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On Schertz's class number formula related to elliptic units for some non-Galois extensions

by

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Introduction. The purpose of this paper is to give a refinement of Schertz's class number formula related to elliptic units. We are going to study some non-Galois cases. Let K/Q be a finite non-Galois extension, k be an imaginary quadratic field, and put $L = Kk$. We denote by h the class number of K and by E the unit group of K. Suppose that L/k is an abelian extension. Then Schertz [6] has shown a class number formula related to elliptic units as follows:

THEOREM (Schertz). *Notations being as above, one can construct a group F of certain elliptic units of K such that*

$$ch/h_0 = (E:FE_0) \tag{0.1}$$

with an explicit constant c depending only on the construction of F. Here, h_0 and E_0 are the class number and the unit group of the maximal absolutely abelian subfield K_0 of K, respectively.

In this formula, h_0 can be known in various ways because K_0 is absolutely abelian. But the constant c is much more complicated and not so small in [6]. Now, to know the class number h, we shall make c as small and explicit as possible. Then we have the best possible construction of F in Schertz's formula. Namely,

THEOREM 1. *Notations being as above, suppose that the Galois closure L of K is dihedral over Q and cyclic over k. Let $n = [K:Q]$. Then we can construct a group F of certain elliptic units such that*

$$h = (E:F) \quad \textit{if } n \textit{ is odd}, \tag{0.2}$$

$$2^b h/h_0 = (E:FE_0) \quad \textit{if } n = 4 \textit{ or } 2l,\ (2, l) = 1. \tag{0.3}$$

In (0.3), *K_0 is a quadratic field and b is a computable positive integer.*

The proof of Theorem 1 is described in Section 3. For that purpose, we prepare some properties of Z-modules in cyclotomic fields in Section 1. Schertz's result above is described in Section 2 precisely. Finally, in Section 4, we discuss a few more cases where L/Q is not dihedral. In particular, in the case where $[K:Q]$ is the product of two primes we prove Theorem 2 which, together with Theorem 1, completely give a formula for $n = pq$ with primes p, q. The class number formula (0.1) is previously studied in detail by H. Hayashi [1], K. Nakamula [2], [3], [4] when $n = 3, 4, 5, 6$.

1. Preliminaries on cyclotomic fields. Let m be a positive integer, > 2. Let $\zeta = \zeta_m$ be a primitive mth root of unity and put $Q_m = Q(\zeta)$. Let J be the complex conjugation of Q_m and let Q_m^+ be the fixed field of Q_m for J. Then Q_m^+ is the maximal real subfield of Q_m. Let $\mathcal{O}$ (resp. $\mathcal{O}^+$) be the ring of integers of Q_m (resp. Q_m^+), and D_m (resp. D_m^+) be the discriminant of Q_m (resp. Q_m^+). Let N be the norm on Q_m/Q. Let $\Phi_m(X)$ be the mth cyclotomic polynomial:

$$\Phi_m(X) = \prod_{(a,m)=1} (X - \zeta_a).$$

Throughout this section, p always denotes a prime number.

Let $n \in Q$. We define the function Φ_n^* from $\{\pm 1\}$ to Z according to the value $\Phi_n(\pm 1)$ as follows

$$\Phi_n^*(\pm 1) = \begin{cases} \Phi_n(\pm 1) & \text{if } n > 2,\ n \in Z, \\ 1 & \text{if } n \in Q - Z. \end{cases}$$

When $n = 1$ or 2, $\Phi_1^*(1) = \Phi_2^*(-1) = 1$ and $\Phi_1^*(-1) = \Phi_2^*(1) = 2$.

We recall that $m > 2$, then $\Phi_m(\pm 1)$ is known as follows:

$$\Phi_m(1) = \begin{cases} p, & m = p^a,\ a \geqslant 1, \\ 1, & \text{otherwise}. \end{cases} \tag{1.1}$$

Since $\Phi_m(-1) = \prod(-1 - \zeta_a) = \prod(1 + \zeta_a)$,

$$\Phi_m(-1) = \begin{cases} \Phi_{m/2}(1) & \text{if } m = 2 \pmod 4, \\ \Phi_m(1) & \text{if } m = 0 \pmod 4, \\ \Phi_{2m}(1) & \text{if } m = \pm 1 \pmod 4. \end{cases}$$

Therefore,

$$\Phi_m(-1) = \begin{cases} p, & m = 2p^a,\ a \geqslant 1, \\ 1, & \text{otherwise}. \end{cases} \tag{1.2}$$

Note that $N(\zeta - 1) = \Phi_m(1)$. We shall only use $\Phi_m^*(\pm 1)$ as the absolute norm of some ideal $\neq (0)$. When we use $\Phi_n^*(\pm 1)$ as the meaning of some positive integer, we can use $\Phi_n(\pm 1)$ instead of $\Phi_n^*(\pm 1)$, without confusion.

Let $\mathfrak{p} = (\zeta - 1)\mathcal{O}$. Then $\mathfrak{p}$ is a prime ideal of $\mathcal{O}$ when m is a prime power, otherwise $\mathfrak{p} = \mathcal{O}$.

The following lemma is known (see Washington [7], Lemma 4.19).

LEMMA 1. *The discriminant of the maximal real subfield* Q_m^+ *is given by*

$$|D_m^+| = \Big(m^{\varphi(m)}/\prod_{p|m} p^{\varphi(m)/(p-1)}\Phi_m(1)\Phi_{m/2}(1)\Big)^{1/2},$$

where p *runs through all prime divisors of* m.

Here, the factor $\Phi_m(1)\Phi_{m/2}(1)$ equals p or 4 according as m or $m/2$ is p^a or 2^a, where p is an odd prime. Otherwise, $\Phi_m(1)\Phi_{m/2}(1) = 1$.

Let ζ_0 be an mth root of unity. We consider a $\boldsymbol{Z}$-module given by the following formula:

$$M = \sum_{j=1}^{m-1} \boldsymbol{Z}\big((\zeta^j-1)+\zeta_0(\zeta^{-j}-1)\big). \tag{1.3}$$

Since $m > 2$, $M \neq 0$. Denote by $d(M)$ the absolute value of the discriminant of the $\boldsymbol{Z}$-module M. Suppose $\zeta_0 = \pm 1$.

LEMMA 2. *Let* $\zeta_0 = 1$. *Then* $M = \mathfrak{p}\cap\mathcal{O}^+$, *and*

$$d(M) = \Phi_m(1)^2|D_m^+|.$$

Proof. Since $\zeta_0 = 1$, we have $M = \sum_{j=1}^{m-1}\boldsymbol{Z}(\zeta^j+\zeta^{-j}-2)$ from (1.3). We shall prove the above in 3 steps.

1. M is an ideal of $\mathcal{O}^+$. Since $\mathcal{O}^+ = \boldsymbol{Z}[\zeta+\zeta^{-1}]$ (see [7], Proposition 2.16), the $\boldsymbol{Z}$-module M is an ideal of $\mathcal{O}^+$. Indeed,

$$(\zeta+\zeta^{-1})(\zeta^j+\zeta^{-j}-2) = (\zeta^{j+1}+\zeta^{-j-1}-2)+(\zeta^{j-1}+\zeta^{-j+1}-2)-2(\zeta+\zeta^{-1}-2).$$

2. $\mathfrak{p}^2\cap\mathcal{O}^+ \subset M \subset \mathfrak{p}\cap\mathcal{O}^+$. The inclusion $M \subset \mathfrak{p}\cap\mathcal{O}^+$ is clear. Since M is an ideal of $\mathcal{O}^+$ and $(\zeta+\zeta^{-1}-2)\in M$,

$$\mathfrak{p}^2\cap\mathcal{O}^+ = (\zeta-1)^2\mathcal{O}\cap\mathcal{O}^+ = (\zeta+\zeta^{-1}-2)\mathcal{O}^+ \subset M.$$

3. $\mathfrak{p}^2\cap\mathcal{O}^+ = \mathfrak{p}\cap\mathcal{O}^+$. If $\mathfrak{p} = \mathcal{O}$ then the equality is trivial. Assume $\mathfrak{p} \neq \mathcal{O}$, then $\mathfrak{p}$ is the prime ideal which totally ramifies in Q_m/Q. Let $\mathfrak{p}^+ = \mathfrak{p}\cap\mathcal{O}^+$. Since $[Q_m:Q_m^+] = 2$, $\mathfrak{p}^+\mathcal{O} = \mathfrak{p}^2$. This implies that $\mathfrak{p}^2\cap\mathcal{O}^+ = \mathfrak{p}\cap\mathcal{O}^+$. Hence $M = \mathfrak{p}\cap\mathcal{O}^+$. In step 3, we saw that $[\mathcal{O}^+:\mathfrak{p}\cap\mathcal{O}^+] = \Phi_m(1)$. Using the formula for the discriminant of an ideal of an algebraic number field, the lemma is proved.

LEMMA 3. *Let* $\zeta_0 = -1$ *and let* m *be even. Then* $M = (\zeta-\zeta^{-1})\mathcal{O}^+$ *and*

$$d(M) = \Phi_{m/2}(1)\Phi_{m/2}(-1)|D_m^+|.$$

Proof. Since $\zeta_0 = -1$, we have $M = \sum_{j=1}^{m-1}\boldsymbol{Z}(\zeta^j-\zeta^{-j})$ from (1.3). Further $\mathcal{O}^+ = \boldsymbol{Z}[\zeta+\zeta^{-1}]$. For any integer k,

$$\zeta^k-\zeta^{-k} = (\zeta-\zeta^{-1})(\zeta^{k-1}+\zeta^{-k+1}+\zeta^{k-3}+\zeta^{-k+3}+\ldots)+e(\zeta-\zeta^{-1})$$

where $e = 1$ or 0. Therefore $M = (\zeta - \zeta^{-1})\mathcal{O}^+$. The discriminant of M is given by

$$d(M) = |N(\zeta - \zeta^{-1})||D_m^+|,$$

and

$$|N(\zeta - \zeta^{-1})| = |N(\zeta^2 - 1)| = |N(\zeta - 1)N(\zeta + 1)| = \Phi_m(1)\Phi_m(-1).$$

The lemma is proved.

Remark 1. The rank of $\mathfrak{p} \cap \mathcal{O}^+$ as $\boldsymbol{Z}$-module is $\varphi(m)/2$. The set $\{\zeta^j + \zeta^{-j} - 2 |\ 1 \leqslant j \leqslant \varphi(m)/2\}$ is an independent system over $\boldsymbol{Z}$. Furthermore the fact that the set is a basis of $\mathfrak{p} \cap \mathcal{O}^+$ is proved in a way similar to the proof of the fact that $\{\zeta^j + \zeta^{-j} |\ 1 \leqslant j \leqslant \varphi(m)/2\}$ is a basis of $\boldsymbol{Z}[\zeta^1 + \zeta^{-1}]$. Similarly, $\{\zeta^j - \zeta^{-j} |\ 1 \leqslant j \leqslant \varphi(m)/2\}$ is a basis of $(\zeta - \zeta^{-1})\mathcal{O}$.

Suppose $\zeta_0 \neq \pm 1$.

Lemma 4. *Let ζ_0 be a primitive d-th root of unity, where d is a divisor of m and $d \neq 1, 2$. Then there is a $\boldsymbol{Z}$-submodule M_0 of M such that the discriminant of M_0 on $\boldsymbol{Q}_m^+$ is given by*

$$d(M_0) = \Phi_d(-1)^{\varphi(m)/\varphi(d)}\Phi_m(1)^2|D_m^+|.$$

Proof. Let $M_0 = (1 + \zeta_0)(\mathfrak{p} \cap \mathcal{O}^+)$. Then M_0 is a $\boldsymbol{Z}$-submodule of M because $\mathfrak{p} \cap \mathcal{O}^+ = \sum_{j=1}^{m-1} \boldsymbol{Z}(\zeta^j + \zeta^{-j} - 2)$ by Lemma 2 and

$$(1 + \zeta_0)(\zeta^j + \zeta^{-j} - 2) = \big((\zeta^j - 1) + \zeta_0(\zeta^{-j} - 1)\big) + \big((\zeta^{-j} - 1) + \zeta_0(\zeta^j - 1)\big) \in M.$$

The discriminant of M_0 is given by $d(M_0) = |N(1 + \zeta_0)||d(\mathfrak{p} \cap \mathcal{O}^+)|$. Since ζ_0 is a primitive dth root of unity, $|N(1 + \zeta_0)| = \Phi_d(-1)^{\varphi(m)/\varphi(d)}$. The lemma is proved.

2. Schertz's results. In this section, we shall describe Schertz's result and give the notations. Using the class field theory, there is a positive integer f such that the ray class field $H_{(f)}$ modulo (f) of k includes L. Let $\mathrm{Cl}(f)$ be the ray class group modulo f of k. The Artin symbol $(c, H_{(f)}/k)$ gives an isomorphism from $\mathrm{Cl}(f)$ to the Galois group $G(H_{(f)}/k)$. Since $f = \bar{f}$, the complex conjugation $c \to \bar{c}$ is an automorphism of $\mathrm{Cl}(f)$. Using this fact, we can prove the next properties. (But the proof is omitted here, see [6].)

The extension $H_{(f)}/\boldsymbol{Q}$ is Galois, the Galois group $G(H_{(f)}/\boldsymbol{Q})$ is the semi-direct product $G(H_{(f)}/k) \cdot \langle J \rangle$, and $G(H_{(f)}/k)$ is a normal subgroup of $G(H_{(f)}/\boldsymbol{Q})$. Since $J^{-1}(c, H_{(f)}/k)J = (\bar{c}, H_{(f)}/k)$, we define c^J by $\bar{c}$. Let U be the subgroup of $\mathrm{Cl}(f)$ corresponding to the field L. Since $[L:K] = 2$, $U^J = U$ can be proved. (See [6, II], pp. 67–68.) Therefore $L/\boldsymbol{Q}$ is Galois. There is an element c_0 of $\mathrm{Cl}(f)$ mod U such that $G(L/K) = \langle (c_0, L/\boldsymbol{Q})J \rangle$, the Galois group $G(L/\boldsymbol{Q})$ is the semi-direct product $G(L/k) \cdot \langle (c_0, L/\boldsymbol{Q})J \rangle$, and $G(L/k)$ is a normal subgroup of $G(L/\boldsymbol{Q})$. For the maximal abelian subfield K_0 of K, the composite $L_0 = K_0 k$ is the maximal abelian subfield of L. Let U_0 be the subgroup of $\mathrm{Cl}(f)$ corresponding to L_0. Then $U_0 = \{c^{1-J} |\ c \in \mathrm{Cl}(f)\}$. Let $A = \mathrm{Cl}(f)/U$ and X

be the character group of A. Since $U^J = U$, we define the action of the automorphism J on X by $\chi^J(c) = \chi(\bar{c})$ for any $\chi \in X$. Let $X_0 = \{\chi \in X \mid \chi^J = \chi\}$. Then X_0 is the character group of U_0. The classes of characters $W = (X - X_0)/\sim$ are defined by the equivalence relation:

$$\chi \sim \chi' \quad \text{if and only if} \quad \langle \chi \rangle = \langle \chi' \rangle \text{ or } \langle \chi^J \rangle = \langle \chi' \rangle.$$

If $L/\boldsymbol{Q}$ is dihedral, then $\langle \chi \rangle = \langle \chi' \rangle$ is equivalent to $\langle \chi^J \rangle = \langle \chi' \rangle$. Therefore, we assume that $\chi \sim \chi'$ satisfies $\langle \chi \rangle = \langle \chi' \rangle$ in this section. Later, we shall treat the case $\langle \chi^J \rangle \neq \langle \chi \rangle$ in Section 4. For any class ω of W, let m_ω be the order of an element of ω. We take the subset ω' of ω defined by

$$\omega' = \{\chi^j \mid 1 \leqslant j \leqslant m_\omega/2, (j, m_\omega) = 1\}.$$

Then $\omega' \cup \omega'^J = \omega$ and $\omega' \cap \omega'^J = \emptyset$. Put $r_\omega = \# \omega' = \varphi(m_\omega)/2$. The rank r of F is found in [6], namely,

$$r = \sum_{\omega \in W} r_\omega + r_0, \qquad r_0 = \# \{\chi \in X_0 \mid \chi(c_0) = 1, \chi \neq 1\}. \tag{2.1}$$

For any class ω, let U_ω, A_ω and k_ω be the following:

$$U_\omega = \{c \in \mathrm{Cl}(f) \mid \chi(c) = 1 \text{ for any } \chi \in \omega\},$$

$$A_\omega = \mathrm{Cl}(f)/U_\omega,$$

k_ω is the field corresponding to U_ω.

Now, F is constructed by canonical elliptic units $\theta_\omega(a)$ of L ($a \in A_\omega$, $\omega \in W$). (For elliptic units in the case where $H_{(f)}$ is the ring class field, see [5].) For each class ω of W, we take integers λ_{ia} ($i = 1, \ldots, r_\omega$; $a \in A_\omega$) such that $d_\omega = |\det(v_{ij})| \neq 0$ where

$$v_{ij} = \sum_{a \in A_\omega} \lambda_{ia}\big((\chi^j(a) - 1) + \chi^j(c_0)(\chi^{-j}(a) - 1)\big) \quad \text{for } i, j = 1, \ldots, r_\omega.$$

Let $\theta_{i\omega} = \prod_{a \in A_\omega} \big(\theta_\omega(a)^{1 + (c_0, L/k)J}\big)^{\lambda_{ia}}$. The group F is generated by $\{\theta_{i\omega} \mid i = 1, \ldots, r_\omega, \omega \in W\}$. We give the constant c in (0.1) as follows. Let

$$c_1 = n^{(r-1)/2}, \qquad c_2 = n_0^{(1-r_0)/2} \quad \text{and} \quad c_3 = \prod_{\omega \in W} d_\omega m_\omega^{r_\omega}$$

where $n_0 = [K_0 : \boldsymbol{Q}]$. Let $c_4 = c_1 c_2 c_3$ and $c_5 = \prod_{\omega \in W} 24 t_\omega$, where $t_\omega = \min\{t \mid t(U_\omega : 1) \equiv 0 \pmod{h_k}\}$, h_k is the class number of k. The constant c is given by $c = c_4 c_5$.

Remark 2. Let b be the number of classes in W which have even order. Using II, Satz 3.2 in [6], we can take $c_5 = 2^b$ by the choice of $\theta_\omega(a)$. For the number d_ω, Schertz [6] showed that we can take d_ω such that $d_\omega \neq 0$ and d_ω divides $(4m_\omega)^{\varphi(m_\omega)/2} \Phi_m(1)|D_m^+|$. But this is not enough for our purpose.

3. Proof of Theorem 1. First, by Remark 2, we can take $c_5 = 2^b$ for some integer b. Therefore, if we prove $c_4 = 1$ then $c = 2^b$. Since L/k is cyclic, we assume that $X = \langle \chi \rangle$. Since L/Q is dihedral, $\chi^J = \chi^{-1}$. Hence, the relation $(\chi^i)^J = \chi^i$ implies that the order of χ^i is 1 or 2. Therefore,

$$X_0 = \begin{cases} \{1, \chi^{n/2}\} & \text{if } n \text{ is even,} \\ \{1\} & \text{if } n \text{ is odd.} \end{cases}$$

Therefore, $n_0 = [K_0 : Q] = 1$ or 2. Let

$$e = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Then we have $n_0 = 2^e$. For the value of $\chi(c_0)$, the next lemma holds.

LEMMA 5. *Let notations be as above. Let* $n = [L:k]$. *Assume that* $n = 2^a u$, $(2, u) = 1$. *Then there is an element* b *of* $\mathrm{Cl}(f)$ *such that* $G(L/K') = \langle (c_0 b^2, L/k) J \rangle$ *and* $\chi(c_0 b^2)$ *is a* 2^a*-th root of unity. Here,* K' *is a conjugate of* K.

Proof. Let $(b, L/k) \in G(L/k)$. Then the conjugate of $(c_0, L/k)J$ by $(b, L/k)$ is $(b^{-2} c_0, L/k)$. Since $(u, 2) = 1$, we can choose an element b of $\mathrm{Cl}(f)$ such that $\chi(b^{-2} c_0)$ is a 2^a-th root of unity.

By the above lemma, we assume that $\chi(c_0)$ is a 2^a-th root of unity. We shall prove the theorem by considering three cases.

Case 1. $\chi(c_0) = 1$. Fix a class ω of W. Since $\chi(c_0) = 1$, $\chi^j(c_0) = 1$ for any j. Therefore $v_{ij} = \sum_{a \in A_\omega} \lambda_{ia} (\psi^j(a) + \psi^{-j}(a) - 2)$. Put $m = m_\omega$ and $r = r_\omega$. Since $v_{ij} \in \mathcal{O}^+$ and $\{v_{ij} |\ \psi^j \in \omega'\}$ are all conjugates of v_{i1} over Q_m^+/Q, $d_\omega = |\det(v_{ij})| = d([v_{11}, \ldots, v_{r1}])^{1/2}$, where $r = \varphi(m)/2$. Let $\zeta_0 = \psi(c_0)$. Then $M = \sum_{j=1}^{m-1} Z(\zeta^j + \zeta^{-j} - 2)$. Using Lemmas 1 and 2, we have

$$d(M) = \Phi_m(1)^2 \Big(m^{\varphi(m)} \Big/ \prod_{p|m} p^{\varphi(m)/(p-1)} \Phi_m(1) \Phi_{m/2}(1)\Big)^{1/2}. \tag{3.1}$$

Assume that $\{v_{11}, \ldots, v_{r1}\}$ is a basis of M. Then, from (3.1),

$$d_\omega = \Phi_m(1) \Big(m^{\varphi(m)} \Big/ \prod_{p|m} p^{\varphi(m)/(p-1)} \Phi_m(1) \Phi_{m/2}(1)\Big)^{1/4}. \tag{3.2}$$

We shall compute the coefficient c_4. Since L/k is cyclic, the correspondence between divisors of n and classes of W is one-to-one. So, put $d_m = d_\omega$. We have $\prod_{\omega \in W} d_\omega m_\omega^{-r_\omega} = \prod'_{m|n} d_m m^{-\varphi(m)/2}$, where the product is taken over all divisors of n except 1 and 2. From (3.2), we have

$$\prod_{m|n}{}' d_m m^{-\varphi(m)/2} = \prod_{m|n}{}' \Phi_m(1)^{3/4} \prod_{m|n}{}' \Big(m^{\varphi(m)} \prod_{p|m} p^{\varphi(m)/(p-1)} \Phi_{m/2}(1)\Big)^{-1/4}. \tag{3.3}$$

For any positive integer n, the formulas

$$\sum_{m|n} \varphi(m) = n, \quad \prod_{m|n} \Phi_m(1) = n \tag{3.4}$$

are known. By comparison of the index of each prime divisor of n, we obtain

$$\prod_{m|n}\left(m^{\varphi(m)}\prod_{p|m}p^{\varphi(m)/(p-1)}\right)=n^n. \tag{3.5}$$

Under the factorization of n in Lemma 5, using (1.1),

$$\prod_{m|n}\Phi_{m/2}(1)=\prod_{m|u}\Phi_m(1)^e\prod_{m|2}\Phi_{m/2}(1)=(n/2)^e. \tag{3.6}$$

Using (3.3)–(3.6),

$$\prod_{m|n}{}' d_m m^{-\varphi(m)/2}=n^{(3-e-n)/4}. \tag{3.7}$$

Since $L/\boldsymbol{Q}$ is cyclic, using (2.1), the rank r of E is given by

$$r-r_0=\left(\sum_{m|n}\varphi(m)-\varphi(2)-\varphi(1)\right)/2. \tag{3.8}$$

By the assumption on the degree n, $\chi(c_0)=1$ yields that

$$\begin{cases} r_0=0,\ K_0=\boldsymbol{Q} & \text{if } n \text{ is odd,}\\ r_0=1,\ K_0 \text{ is a real quadratic field} & \text{if } n \text{ is even.}\end{cases}$$

We obtain $r_0=e$. From (3.4) and (3.8), the rank of E is

$$r=(n-1-e)/2.$$

From the definition of the constants c_1, c_2, c_3 in Section 2, we have

$$c_1=n^{(r-1)/2}=n^{(n-3+e)/4}.$$

Since $r_0=e=0$ or 1, and $n_0=2^e$,

$$c_2=2^{e(1-e)/2}=1.$$

From (3.7), $c_3=n^{(3-e-n)/4}$. Hence $c_4=c_1c_2c_3=1$. In this case, the theorem is proved.

Case 2. $\chi(c_0)=-1$. If m_ω is odd then d_ω is the same as in case 1. Therefore we shall compare cases 1 and 2 for the value of d_ω only in the case where m_ω is even. For that purpose, we write c_1', d_ω' instead of c_1, d_ω in case 1 and so on. If $(j, n)=1$, then j is odd and $\chi^j(c_0)=-1$. Therefore,

$$v_{ij}=\sum_{a\in A_\omega}\lambda_{ia}(\psi^j(a)-\psi^{-j}(a)) \quad \text{and} \quad v_{ij}\in M=\sum_{j=1}^{m-1}\boldsymbol{Z}(\zeta^j-\zeta^{-j}).$$

Suppose that $\{v_{11}, \ldots, v_{r1}\}$ is a basis of M. Then using Lemma 3,

$$d_\omega=(\Phi_m(1)\Phi_m(-1))^{1/2}|D_m^+|^{1/2}. \tag{3.9}$$

By comparison of (3.2) and (3.9),

$$d_\omega'/d_\omega=\Phi_m(1)/(\Phi_m(1)\Phi_m(-1))^{1/2}.$$

Put $m = 2^a u$, $(2, u) = 1$, where $a \geqslant 1$. By the assumption of the degree n, if $a \geqslant 2$ then $n = 4$ and $\Phi_m(1) = \Phi_m(-1) = 2$. If $a = 1$ then $\Phi_m(1) \doteq 1$. Therefore,

$$d'_\omega/d_\omega = \begin{cases} \Phi_m(-1)^{-1/2} & \text{if } a = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Assume $a > 1$. Then $d'_\omega/d_\omega = 1$. Since $\chi^2(c_0) = 1$, $r_0 = 1$. Therefore, $c_4 = c'_4 = 1$. Assume $a = 1$. Then $d'_\omega/d_\omega = \Phi_m(-1)^{1/2}$. Therefore, $\prod_{\omega \in W}(d'_\omega/d_\omega) = u^{1/2}$. Since $r_0 = 0$, the rank $r = r' - 1$. Then $c'_1 c'_2/c_1 c_2 = (n/2)^{1/2}$. Therefore $c'_4/c_4 = 1$. In this case, the theorem is proved.

Case 3. $\chi(c_0) \neq \pm 1$. In this case, by the assumption of n, we have $n = 4$ and $W = \{\chi\}$. We compute d_ω immediately using Lemma 4. Then $d_\omega = 2^{3/2}$. Since $n_0 = 2$, $r_0 = 0$ and $r = 1$. So $c_4 = 1$. Now, the proof of Theorem 1 is complete.

COROLLARY. *Let L be cyclic over k. If the maximal real subfield K of L is non-Galois over Q then*

$$h = 2^b h_0(E : FE_0).$$

Proof. Since K is the maximal real subfield of L, $\chi(c_0) = 1$. Therefore, the corollary can be proved as in the above proof of case 1.

4. Non-dihedral cases. In this section, we consider the case where $n = p$ or $n = pq$, both p and q are primes, and the case is not included in Theorem 1. We consider the automorphism τ of order 2 instead of the complex conjugation J on X. When $\langle \chi^\tau \rangle \neq \langle \chi \rangle$, we choose representatives of the class ω of W and give v_{ij}'s which are different from those of Section 2. (More details in [6].) If $n = p$, then L has only one character class, therefore, L/Q is dihedral. Suppose $n = pq$. Similarly, if $p = 2$ and $q \neq 2$ then L/Q is dihedral. If $p = q = 2$ then L/Q is dihedral because $G(L/Q)$ is a non-abelian group of order 8. Suppose that both p and q are odd primes. Then the next theorem holds.

THEOREM 2. *Let K/Q be non-Galois and suppose the Galois closure L of K is abelian over k. Suppose $n = [K : Q] = pq$ (both p and q are odd primes), and L/Q is not dihedral. Then we can construct a group F of elliptic units such that if $p = q$ then $G(L/k)$ is an abelian group of type (p, p) and,*

$$p^{(p^2+2p-3)/4} h/h_0 = (E : FE_0);$$

if $p \neq q$ and $K_0 \neq Q$ then,

$$2^{(p-1)(q-1)/2} h/h_0 = (E : FE_0).$$

Proof. In both cases above, the character group X is the direct product $\langle \chi \rangle \times \langle \psi \rangle$ where χ and ψ are the characters of order p and q, respectively. Since $\tau^2 = 1$ and L/Q is not abelian, we assume that $\psi^\tau \neq \psi^{-1}$. We denote by $\langle \psi \rangle^*$ the subset of $\langle \psi \rangle$ whose element has order m where m is the order of ψ. Let $p = q$. Then we have two cases: (1) $\chi^\tau = \chi^{-1}$, (2) $\chi^\tau = \chi$.

Case 1: W has $p+1$ classes $\langle\chi\rangle^*$ and $\langle\chi^i\psi\rangle^*$ ($i = 1, \ldots, p$). In this case, the construction of F is the same as in Theorem 1. For any class ω, $d_\omega = p^{(p+1)/4}$ and $r_\omega = \varphi(p)/2$. Therefore $c_4 = p^{(p^2+2p-3)/4}$.

Case 2: $X_0 = \langle\chi\rangle$. $\langle(\psi^i\chi)^\tau\rangle = \langle\psi^{-i}\chi\rangle \neq \langle\psi^i\chi\rangle$ ($i = 1, \ldots, (p-1)/2$). Then W has $(p+1)/2$ classes $\langle\psi\rangle^*$ and $\langle\psi^i\chi\rangle^* \cup \langle\psi^{-i}\chi\rangle^*$ ($i = 1, \ldots, (p-1)/2$). If $\omega = \langle\psi\rangle^*$ then d_ω is the same as in case 1. Otherwise, we take v_{ij} in Section 2 as

$$v_{ij} = \sum_{a \in A_\omega} \lambda_{ia}(\phi^j(a) - 1)$$

for $i, j = 1, \ldots, p$ where $\phi \in \langle\psi^k\chi\rangle^*$ for some k. Then $m_\omega = p$ and d_ω is the discriminant of the $\boldsymbol{Z}$-module constructed by v_{ij}'s. We can take $d_\omega = \Phi_p(1)|D_p|^{1/2}$ in [6], I, Satz 1.4. Therefore, $c_4 = \prod_{\omega \in W} d_\omega m_\omega^{-r_\omega} = p^{(p^2+2p-3)/4}$ where the product is taken over all classes of W. In the former case, the theorem is proved.

Let $p \neq q$. Then $X_0 = \langle\chi\rangle$. Since L/k is cyclic, W has two classes $\langle\chi\rangle^*$ and $\langle\chi\psi\rangle^*$. If $\omega = \langle\chi\rangle^*$ then d_ω is the same as in Theorem 1. Let $\boldsymbol{Q}_{pq}^*$ be the fixed field of $\boldsymbol{Q}_{pq}$ for τ. Let $\omega = \langle\chi\psi\rangle^*$ and M be the $\boldsymbol{Z}$-module constructed by v_{ij} in Section 2. Then M includes the $\boldsymbol{Z}$-module $2(\boldsymbol{p} \cap \mathcal{O}^*)$ where $\mathcal{O}^*$ is the ring of integers of $\boldsymbol{Q}_{pq}^*$. The discriminant D^* of $\mathcal{O}^*$ is $(pq)^{\varphi(pq)/2} q^{1-p}$ which is easily shown by examining the ramification of $\boldsymbol{Q}_{pq}/\boldsymbol{Q}_{pq}^*$. If we take v_{ij} as a basis of $2(\boldsymbol{p} \cap \mathcal{O}^*)$, then $d_\omega = 2^{\varphi(pq)/2}\Phi_{pq}(1)|D^*|^{1/2}$. Therefore, $c_4 = 2^{(p-1)(q-1)/2}$. The proof is completed.

Example. Let K_0 be abelian of degree p. Let $K \supset K_0$ and $K/\boldsymbol{Q}$ be non-Galois of degree pq. Suppose the Galois closure L of K to be abelian over an imaginary quadratic field k and $[L:k] = pq$. Then $L/\boldsymbol{Q}$ is not dihedral.

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