problems, as usual, one of these being the following question of Turán: Is it true that for almost all pairs of conjugacy classes of permutations of S_n, the cycle representations of the permutations in these classes contain cycles of the same length? From the one-to-one correspondence between conjugacy classes of S_n and ordinary partitions of n, this amounts to the question whether or not almost all pairs of ordinary partitions of n contain common summands.

Let $p(n)$ denote the number of ordinary partitions of n and let Π be a generic partition of n. For a partition Π, the set of its summands (with multiplicity) will be denoted by $\overline{\Pi}$ and the cardinality of $\overline{\Pi}$ by $|\overline{\Pi}|$. Turán [7] obtained the following result:

Theorem 1. *Let $\varepsilon > 0$ be an arbitrarily small real number and $k \geq 2$ be a fixed integer. Suppose $n \leq n_1 \leq n_2 \leq \ldots \leq n_k \leq n(1+o(1))$ with $n \to \infty$. For sufficiently large n, the inequality*

$$|\overline{\Pi}_1 \cap \ldots \cap \overline{\Pi}_k| \geq \left(\frac{1}{k} - \varepsilon\right) \max(|\overline{\Pi}_1|, \ldots, |\overline{\Pi}_k|) \tag{1}$$

holds for almost all k-tuples $\Pi_1, \Pi_2, \ldots, \Pi_k$ of ordinary partitions of n_1, $n_2, \ldots, n_k$ respectively (*that is with the exception of $o(p(n_1)p(n_2)\ldots p(n_k))$ such k-tuples at most*).

Essentially, the above theorem asserts that, for fixed k and for almost all of the k-tuples of partitions in question, a positive percentage of summands occurs in all the k partitions (independently of n). It is easy to see that for almost all partitions (that is, with the exception of $o(p(n))$ at most), the summand 1 appears at least $[\sqrt{n}/\omega(n)]$ times, where $\omega(n)$ is any function

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which tends to infinity with n. It seems reasonable to conclude that the above phenomenon is due to the presence of a large number of repeated small parts. However, this is not correct. For restricted partitions, Turán obtained a completely analogous result. Here, $q(n)$ denotes the number of restricted partitions of n.

THEOREM 2. *Let $\varepsilon, k, n, n_1, \ldots, n_k$ be as in Theorem 1. For $n \to \infty$, the inequality*

$$|\overline{Q}_1 \cap \ldots \cap \overline{Q}_k| \ge \left(\frac{1}{k2^k \log 2} - \varepsilon\right) \max(|\overline{Q}_1|, \ldots, |\overline{Q}_k|) \tag{2}$$

holds for almost all k-tuples $Q_1, Q_2, \ldots, Q_k$ of restricted partitions of n_1, $n_2, \ldots, n_k$ respectively (that is, with the exception of $o(q(n_1)q(n_2)\ldots q(n_k))$ such k-tuples at most).

In [7], Turán claimed that inequalities like (1) and (2) can be obtained for partitions with summands taken from a general sequence of natural numbers. However, no extension in this direction has appeared in the literature. In this paper, we extend Theorem 2 to restricted partitions with summands taken from a wider class of sequences which includes the set of sth powers. We go on to determine the distribution of the number of common parts in k-tuples of ordinary partitions under slightly more stringent conditions on the n_i's, namely $n \le n_1 \le \ldots \le n_k \le n(1 + o((\log n)^{-1-\delta}))$ for some $\delta > 0$. This resolves another problem of Turán under these conditions and shows that the constant $1/k$ appearing in (1) is optimal.

2. Generalisation of the result of Turán. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ be a strictly increasing sequence of positive integers. A *restricted partition* of n is a partition of n into distinct parts. A *restricted Λ-partition* of n is a restricted partition of n whose summands are taken from Λ. We denote the total number of restricted Λ-partitions of n by $Q(n;\Lambda)$.

In order to give asymptotic results for $Q(n;\Lambda)$ and related quantities, it is necessary to put some restrictions on the sequence Λ. We shall suppose that Λ satisfies the following two conditions (compare [3] and [5]):

$$D_\Lambda(x) = \sum_{\substack{\lambda \text{ in } \Lambda \\ \lambda \le x}} 1 = \frac{Ax^\alpha}{\log^\beta x}\left(1 + O\left(\frac{1}{\log x}\right)\right), \tag{I}$$

where $0 < \alpha \le 1$ and β is real, and

$$J_k = \inf\left\{\frac{1}{\log k}\sum_{\nu=1}^{k} ||\lambda_\nu \theta||^2\right\} \to \infty \tag{II}$$

as $k \to \infty$, where the infimum is taken over those θ satisfying $\frac{1}{2}\lambda_k^{-1} < \theta \le \frac{1}{2}$.

Notice that (I) implies that $\lim_{k\to\infty}\log\lambda_k/\log k = 1/\alpha$. Thus all the results obtained in [5] are applicable to a sequence Λ satisfying (I) and (II). In particular, from equation (1) of [5], by a routine calculation,

$$Q(n;\Lambda) = \exp\{(1+o(1))c_1 n^{\alpha/(\alpha+1)}\log^{-\beta/(\alpha+1)} n\}, \tag{3}$$

with $c_1 = \{A\Gamma(\alpha+1)\zeta(\alpha+1)(1-2^{-\alpha})(\alpha+1)^{\alpha+\beta+1}\alpha^{-\alpha}\}^{1/(\alpha+1)}$ and A is the constant in (I).

The main point of the first part of this paper is to establish the following theorems generalising Theorem 2 above.

Theorem 3. *Let Λ be a sequence of positive integers satisfying conditions* (I) *and* (II) *above. Let $k, n, n_1, \ldots, n_k$ be as in Theorem* 1. *For sufficiently large n, almost all k-tuples $Q_1, \ldots, Q_k$ of restricted Λ-partitions of $n_1, \ldots, n_k$ have at least*

$$\frac{1-o(1)}{2^k k^\alpha} c_2 n^{\alpha/(\alpha+1)} \log^{-\beta/(\alpha+1)} n$$

common summands. Here $c_2 = \left(\dfrac{A(\alpha+1)^\beta \Gamma(\alpha+1)}{\alpha^\alpha \zeta^\alpha(\alpha+1)(1-2^{-\alpha})^\alpha}\right)^{1/(\alpha+1)}$.

If, in addition, the partition function satisfies the inequality

$$\text{(III)}\quad \log Q(n;\Lambda) > c_1 n^{\alpha/(\alpha+1)} \log^{-\beta/(\alpha+1)} n \left(1 - \frac{1}{\log^{1/(2\alpha+2)} n \log\log n}\right),$$

then Erdős and Turán [3] have shown that almost all restricted Λ-partitions of n contain

$$c_3 n^{\alpha/(\alpha+1)} \log^{-\beta/(\alpha+1)} n(1 + O(\log^{-1/(4\alpha+4)} n))$$

summands, where

$$c_3 = \frac{A^{1/(\alpha+1)}\Gamma(\alpha+1)^{1/(\alpha+1)}(1-2^{1-\alpha})\zeta(\alpha)(\alpha+1)^{\beta/(\alpha+1)}}{(\alpha(1-2^{-\alpha})\zeta(\alpha+1))^{\alpha/(\alpha+1)}}.$$

(Note that, when $\alpha = 1$, the indeterminate $(1-2^{1-\alpha})\zeta(\alpha)$ is equal to $\log 2$.) In view of this, the following is an immediate consequence of Theorem 3.

Theorem 4. *Let Λ be a sequence of positive integers satisfying the conditions* (I), (II) *and* (III). *Then, for any $\varepsilon > 0$, $k \ge 2$ and for $n \to \infty$, almost all k-tuples $Q_1, \ldots, Q_k$ of restricted Λ-partitions of $n_1, \ldots, n_k$ with $n \le n_1 \le \ldots \le n_k \le n(1+o(1))$ have at least*

$$\left(\frac{1}{2^k k^\alpha (1-2^{1-\alpha})\zeta(\alpha)} - \varepsilon\right) \max(|\overline{Q}_1|, \ldots, |\overline{Q}_k|)$$

common summands.

It is known that the set of sth powers satisfies the three conditions (I), (II) and (III) and so provides a concrete example for Theorem 4. (See [3], p. 55.)

It is possible to work out the analogue of Theorem 3 for unrestricted Λ-partitions. However, even in the case of unrestricted partitions into squares, no analogue of Theorem 4 is known. The following question was put to the authors by Erdős:

PROBLEM 1. *Let $p_2(n)$ denote the number of unrestricted partitions of n into squares. Does there exist a function $f(c)$ such that the number of unrestricted partitions of n into squares in which the number of summands is less than $cn^{2/3}\log n$ is asymptotic to $f(c)p_2(n)$?*

3. Variation on a problem of Turán concerning common summands. Asymptotically, almost all ordinary partitions of n have $\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n$ summands. Consequently, by Theorem 1, the typical k-tuple of ordinary partitions of n has asymptotically at least $\frac{\sqrt{6}}{2\pi k}\sqrt{n}\log n$ common summands. This leads to the following problem of Turán.

PROBLEM 2 (Turán [9]). *Let $k \ge 2$ be a fixed integer and let λ be a fixed real number. Suppose $n \le n_1 \le n_2 \le \ldots \le n_k \le n(1+o(1))$. Denote by $K(n_1,\ldots,n_k;\lambda)$ the number of k-tuples of ordinary partitions $\Pi_1,\ldots,\Pi_k$ of $n_1,\ldots,n_k$ with the property*

$$|\overline{\Pi}_1 \cap \ldots \cap \overline{\Pi}_k| \le \frac{\sqrt{6}}{2\pi k}\sqrt{n}\log n + \lambda\sqrt{n}.$$

Does there exist a distribution function $\Phi(\lambda)$ such that

$$\lim_{n\to\infty}\frac{K(n_1,\ldots,n_k;\lambda)}{p(n_1)\ldots p(n_k)} = \Phi(\lambda)?$$

In this section, we give an affirmative answer to a slight variation on this question. Our theorem settles the original problem when $\lambda = o(\log\log n)$ and $n \le n_1 \le n_2 \le \ldots \le n_k \le n(1+\theta(n))$, where $\theta(n) = o((\log n)^{-1-\delta})$ for some $\delta > 0$.

THEOREM 5. *Let $k \ge 1$ be a fixed integer and $K(n_1,\ldots,n_k;\lambda)$ be defined as above. Suppose $n \le n_1 \le n_2 \le \ldots \le n_k \le n(1+\theta(n))$, where $\theta(n) = o((\log n)^{-1-\delta})$ for some $\delta > 0$. Then, as $n \to \infty$,*

$$\frac{K(n_1,\ldots,n_k;\lambda)}{p(n_1)\ldots p(n_k)} \sim \Phi(\lambda) = \exp\left(-\frac{1}{kd}e^{-kd\lambda}\right),$$

where $d = \pi/\sqrt{6}$ and $\lambda = o(\log\log n)$.

Note that the case $k = 1$ is the classical result of Erdős and Lehner [1]. Notice further that $\Phi(\lambda) \to 1$ as $\lambda \to \infty$ and $\Phi(\lambda) \to 0$ as $\lambda \to -\infty$.

It follows from Theorem 5 that the number of common parts in almost all k-tuples of partitions of $n_1, \ldots, n_k$ (that is, with the exception of at most $o(p(n_1)\ldots p(n_k))$ partitions) lies between the extremes $\frac{\sqrt{6}}{2\pi k}\sqrt{n}\log n \pm \omega(n)\sqrt{n}$, where $\omega(n) \to \infty$ with n arbitrarily slowly. This, together with equation (1.4) in [1], shows that Theorem 1 is best possible in the sense that we cannot replace $1/k$ in the theorem by a larger constant.

We are unable to obtain the analogue of the above theorem for restricted partitions and we propose the following problem:

PROBLEM 3. *Is the lower bound in* (2) *optimal*? *In particular*, *let* $N(n_1, n_2; \lambda)$ *be the number of pairs of unequal partitions* Q_1, Q_2 *of* n_1, n_2 *such that* $n \le n_1 \le n_2 \le n(1+o(1))$ *and* $|\overline{Q}_1 \cap \overline{Q}_2| \le \frac{\sqrt{3}}{4\pi}\sqrt{n} + \lambda n^{1/4}$. *Is there a distribution function* $\Psi(\lambda)$ *such that*

$$\lim_{n\to\infty} \frac{N(n_1, n_2; \lambda)}{q(n_1)q(n_2)} = \Psi(\lambda)?$$

In view of Problem 1, it would be interesting to extend Theorem 5 to partitions with parts drawn from more general sequences as in Section 2. Our method fails because of the lack of a suitable generating function. (See equation (9) below.)

4. Proof of Theorem 3. In the course of the proof, we will need two simple lemmas which are consequences of hypothesis (I).

LEMMA 1. *Let* Λ *be a sequence of positive integers satisfying* (I). *For* $r \to 0^+$,

$$\sum_{\lambda \text{ in } \Lambda} e^{-r\lambda} \sim A\Gamma(\alpha+1) r^{-\alpha} \log^{-\beta}\left(\frac{1}{r}\right). \tag{4}$$

Proof. For $r \to 0^+$, we can estimate

$$\sum_{\lambda \text{ in } \Lambda} e^{-r\lambda} = \int_0^\infty e^{-rx} d\, D_\Lambda(x) = [e^{-rx} D_\Lambda(x)]_0^\infty + r\int_0^\infty e^{-rx} D_\Lambda(x)\, dx$$
$$= r\int_2^\infty \frac{Ax^\alpha e^{-rx}}{\log^\beta x}\left(1 + O\left(\frac{1}{\log x}\right)\right) dx + O(r).$$

After substituting $y = rx$ in the integral, we see that the above is asymptotic to

$$Ar^{-\alpha}\log^{-\beta}\left(\frac{1}{r}\right)\int_0^\infty e^{-y} y^\alpha\, dy$$

and this is the required result.

By a similar calculation, we also obtain the second lemma.

Lemma 2. *Let* $g_\Lambda(r) = \prod_{\lambda \text{ in } \Lambda}(1+e^{-r\lambda})$, *with* Λ *as in Lemma* 1. *For* $r \to 0^+$,

$$\log g_\Lambda(r) \sim c_4 r^{-\alpha} \log^{-\beta}\left(\frac{1}{r}\right), \tag{5}$$

where $c_4 = A(1-2^{-\alpha})\Gamma(\alpha+1)\zeta(\alpha+1)$.

Now we proceed to the proof proper. Let $h_\Lambda(n_1, \ldots, n_k, n_{k+1})$ denote the number of k-tuples of restricted Λ-partitions $Q_1, \ldots, Q_k$ of $n_1, \ldots, n_k$ respectively such that $|\overline{Q}_1 \cap \ldots \cap \overline{Q}_k| = n_{k+1}$. We are done if we can show that whenever

$$n_{k+1} < \frac{1-\varepsilon}{2^k k^\alpha} c_2 n^{\alpha/(\alpha+1)} \log^{-\beta/(\alpha+1)} n,$$

we have

$$\sum_{n=0}^{n_{k+1}} h_\Lambda(n_1, \ldots, n_k, n) = o(Q(n_1; \Lambda) \ldots Q(n_k; \Lambda)).$$

The generating function for h_Λ is

$$\sum_{n_1, \ldots, n_{k+1}=0}^{\infty} h_\Lambda(n_1, \ldots, n_k, n_{k+1}) x_1^{n_1} \ldots x_k^{n_k} t^{n_{k+1}} = \prod_\lambda \{(1+x_1^\lambda) \ldots (1+x_k^\lambda) - (1-t) x_1^\lambda \ldots x_k^\lambda\}.$$

On putting $x_i = e^{-r_i}$ for $1 \le i \le k$ and $t = e^{-r_{k+1}}$, with each $r_i > 0$, the above expression becomes

$$\sum_{n_1, \ldots, n_{k+1}=0}^{\infty} h_\Lambda(n_1, \ldots, n_k, n_{k+1}) e^{-(n_1 r_1 + \ldots + n_{k+1} r_{k+1})} = g_\Lambda(r_1) \ldots g_\Lambda(r_k) \prod_\lambda \left(1 - \frac{(1-e^{-r_{k+1}}) e^{-\lambda(r_1+\ldots+r_k)}}{(1+e^{-\lambda r_1}) \ldots (1+e^{-\lambda r_k})}\right). \tag{6}$$

Let us denote the last infinite product in (6) by $T(\boldsymbol{r})$. For $r_i \to 0^+$, we have

$$\begin{aligned} \log T(\boldsymbol{r}) &< -(1-e^{-r_{k+1}}) \sum_\lambda \frac{e^{-\lambda(r_1+\ldots+r_k)}}{(1+e^{-\lambda r_1}) \ldots (1+e^{-\lambda r_k})} \\ &\le -\frac{1-e^{-r_{k+1}}}{2^k} \sum_\lambda e^{-\lambda(r_1+\ldots+r_k)} \\ &< -\frac{(1+o(1)) r_{k+1}}{2^k} A\Gamma(\alpha+1)\Big(\sum_{i=1}^k r_i\Big)^{-\alpha} \log^{-\beta}\Big(\sum_{i=1}^k r_i\Big)^{-1}, \end{aligned} \tag{7}$$

using Lemma 1. In (6), the coefficients $h_\Lambda(n_1,\ldots,n_k,n_{k+1})$ are non-negative and so the sum of any group of terms from the left-hand side is less than the product on the right. In particular,

$$\begin{aligned}(8)\qquad \sum_{n=0}^{n_{k+1}} & h_\Lambda(n_1,\ldots,n_k,n)e^{-(n_1r_1+\ldots+n_{k+1}r_{k+1})}\\ &< \exp\left\{c_4(1+o(1))\Big(\sum r_i^{-\alpha}\log^{-\beta}\left(\frac{1}{r_i}\right)\Big)\right.\\ &\quad \left.-\frac{(1+o(1))r_{k+1}}{2^k}A\Gamma(1+\alpha)\Big(\sum r_i\Big)^{-\alpha}\log^{-\beta}\left(\frac{1}{\sum r_i}\right)\right\}.\end{aligned}$$

(All the sums on the right run from $i=1$ to k.) Choose

$$r_i=\frac{\alpha}{\alpha+1}c_1n_i^{-1/(\alpha+1)}\log^{-\beta/(\alpha+1)}n_i\qquad(1\le i\le k).$$

Rearranging (8) then gives

$$\begin{aligned}\sum_{n=0}^{n_{k+1}} & h_\Lambda(n_1,\ldots,n_k,n)\\ &< \exp\Big\{\sum c_1(1+o(1))n_i^{\alpha/(\alpha+1)}\log^{-\beta/(\alpha+1)}n_i\\ &\quad + r_{k+1}\Big(n_{k+1}-\frac{1+o(1)}{2^kk^\alpha}c_2n^{\alpha/(\alpha+1)}\log^{-\beta/(\alpha+1)}n\Big)\Big\}.\end{aligned}$$

If

$$n_{k+1}<\frac{1-\varepsilon}{2^kk^\alpha}c_2n^{\alpha/(\alpha+1)}\log^{-\beta/(\alpha+1)}n,$$

r_{k+1} is sufficiently small and fixed and n is sufficiently large, then (3) gives

$$\sum_{n=0}^{n_{k+1}} h_\Lambda(n_1,\ldots,n_k,n_{k+1},n)=o(Q(n_1;\Lambda)\ldots Q(n_k;\Lambda)).$$

This completes the proof of Theorem 3.

5. Common summands in a pair of partitions. We now begin the proof of Theorem 5. For simplicity, we give the details for the case $k=2$ and indicate the changes required for the general case later. More precisely, we shall prove the following theorem.

THEOREM 6. *Let $K(n_1,n_2;\lambda)$ denote the number of pairs of ordinary partitions Π_1,Π_2 of n_1,n_2 with the property*

$$|\overline{\Pi}_1\cap\overline{\Pi}_2|\le\frac{\sqrt6}{4\pi}\sqrt n\log n+\lambda\sqrt n\,,$$

where $\lambda=o(\log\log n)$, $n\le n_1\le n_2\le n(1+\theta(n))$ and $\theta(n)=o((\log n)^{-1-\delta})$

for some $\delta > 0$. *Then, for* $n \to \infty$,

$$\frac{K(n_1,n_2;\lambda)}{p(n_1)p(n_2)} \sim \exp\left(-\frac{1}{2d}e^{-2d\lambda}\right), \qquad d=\frac{\pi}{\sqrt{6}}.$$

Let $f(x)=\prod_{\nu=1}^{\infty}(1-x^\nu)^{-1}$ and $L=[\frac{\sqrt{6}}{4\pi}\sqrt{n}\log n+\lambda\sqrt{n}\,]$. As shown in [7], pp. 193–195, $K(n_1,n_2;\lambda)$ is the coefficient of $x^{n_1}y^{n_2}$ in

$$G(x,y)=f(x)f(y)\prod_{\nu=L+1}^{\infty}(1-(xy)^\nu)=\frac{f(x)f(y)}{f(xy)}\prod_{\nu=1}^{L}\frac{1}{1-(xy)^\nu}. \tag{9}$$

Therefore, by Cauchy's theorem,

$$K(n_1,n_2;\lambda)=-\frac{1}{4\pi^2}\iint\frac{G(z,w)}{z^{n_1+1}w^{n_2+1}}\,dzdw, \tag{10}$$

where the integral is taken over the product of two circles $z=e^{-\alpha+i\theta}, w=e^{-\beta+i\phi}$ with $-\pi<\theta,\phi\le\pi$. Here, α and β are chosen to satisfy the saddle-point conditions

$$\begin{aligned}\sum_{\nu=1}^{\infty}\frac{\nu}{e^{\alpha\nu}-1}-\sum_{\nu=L+1}^{\infty}\frac{\nu}{e^{(\alpha+\beta)\nu}-1}&=n_1\,,\\ \sum_{\nu=1}^{\infty}\frac{\nu}{e^{\beta\nu}-1}-\sum_{\nu=L+1}^{\infty}\frac{\nu}{e^{(\alpha+\beta)\nu}-1}&=n_2\,.\end{aligned} \tag{11}$$

We choose

$$\alpha=\frac{\pi}{\sqrt{6n_1}}, \qquad \beta=\frac{\pi}{\sqrt{6n_2}}.$$

By the Euler–Maclaurin formula, the two saddle-point equations hold up to errors of $O(\sqrt{n_1}\log^2 n_1)$ and $O(\sqrt{n_2}\log^2 n_2)$ respectively.

Now (10) can be written in the form

$$K(n_1,n_2;\lambda)=\frac{e^{\alpha n_1+\beta n_2}}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}G(e^{-\alpha+i\theta},e^{-\beta+i\phi})e^{-in_1\theta-in_2\phi}d\theta\,d\phi\,. \tag{12}$$

Set $\theta_0=n^{-5/7}$. We split the double integral into four pieces: the main term I_1 corresponds to the major arc $|\theta|\le\theta_0, |\phi|\le\theta_0$ and the error terms I_2, I_3 and I_4 correspond to the respective minor arcs $\theta_0<|\theta|\le\pi$, $\theta_0<|\phi|\le\pi$ for I_2, $\theta_0<|\theta|\le\pi$, $|\phi|\le\theta_0$ for I_3 and $|\theta|\le\theta_0$, $\theta_0<|\phi|\le\pi$ for I_4. The integrand in these double integrals can be written as

$$\begin{aligned}H(\theta,\phi)=\ &\exp\Big\{-\sum_{\nu=1}^{\infty}\log(1-e^{-\alpha\nu})-\sum_{\nu=1}^{\infty}\log(1-e^{-\beta\nu})\\ &+\sum_{\nu=L+1}^{\infty}\log(1-e^{-(\alpha+\beta)\nu})\Big\}\exp\{S_1+S_2-in_1\theta-in_2\phi\},\end{aligned}$$

where

$$S_1 = -\sum_{\nu=1}^{\infty} \log\left(\frac{1-e^{-\alpha\nu+i\nu\theta}}{1-e^{-\alpha\nu}}\right) - \sum_{\nu=1}^{\infty} \log\left(\frac{1-e^{-\beta\nu+i\nu\phi}}{1-e^{-\beta\nu}}\right),$$

$$S_2 = \sum_{\nu=L+1}^{\infty} \log\left(\frac{1-e^{-(\alpha+\beta)\nu+i\nu(\theta+\phi)}}{1-e^{-(\alpha+\beta)\nu}}\right).$$

6. The major arc. Suppose (θ,ϕ) lies in the region of integration of I_1. By a Taylor expansion,

$$S_1 = i\theta\sum_{\nu=1}^{\infty}\frac{\nu}{e^{\alpha\nu}-1} + i\phi\sum_{\nu=1}^{\infty}\frac{\nu}{e^{\beta\nu}-1} - \frac{\theta^2}{2}\sum_{\nu=1}^{\infty}\frac{\nu^2 e^{\alpha\nu}}{(e^{\alpha\nu}-1)^2}$$
$$-\frac{\phi^2}{2}\sum_{\nu=1}^{\infty}\frac{\nu^2 e^{\beta\nu}}{(e^{\beta\nu}-1)^2} + O\left(\frac{|\theta|^3}{\alpha^4}+\frac{|\phi|^3}{\beta^4}\right),$$

$$S_2 = -i(\theta+\phi)\sum_{\nu=L+1}^{\infty}\frac{\nu}{e^{(\alpha+\beta)\nu}-1}$$
$$+\frac{(\theta+\phi)^2}{2}\sum_{\nu=L+1}^{\infty}\frac{\nu^2 e^{(\alpha+\beta)\nu}}{(e^{(\alpha+\beta)\nu}-1)^2} + O\left(\frac{|\theta+\phi|^3}{(\alpha+\beta)^4}\right).$$

The contributions of the two error terms are at most $o(1)$. The saddle-point equations (11) are satisfied with error at most $O(\sqrt{n}\log^2 n)$, so the above expansion gives

$$H(\theta,\phi) = \exp\Bigg\{-\sum_{\nu=1}^{\infty}\log(1-e^{-\alpha\nu}) - \sum_{\nu=1}^{\infty}\log(1-e^{-\beta\nu})$$
$$+\sum_{\nu=L+1}^{\infty}\log(1-e^{-(\alpha+\beta)\nu})$$
$$-\frac{\theta^2}{2}\sum_{\nu=1}^{\infty}\frac{\nu^2 e^{\alpha\nu}}{(e^{\alpha\nu}-1)^2} - \frac{\phi^2}{2}\sum_{\nu=1}^{\infty}\frac{\nu^2 e^{\beta\nu}}{(e^{\beta\nu}-1)^2}$$
$$+\frac{(\theta+\phi)^2}{2}\sum_{\nu=L+1}^{\infty}\frac{\nu^2 e^{(\alpha+\beta)\nu}}{(e^{(\alpha+\beta)\nu}-1)^2} + o(1)\Bigg\}.$$

Now

$$\sum_{\nu=L+1}^{\infty}\frac{\nu^2 e^{(\alpha+\beta)\nu}}{(e^{(\alpha+\beta)\nu}-1)^2} \le 2\sum_{\nu=L+1}^{\infty}\frac{\nu^2}{e^{(\alpha+\beta)\nu}} \sim 2\int_L^{\infty} t^2 e^{-(\alpha+\beta)t}\,dt$$
$$\sim 2e^{-(\alpha+\beta)L}\left(\frac{L^2}{\alpha+\beta}+\frac{2L}{(\alpha+\beta)^2}+\frac{2}{(\alpha+\beta)^3}\right).$$

Since both α and β are equal to

$$\frac{\pi}{\sqrt{6n}}(1+o((\log n)^{-1-\delta})),$$

we have

$$(\alpha+\beta)L=\frac{1}{2}\log n+\frac{2\pi}{\sqrt{6}}\lambda+o((\log n)^{-\delta}). \tag{13}$$

Also

$$\frac{L^2}{\alpha+\beta}\sim cn^{3/2}\log^2 n,\quad \frac{L}{(\alpha+\beta)^2}\sim cn^{3/2}\log n,\quad \frac{1}{(\alpha+\beta)^3}\sim cn^{3/2}$$

and so

$$(\theta+\phi)^2\sum_{\nu=L+1}^{\infty}\frac{\nu^2 e^{(\alpha+\beta)\nu}}{(e^{(\alpha+\beta)\nu}-1)^2}\le cn^{-10/7}\cdot n^{-1/2}e^{-2\pi\lambda/\sqrt{6}}\cdot n^{3/2}\log^2 n=o(1).$$

Here, and later, c denotes an absolute positive constant which may vary from instance to instance. Hence, within the region of integration of I_1,

$$H(\theta,\phi)=\exp\Big\{-\sum_{\nu=1}^{\infty}\log(1-e^{-\alpha\nu})-\sum_{\nu=1}^{\infty}\log(1-e^{-\beta\nu})+\sum_{\nu=L+1}^{\infty}\log(1-e^{-(\alpha+\beta)\nu})-\frac{A_2\theta^2}{2}-\frac{B_2\phi^2}{2}+o(1)\Big\},$$

where

$$A_2=\sum_{\nu=1}^{\infty}\frac{\nu^2e^{\alpha\nu}}{(e^{\alpha\nu}-1)^2}\sim\frac{\pi^2}{3\alpha^3},\quad B_2=\sum_{\nu=1}^{\infty}\frac{\nu^2e^{\beta\nu}}{(e^{\beta\nu}-1)^2}\sim\frac{\pi^2}{3\beta^3}.$$

From all this, $A_2\theta_0^2\sim cn^{1/14}$ and $B_2\theta_0^2\sim cn^{1/14}$, so we can replace the limits of integration $\pm\theta_0$ in I_1 by $\pm\infty$ without altering the asymptotic estimation of I_1. Observe that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\Big(-\frac{A_2\theta^2}{2}-\frac{B_2\phi^2}{2}\Big)\,d\theta\,d\phi\sim\frac{\pi^2}{\sqrt{6}n_1^{3/4}n_2^{3/4}}.$$

Thus, finally,

$$I_1\sim\frac{\pi^2}{\sqrt{6}n_1^{3/4}n_2^{3/4}}\exp\Big\{-\sum_{\nu=1}^{\infty}\log(1-e^{-\alpha\nu})-\sum_{\nu=1}^{\infty}\log(1-e^{-\beta\nu})+\sum_{\nu=L+1}^{\infty}\log(1-e^{-(\alpha+\beta)\nu})\Big\}. \tag{14}$$

7. The minor arcs. We now show that I_2, I_3 and I_4 are negligible compared to I_1.

To deal first with I_2, it suffices to show that

$$J(\theta,\phi) = \exp\{S_1 + S_2 - in_1\theta - in_2\phi\} = o(n_1^{-3/4} n_2^{-3/4}) = o(n^{-3/2})$$

when $\theta_0 \le |\theta|, |\phi| \le \pi$. The real parts of the sums S_1 and S_2 are given by

$$\operatorname{Re} S_1 = -\frac{1}{2}\sum_{\nu=1}^{\infty} \log\left(1 + \frac{4e^{\alpha\nu}\sin^2(\nu\theta/2)}{(e^{\alpha\nu}-1)^2}\right) - \frac{1}{2}\sum_{\nu=1}^{\infty} \log\left(1 + \frac{4e^{\beta\nu}\sin^2(\nu\phi/2)}{(e^{\beta\nu}-1)^2}\right)$$

and

$$\operatorname{Re} S_2 = \frac{1}{2}\sum_{\nu=L+1}^{\infty} \log\left(1 + \frac{4e^{(\alpha+\beta)\nu}\sin^2(\nu(\theta+\phi)/2)}{(e^{(\alpha+\beta)\nu}-1)^2}\right).$$

The last sum is positive and bounded by

$$\frac{1}{2}\sum_{\nu=L+1}^{\infty} \frac{4e^{(\alpha+\beta)\nu}}{(e^{(\alpha+\beta)\nu}-1)^2} \le 8 \sum_{\nu=L+1}^{\infty} e^{-(\alpha+\beta)\nu} \le c(\alpha+\beta)^{-1} e^{-(\alpha+\beta)L}.$$

But $e^{-(\alpha+\beta)L} \sim n^{-1/2} e^{-c\lambda}$ and $\lambda = o(\log\log n)$ by hypothesis, so $|\operatorname{Re} S_2| \le ce^{-c\lambda} = o(n^{\varepsilon})$, for every $\varepsilon > 0$. Thus

$$|J(\theta,\phi)| \le \exp\left\{-\frac{1}{2}\sum_{\nu\le\sqrt{n}} \log\left(1 + \frac{4e^{\alpha\nu}\sin^2(\nu\theta/2)}{(e^{\alpha\nu}-1)^2}\right) - \frac{1}{2}\sum_{\nu\le\sqrt{n}} \log\left(1 + \frac{4e^{\beta\nu}\sin^2(\nu\phi/2)}{(e^{\beta\nu}-1)^2}\right) + o(n^{\varepsilon})\right\}.$$

Following the argument in [6] or [4], pp. 267–269,

$$|J(\theta,\phi)| \le \exp(-cn_1^{1/14} - cn_2^{1/14} + o(n^{\varepsilon})) = o(n_1^{-3/4} n_2^{-3/4})$$

when $\theta_0 \le |\theta|, |\phi| \le \pi$, as required. This gives $|I_2| = o(I_1)$.

For I_3 and I_4, we can combine the techniques used for I_1 and I_2. Let S_α and S_β denote the two constituent sums in S_1, that is

$$S_1 = -S_\alpha - S_\beta\,.$$

Over the region of integration for I_3, namely $\theta_0 < |\theta| \le \pi, |\phi| \le \theta_0$, we have

$$|\exp(-S_\alpha + S_2 - in_1\theta)| \le \exp(-cn^{1/14}),$$

as in the treatment of I_2, and

$$\int_{-\theta_0}^{\theta_0} |\exp(-S_\beta - in_2\phi)| d\phi \sim \frac{\pi}{6^{1/4} n_2^{3/4}},$$

as in the treatment of I_1. So $I_3 = o(I_1)$ and similarly for I_4.

8. Return to the major arc. By (12), (14) and the estimates of the last section,

$$(15)\quad K(n_1,n_2;\lambda) \sim \frac{e^{\alpha n_1+\beta n_2}}{4\sqrt{6}n_1^{3/4}n_2^{3/4}}\,\frac{f(e^{-\alpha})f(e^{-\beta})}{f(e^{-\alpha-\beta})}\,\exp\Big\{-\sum_{\nu=1}^{L}\log(1-e^{-(\alpha+\beta)\nu})\Big\},$$

where $f(x)=\prod_{\nu=1}^{\infty}(1-x^{\nu})^{-1}$. To estimate f, we use the well-known estimate

$$(16)\qquad f(e^{-x})=\exp\bigg(\frac{\pi^2}{6x}+\frac{1}{2}\log\frac{x}{2\pi}+o(1)\bigg)$$

as $x\to 0^+$. This gives

$$(17)\quad K(n_1,n_2;\lambda) \sim \frac{e^{\alpha n_1+\beta n_2}}{4\sqrt{6}n_1^{3/4}n_2^{3/4}}\exp\bigg\{\frac{\pi^2}{6\alpha}+\frac{1}{2}\log\frac{\alpha}{2\pi}+\frac{\pi^2}{6\beta}+\frac{1}{2}\log\frac{\beta}{2\pi}-\frac{\pi^2}{6(\alpha+\beta)} -\frac{1}{2}\log\bigg(\frac{\alpha+\beta}{2\pi}\bigg)-\sum_{\nu=1}^{L}\log(1-e^{-(\alpha+\beta)\nu})\bigg\}.$$

By the Euler–Maclaurin formula,

$$(18)\quad \sum_{\nu=1}^{L}\log(1-e^{-(\alpha+\beta)\nu}) = -\frac{\pi^2}{6(\alpha+\beta)}+\frac{e^{-(\alpha+\beta)L}}{\alpha+\beta}-\frac{1}{2}\log\frac{\alpha+\beta}{2\pi}+o(1)\,.$$

Also, from the choice of α and β and the hypotheses on n_1 and n_2,

$$\frac{1}{\alpha+\beta}=\bigg(\frac{\pi}{\sqrt{6n_1}}+\frac{\pi}{\sqrt{6n_2}}\bigg)^{-1}=\frac{\sqrt{6n}}{2\pi}(1+o((\log n)^{-1-\delta}))$$

and, by (13),

$$\frac{e^{-(\alpha+\beta)L}}{\alpha+\beta}=\frac{\sqrt{6n}}{2\pi}\cdot\frac{e^{-2\pi\lambda/\sqrt{6}}}{\sqrt{n}}(1+o((\log n)^{-\delta})) = \frac{e^{-\pi\sqrt{2/3}\lambda}}{\pi\sqrt{2/3}}+o(1),$$

since $\lambda=o(\log\log n)$. Substituting into (17) gives

$$K(n_1,n_2;\lambda)\sim\frac{1}{48n_1n_2}\exp\bigg\{\frac{2\pi}{\sqrt{6}}(\sqrt{n_1}+\sqrt{n_2})-\frac{e^{-\pi\sqrt{2/3}\lambda}}{\pi\sqrt{2/3}}\bigg\} \sim p(n_1)p(n_2)\exp\bigg\{-\frac{e^{-\pi\sqrt{2/3}\lambda}}{\pi\sqrt{2/3}}\bigg\},$$

which proves Theorem 6.

9. Common summands in k-tuples of partitions. To obtain Theorem 5, we modify the argument of the preceding sections as follows. Set

$$L_k = \left[\frac{\sqrt{6}}{2\pi k}\sqrt{n}\log n + \lambda\sqrt{n}\right].$$

The generating function for $K(n_1,\ldots,n_k;\lambda)$ is

$$G_k(x_1,\ldots,x_k) = \frac{f(x_1)\ldots f(x_k)}{f(x_1\ldots x_k)}\prod_{\nu=1}^{L_k}\frac{1}{1-(x_1\ldots x_k)^\nu}$$

and Cauchy's theorem yields

$$K(n_1,\ldots,n_k;\lambda) = \frac{1}{(2\pi i)^k}\int\ldots\int\frac{G_k(z_1,\ldots,z_k)}{z_1^{n_1+1}\ldots z_k^{n_k+1}}\,dz_1\ldots dz_k,$$

where the integral is taken over the product of k circles

$$z_i = e^{-\alpha_i+i\theta_i},\qquad -\pi<\theta_i\le\pi\qquad(1\le i\le k)$$

and $\alpha_1,\ldots,\alpha_k$ are chosen to satisfy the saddle-point conditions

$$\sum_{\nu=1}^{\infty}\frac{\nu}{e^{\alpha_i\nu}-1} - \sum_{\nu=L_k+1}^{\infty}\frac{\nu}{e^{\alpha\nu}-1} = n_i\qquad(1\le i\le k)$$

and

$$\alpha = \alpha_1+\ldots+\alpha_k.$$

We choose $\alpha_i = \pi/\sqrt{6n_i}$, so that the saddle-point conditions are satisfied up to an error $O(\sqrt{n}\log^2 n)$. This leads, as before, to

$$K(n_1,\ldots,n_k;\lambda) \sim \frac{1}{(2\pi)^k}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\exp\Big(-\frac{1}{2}\sum_{i=1}^{k}A_i\theta_i^2\Big)\,d\theta_1\ldots d\theta_k$$

$$\times\exp\Big\{\sum_{i=1}^{k}\alpha_i n_i - \sum_{i=1}^{k}\sum_{\nu=1}^{\infty}\log(1-e^{-\alpha_i\nu})$$

$$+\sum_{\nu=1}^{\infty}\log(1-e^{-\alpha\nu}) - \sum_{\nu=1}^{L_k}\log(1-e^{-\alpha\nu})\Big\},$$

where

$$A_i = \sum_{\nu=1}^{\infty}\frac{\nu^2 e^{\alpha_i\nu}}{(e^{\alpha_i\nu}-1)^2}\sim\frac{\pi^2}{3\alpha_i^3}\qquad(1\le i\le k).$$

The various sums here can be estimated by (16) and (18), leading finally to

$$K(n_1,\ldots,n_k;\lambda)\sim p(n_1)\ldots p(n_k)\exp\left(-\frac{e^{-\pi k\lambda/\sqrt{6}}}{\pi k/\sqrt{6}}\right).$$

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SCHOOL OF MATHEMATICS, PHYSICS, COMPUTING AND ELECTRONICS
MACQUARIE UNIVERSITY
SYDNEY, NEW SOUTH WALES 2109
AUSTRALIA

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