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Changes of sign of error terms related to Euler's function and to divisor functions II

by

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1. Introduction. Let $f\colon [1, \infty] \to \boldsymbol{R}$. In [6] we define what we mean by the number of changes of sign of f in $(1, x)$, and denote this number by $X_f(x)$. Since the functions we consider there are error terms for summatory functions of arithmetical functions it is also of interest, when investigating their changes of sign, to ask how many there are on the integers.

We say that f *has* N *changes of sign on the integers in the interval* $(1, x)$, if there are exactly N integers m, $1 < m < x$, such that $f(m-1)f(m) < 0$. This number N we denote by $N_f(x)$.

We consider the four error terms

$$R(x) := \sum_{n \leqslant x} \varphi(n) - \frac{3}{\pi^2} x^2, \tag{1}$$

$$H(x) := \sum_{n \leqslant x} \frac{\varphi(n)}{n} - \frac{6}{\pi^2} x, \tag{2}$$

$$F_1(x) := \sum_{n \leqslant x} \sigma(n) - \frac{\pi^2}{12} x^2 + \frac{x}{2} + \frac{\zeta(-1)}{2}, \tag{3}$$

$$F_{-1}(x) := \sum_{n \leqslant x} \frac{\sigma(n)}{n} - \frac{\pi^2}{6} x - \frac{\log x}{2} + \frac{\gamma + \log 2\pi}{2}, \tag{4}$$

where φ is Euler's function, $\sigma(n)$ denotes the sum of the positive divisors of n, γ is Euler's constant and ζ is Riemann's zeta function (see Remark 3 in [6] concerning the choice of $F_{\pm 1}$).

In 1967 Erdős conjectured [1] that

$$N_R(x) = Cx + o(x) \quad (x \to \infty) \tag{5a}$$

for some positive constant C; in 1985 he proposed [2] the weaker

$$N_R(x) = \Omega(x) \quad (x \to \infty). \tag{5b}$$

In 1951 Erdős and Shapiro [3] proved that

$$R(x) = \Omega_{\pm}(x \log\log\log\log x), \tag{6}$$

and hence that

$$N_R(x) \to \infty \quad (x \to \infty). \tag{7}$$

The only other result in the literature is due to Proschan (1971, [9]):

$$N_R(x) \geqslant \mathrm{IL}(x) + O(1) \quad (x \to \infty), \tag{8}$$

where $\mathrm{IL}(x)$ is the smallest integer k such that the $4k$-fold iterated logarithm of x in a sufficiently large basis is either smaller than 2 or undefined.

In [6] we prove that

$$X_f(x) = Cx + o(x) \quad (x \to \infty), \quad \text{if } f = R \text{ or } H, \tag{9}$$

where

$$C \geqslant \frac{8}{3}\left(1 - \frac{\pi^2}{24}\right) = 1.57004\ldots, \tag{10}$$

and that

$$X_f(x) \geqslant C'x + o(x) \quad (x \to \infty), \quad \text{if } f = F_{-1} \text{ or } F_1, \tag{11}$$

where

$$C' = \frac{8}{3}\left(1 - \frac{15}{4\pi^2}\right) = 1.65345\ldots \tag{12}$$

In Sections 2 and 3 of this article we show that

$$N_f(x) \geqslant \left(\frac{2}{\log 2} - \varepsilon\right)\log\log x + O_\varepsilon(1), \quad \text{for any } \varepsilon > 0, \tag{13}$$

if f is any one of the four error terms R, H, F_{-1}, F_1.

To look at sign changes of a function is to consider its oscillations about the value zero. In the hope of improving (13) we consider more generally the oscillations of f about any real value r and seek estimates of $N_{f*}(x)$, where

$$f^*(n) := \begin{cases} f(n) - r & \text{if } f = H, F_{-1}, \\ f(n) - rn & \text{if } f = R, F_{+1}. \end{cases} \tag{14}$$

The results we obtain in Section 4 can be summarized as follows. Let

$$C = C_f := \begin{cases} 3/\pi^2 & \text{if } f = H, R, \\ \pi^2/12 & \text{if } f = F_{\pm 1}, \end{cases} \tag{15}$$

and

$$g(x) = g_f(x) := \begin{cases} \exp\left(A(\log x)^{3/5}(\log\log x)^{-1/5}\right) & \text{if } f = H, R, \\ x^{273/382} & \text{if } f = F_{\pm 1}, \end{cases} \tag{16}$$

where A is a certain positive constant. Then, with an implied constant independent of r, the estimate

$$N_{f*}(x) = \Omega(g(x)) \tag{17}$$

is true for all $r > C$ or (non-exclusively) for all $r < C$. (In the latter case, (17) would add significantly to the information provided by (13) for the case $r = 0$.)

Moreover, when $f = H$ or R, and under the assumption of the Riemann hypothesis, (17) remains true for

$$g(x) := x^{1/2-\varepsilon}, \quad \text{for any } \varepsilon > 0. \tag{18}$$

2. F_{-1} and F_1. Let

$$F_{-1,K}(n) := F_{-1}(n) - K \tag{19}$$

and

$$F_{1,K,L}(n) := F_1(n) - Kn + L, \tag{20}$$

where the error terms F_{-1} and F_1 are as in (3) and (4).

THEOREM 1. *Let $K < \pi^2/12$ and L be any real number. Then*

$$N_{F_{-1,K}}(x) \geqslant C \log\log x + O_{C,K}(1) \tag{21}$$

and

$$N_{F_{1,K,L}}(x) \geqslant C \log\log x + O_{C,K,L}(1), \tag{22}$$

for any positive constant C smaller than $2/\log 2$.

We will use the relation

$$\sum_{n \leqslant x} F_{-1}(n) \sim \frac{\pi^2}{12}x, \tag{23}$$

due to Pillai and Chowla [8]. The general idea is to use the proof [5] of

$$F_{-1}(x) = \Omega_-(\log\log x) \tag{24}$$

to exhibit many integers n for which $F_{-1}(n) < 0$, and then with (23) to find after each such n a not too distant integer m such that $F_{-1}(m) > 0$.

We will prove that (21) and (22) are true for some constant $C > 0$. With Remark 4 in [5] and straightforward refinements in the proof, one can then easily see that they are also true for any $C < 2/\log 2$ (see [4], pp. 62–63).

We observe that (22) applies to $E_1(u) := F_1(u) - u/2 - \zeta(-1)/2$, since $1/2 < \pi^2/12$, but that (21) does not apply to $E_{-1}(u) := F_{-1}(u) - (\gamma + \log 2\pi)/2$, since $(\gamma + \log 2\pi)/2 > \pi^2/12$ (see [6], Remark 3).

It follows from (24) that there is a positive constant C such that

$$F_{-1}(y) - K \leqslant -C \log\log y \tag{25}$$

for arbitrarily large integers y. We require a lemma concerning these integers.

LEMMA 1. *If $K < \pi^2/12$ and if the integer y satisfies (25) and is sufficiently large, then there are integers y_+ and y_- such that*

1) $y < y_+ < 2y$ *and* $F_{-1}(y_+)-K > \frac{1}{2}\left(\frac{\pi^2}{12}-K\right)$;
2) $2y < y_- < y^4$ *and* y_- *satisfies* (25).

We first deduce Theorem 1 from Lemma 1. Let y_0 satisfy (25), and be sufficiently large to ensure that the lemma applies to all $y \geqslant y_0$ satisfying (25). We then have at least two changes of sign of F_{-1} on the integers in (y_0, y_0^4), two more in $(y_0^4, y_0^{4^2})$, and in general at least $2l$ in $(y_0, y_0^{4^l})$. Now let x be such that $\log\log x \geqslant 2\log\log y_0$ and define the integer l by

$$y_0^{4^l} \leqslant x < y_0^{4^{l+1}}. \tag{26}$$

Then

$$(l+1)\log 4 > \log\log x - \log\log y_0, \tag{27}$$

so that

$$l > \frac{\log\log x}{2\log 4} - 1. \tag{28}$$

We have therefore obtained (21) with $C = 1/\log 4$. Then (22) for the same C follows from the known

$$F_1(y) - Ky + L = y(F_{-1}(y) - K) + o(y) + L \tag{29}$$

(the best estimate to date of the error-term is due to Recknagel [10]), and from the fact that the y_+ of the lemma are such that $F_{-1}(y_+) - K > K' > 0$, where $K' := \frac{1}{2}\left(\frac{\pi^2}{12} - K\right)$ depends only on K. ∎

Proof of Lemma 1. The existence of y_+ is an easy consequence of (23). We prove the existence of y_-. Let p_k denote the kth prime, and let

$$A_k := \prod_{p \leqslant p_k} p^{a_k}, \quad \text{where } a_k := \left[\frac{\log p_k}{\log 2}\right]. \tag{30}$$

If y is fixed, define n by

$$A_n > 4y \geqslant A_{n-1}. \tag{31}$$

For simplicity we adopt the convention that in this proof, inequalities involving y are meant to hold for all sufficiently large y. A similar remark applies to inequalities involving A_n.

The proof of (24) provides an integer m^*, $1 \leqslant m^* \leqslant A_n^2$, such that $y_- := A_n m^* - p_n$ satisfies (25) (see [5], (52)); and $y_- \geqslant A_n - p_n > A_n/2 > 2y$. We proceed to verify that $y_- < y^4$. By (30) and the prime number theorem we have

$$\log A_n = \left[\frac{\log p_n}{\log 2}\right] \sum_{p \leqslant p_n} \log p \sim \frac{n \log^2 n}{\log 2}, \tag{32}$$

whence

$$A_n = A_{n-1}^{(1+o(1))} \tag{33}$$

and by (31)

$$A_n < (4y)^{1+o(1)} < y^{4/3}. \tag{34}$$

Consequently,

$$y_- = A_n m^* - p_n < A_n^3 < y^4. \ ∎ \tag{35}$$

3. H and R. Let $N_f(x)$ be defined as in Section 1, R and H as in (1) and (2), and set

$$H_K(n) := H(n) - K \tag{36}$$

and

$$R_K(n) := R(n) - Kn. \tag{37}$$

We proceed to prove

THEOREM 2. *If $K \neq 3/\pi^2$, we have*

$$N_{H_K}(x) \geqslant C\log\log x + O_{C,K}(1), \tag{38}$$

and

$$N_{R_K}(x) \geqslant C\log\log x + O_{C,K}(1), \tag{39}$$

where the positive constant C may be chosen arbitrarily close to and smaller than $2/\log 2$.

The proof combines the method of Erdős and Shapiro [3] and a refinement of the method used by Pillai and Chowla in [7] to show that $H(x) = \Omega(\log\log\log x)$. The general idea is basically the same as in Section 2 but the proof is somewhat more involved. The main reason for this is that unlike the sequence used in [5] to prove $F_{-1}(y) = \Omega_-(\log\log y)$, the sequence used in [3] to prove $H(y) = \Omega_-(\log\log\log\log y)$ is a non-explicit subsequence of a sequence which also contains the y's used to prove $H(y) = \Omega_+(\log\log\log\log y)$, and might be very sparse (in view of [9]).

We shall prove (38) and (39) (and also Lemma 5 below) only in the case $K < 3/\pi^2$, the case $K > 3/\pi^2$ being very similar. The reason why our method does not apply to $K = 3/\pi^2$ will be clear in view of (53) below.

The result in [7] is in fact much more precise than

$$H(x) = \Omega(\log\log\log x),$$

namely

(40) $$H(x+\log\log\log x)-H(x) = \Omega_-(\log\log\log x).$$

We show in Lemma 2 that there are many x after which H decreases rapidly.

LEMMA 2. *Let f: $\mathbf{R}^+ \to \mathbf{R}^+$ be increasing and unbounded. Then there is a function m: $\mathbf{R}^+ \to \mathbf{Z}^+$ with*

(41) $$m(x) \sim \log\log\log f(x) \quad (x \to \infty),$$

such that for every sufficiently large x, there is a positive integer $\bar{x}$ satisfying

(42) $$|x-\bar{x}| < f(x)$$

and

(43) $$H(\bar{x}+m(x))-H(\bar{x}) \leqslant -Cm(x),$$

for some positive absolute constant C.

Proof. Let

(44) $$P(u, v) := \prod_{u<p\leqslant v} p \quad (v > u \geqslant 1).$$

For each sufficiently large x, there is an integer $m \geqslant 1$ such that

(45) $$P(1, e^{e^m}) \leqslant f(x) < P(1, e^{e^{m+1}});$$

by the prime number theorem this implies that

(46) $$m \sim \log\log\log f(x) \quad (x \to \infty).$$

Consider now the system of congruences

(47) $$\begin{cases} z+1 \equiv 0(P(1, e^{e^1})), \\ z+2 \equiv 0(P(e^{e^1}, e^{e^2})), \\ \dots\dots\dots\dots\dots \\ z+m \equiv 0(P(e^{e^{m-1}}, e^{e^m})). \end{cases}$$

We use the method of [7] to produce an integer $\bar{x}$ satisfying (42) and (43). Let x_0 be any positive solution of system (47). Using a classical theorem of Mertens, we obtain the existence of a positive integer k such that

(48) $$\prod_{x<p\leqslant x^e} \left(1-\frac{1}{p}\right) < \frac{1}{2}, \quad \text{if } x^e \geqslant e^{e^k}.$$

This in turn implies, since x_0 is a solution of (47), that

(49) $$\frac{\varphi(x_0+t)}{x_0+t} < \frac{1}{2} \quad (k \leqslant t \leqslant m);$$

therefore, as $m \to \infty$,

(50) $$H(x_0+m)-H(x_0) = \sum_{t=k}^{m} \frac{\varphi(x_0+t)}{x_0+t} - \frac{6}{\pi^2}m + O(1) < \left(\frac{1}{2}-\frac{6}{\pi^2}\right)m + O(1).$$

So if m is large enough, and x_0 is any positive solution of (47), we have

(51) $$H(x_0+m)-H(x_0) \leqslant -Cm,$$

for some positive absolute constant C. And since $m = m(x)$ is defined by (45), we have $m \to \infty$ as $x \to \infty$, so that (51) holds for all sufficiently large x and for any solution $x_0 > 0$ of (47). We also note that if n runs through $\mathbf{Z}$, then $x_n := x_0 + nP(1, e^{e^m})$ runs through the solutions of (47). Hence if $x > 0$ and x_N is the solution of (47) such that $x_N < x \leqslant x_{N+1}$, then $x_{N+1} > 0$ and

(52) $$|x-x_{N+1}| < P(1, e^{e^m}) \leqslant f(x).$$

This concludes the proof, if we set $x_{N+1} = \bar{x}$. ∎

LEMMA 3. *Let $K < 3/\pi^2$. Then for some positive constant $C = C_K$ and for every sufficiently large real number x there is an integer $y \in (x, 2x)$ satisfying $H_K(y) > C$.*

Proof. This follows easily from [7], (3.10), which implies that

(53) $$\sum_{n\leqslant x} H_K(n) \sim \left(\frac{3}{\pi^2}-K\right)x \quad (x \to \infty). \ ∎$$

LEMMA 4. *Let K be any real number. If $H(B)$ is sufficiently large and if $A := \prod_{p\leqslant B} p$, then there is an integer m such that*

$$A-B \leqslant m \leqslant A^3-B \quad \text{and} \quad H_K(m) < -1.$$

Proof. This follows easily from the main result of [3] which implies that if $A := \prod_{p\leqslant B} p$ and $x = A^2$, then

(54) $$\frac{1}{x}\sum_{n\leqslant x} H(An-B) = -H(B)+O(1)$$

uniformly in B as $B \to \infty$ (see [5], Remark 1). ∎

LEMMA 5. *Let $K \neq 3/\pi^2$. If x_0 is sufficiently large, and if*

(55) $$x_k = 18x_{k-1} \quad (k = 1, \dots, n),$$

where

(56) $$n = o(x_0) \quad (x_0 \to \infty),$$

then

(57) $$N_{H_K}(e^{x_n}) \geqslant 2n.$$

Proof. As announced, we give the proof only in the case $K < 3/\pi^2$. Some of the estimates below are valid only for x_0 large enough; for brevity, we will not point this out at each occurrence.

Take $f(x) = x/3$ in Lemma 2; then $m \sim \log\log\log x$ as $x \to \infty$. By n applications of Lemma 2 we obtain a sequence of integers $x'_0 < x'_1 < \ldots < x'_{n-1}$ satisfying

(58) $$x_0/2 \leqslant x'_0, \quad x'_{n-1} < 2x_{n-1} = x_n/9$$

and

(59) $$x'_{k+1} \geqslant x_{k+1}/2 = 9x_k > 4x'_k \quad (k = 0, \ldots, n-2),$$

and such that

(60) $$|H(x'_k)| > C' \log\log\log x_k \quad (k = 0, \ldots, n-1)$$

for some positive constant C' (x_k is the x of (42); x'_k is either $\bar{x}_k$ or $\bar{x}_k + m(x_k)$ of (43); and we may take for C' any constant smaller than $C/2$, where C satisfies (43)). Because of (60) there is a positive constant M such that

(61) $$|H_K(x'_k)| > M \quad (k = 0, \ldots, n-1).$$

Now let ε $(0 \leqslant \varepsilon \leqslant 1)$ be such that

(62) εn of the x'_k satisfy $H_K(x'_k) < -M$,

(63) the $(1-\varepsilon)n$ others satisfy $H_K(x'_k) > M$.

The constant M can be chosen sufficiently large to ensure that Lemma 4 applies to $B = x'_k$ if x'_k satisfies (63).

By Lemma 3, each x'_k in (62) provides a change of sign of H_K from $-$ to $+$ in $(x'_k, 2x'_k)$. Since $2x'_k < x'_{k+1}$ by (59), and $H_K(1) > 0$ for $K < 3/\pi^2$, the x'_k of (62) ensure the existence of $2\varepsilon n$ changes of sign in $(1, 2x'_{n-1}) \subset (1, x_n)$.

Consider now the x'_k in (63). Let $A_k := \prod_{p \leqslant x'_k} p$; then

$$A_k = e^{(1+o(1))x'_k}, \quad \text{as } x_0 \to \infty.$$

By Lemma 4 there is an m_k such that

(64) $$A_k - x'_k \leqslant m_k \leqslant A_k^3 - x'_k$$

and

(65) $$H_K(m_k) < -1;$$

because of (64), we have

(66) $$e^{(1+o(1))x'_k} \leqslant m_k \leqslant e^{3(1+o(1))x'_k}.$$

By Lemma 3 again, there is a change of sign of H_K from $-$ to $+$ in $(m_k, 2m_k)$. By (59) and (66), $2m_k < m_{k+1}$ since

$$2m_k \leqslant 2e^{3(1+o(1))x'_k} < A_{k+1} - x'_{k+1} \leqslant m_{k+1}.$$

Therefore, the x'_k in (63) provide us with $2(1-\varepsilon)n$ changes of sign in $(m_0, 2A^3_{n-1})$, that are distinct from those provided by the x'_k of (62), since $x_n < m_0$.

Finally,

$$2A^3_{n-1} = e^{3(1+o(1))x'_{n-1}} < e^{8x_{n-1}} < e^{x_n}$$

concludes the proof of Lemma 5. ∎

Proof of Theorem 2. As in the proof of (21), we establish (38) for some $C > 0$, and leave it to the reader to verify that it is true for any $C < 2/\log 2$ by refining Lemmata 4 and 5 (see [4], p. 72).

Choose a constant M, $0 < M < 1/\log 18$. Let $X > e$; then set

(67) $$n = [M \log\log X],$$

(68) $$x_0 = \frac{1}{18^n} \log X,$$

and

(69) $$x_k = 18x_{k-1} \quad (k = 1, \ldots, n).$$

Lemma 5 can be applied if X is large enough; we obtain

(70) $$N_{H_K}(X) \geqslant 2[M \log\log X],$$

and (38) is proved for any $C < 2/\log 18$ and for $K < 3/\pi^2$.

To prove (39) we note that the values of x used in the proof of Lemma 5 to exhibit sign changes of H_K in $(1, e^{x_n})$ are such that either

(71) $$H_K(x) > C > 0$$

by Lemma 3, or

(72) $$H_K(x) < -1$$

by Lemma 4; this together with the familiar [7]

(73) $$R(x) = xH(x) + o(x)$$

proves (39) for each K and C for which (38) holds. ∎

4. Improvements for many unidentifiable K. Let H_K and $F_{-1,K}$ be defined as in (36) and (19). We proceed to prove the following results concerning their changes of sign on the integers.

THEOREM 3. *If*

(74) $$K < 3/\pi^2 < L,$$

then

(75) $$N_{H_K}(x) + N_{H_L}(x) > Cg(x) + O_{K,L}(1),$$

where

$$g(x) := \exp\big(C'(\log x)^{3/5}(\log\log x)^{-1/5}\big), \tag{76}$$

and where C and C' are positive absolute constants.

Corollary. *For g as in (76), we have*

$$N_{H_K}(x) = \Omega(g(x)) \tag{77}$$

for every $K < 3/\pi^2$ or every $K > 3/\pi^2$ (or both). For the constant implied by Ω in (77) we may take the C of (75).

Theorem 3RH. *If K and L are as in (74), and if the Riemann hypothesis holds, then (75) and (77) remain true if we replace g by*

$$g^*(x) := x^{1/2-\varepsilon}, \tag{78}$$

for any $\varepsilon > 0$.

Theorem 4. *If*

$$K < \pi^2/12 < L, \tag{79}$$

then

$$N_{F_{-1,K}}(x) + N_{F_{-1,L}}(x) > Cx^{273/382} + O_{K,L}(1), \tag{80}$$

where C is a positive absolute constant.

Corollary. *We have*

$$N_{F_{-1,K}}(x) = \Omega(x^{273/382}) \tag{81}$$

for every $K < \pi^2/12$ or every $K > \pi^2/12$ (or both). And the constant implied by Ω may be taken equal to the C of (80).

These results can be extended to the functions R_K of (37) and $F_{1,K,L}$ of (20) by using (73) and (29).

Proof of Theorem 3. From [10], (3.2), it follows that

$$\sum_{n \leqslant x} H_K(n) = (3/\pi^2 - K)x + O\big(x/g(x)\big), \tag{82}$$

where g is as in (76) and K is an arbitrary constant. This implies, if K and L are as in (74), that there are positive constants A_+, A_-, B and M such that whenever

$$m \geqslant M, \tag{83}$$

there are integers n_+ and n_- satisfying

$$n_+ \in \big(m,\ m + Bm/g(m)\big), \tag{84}$$

$$n_- \in \big(m,\ m + Bm/g(m)\big), \tag{85}$$

$$H_K(n_+) > A_+, \tag{86}$$

and

$$H_L(n_-) < -A_-. \tag{87}$$

On the other hand by Lemma 2, if x is large enough, we can find an integer m such that

$$m \in \big(x,\ x + Bx/g(x)\big) \tag{88}$$

and

$$|H(m)| > C\log\log\log m > \max(|K|+1,\ L+1), \tag{89}$$

for some positive constant C.

Now let x be large enough to ensure the existence of m_1 satisfying (83), (88) and (89). If $H(m_1) > 0$, a change of sign of H_L in the interval $\big(m_1,\ m_1 + Bm_1/g(m_1)\big)$ is guaranteed by the existence of n_- satisfying (85) and (87). If $H(m_1) < 0$, H_K changes sign there because of n_+ as in (84) and (86). Then if we let N be the larger of n_- and n_+, Lemma 2 provides an integer m_2 in $\big(N,\ N + BN/g(N)\big)$ satisfying (89).

We then proceed with m_2 as with m_1, obtaining another change in sign of H_K or H_L, closely followed by an integer m_3 satisfying (89).

We continue until we reach m_t, where t is defined by $m_t \leqslant 2x < m_{t+1}$. It is easy to see that if x is large enough,

$$t > \left[\frac{g(x)}{2B}\right]; \tag{90}$$

the fact that B does not depend on K or L completes the proof of the theorem. ∎

We now prove the corollary. It is clear by (75) that (77) is true for H_K or H_L (or both) if K and L satisfy (74). Now on the one hand, the m_i used in the proof satisfy (89), whence

$$|H(m_i)| \to \infty \quad \text{as } x \to \infty. \tag{91}$$

On the other hand, the constants A_+ and A_- in (86) and (87) may be chosen arbitrarily close to, but smaller than, $3/\pi^2 - K$ and $L - 3/\pi^2$, respectively.

The latter shows that if (77) is true for some $K < 3/\pi^2$, then it is also true for all $K' \in (K, 3/\pi^2)$; that it is also true for all $K' < K$ is a consequence of (91). Similarly if (77) is true for some $K > 3/\pi^2$, then it is true for all $K > 3/\pi^2$. ∎

Theorem 3RH now follows easily from a result obtained by Suryanarayana [11] under the assumption of the Riemann hypothesis,

$$\sum_{n \leqslant x} H(n) = \frac{3}{\pi^2}x + O(x^{1/2+\varepsilon}) \quad \text{for any } \varepsilon > 0. \tag{92}$$

The proof of Theorem 4 is also very similar to that of Theorem 3. We use Recknagel's [10]

(93) $$\sum_{n\leqslant x} F_{-1}(n) = \frac{\pi^2}{12}x + O(x^{109/382}),$$

and the following remark instead of (88) and (89). If

(94) $$C_a := \prod_{p\leqslant a} p \qquad (a>0)$$

and $x_n := nC_a$, then

(95) $$F_{-1}(x_n) - F_{-1}(x_n - 1) \geqslant C\log\log C_a$$

for some absolute positive constant C and for any positive integer n, provided that a is large enough. Hence, if B is a positive constant, we can for any large enough x find an integer m such that

(96) $$m\in(x,\ x+Bx^{273/382})$$

and

(97) $$|F_{-1}(m)|\to\infty \qquad \text{as } x\to\infty.$$

Added in proof. I have recently obtained improvements on this paper's estimates in *On the distribution of values of an error term related to the Euler function*, which is to appear in the Proceedings of the Number theory conference held at Laval University (Québec) on 5–18 July 1987.

References

[1] P. Erdős, Letter to J. Steinig (December 3, 1967).
[2] — Letter to J. Steinig (April 15, 1985).
[3] P. Erdős and H. N. Shapiro, *On the changes of sign of a certain error function*, Canad. J. Math. 3 (1951), 375–385.
[4] Y.-F. S. Pétermann, *Oscillations and changes of sign of error terms related to Euler's function and to the sum-of-divisors function*, Thèse de doctorat, Université de Genève, 1985.
[5] — *An Ω-theorem for an error term related to the sum-of-divisors function*, Mh. Math. 103(1987), 145–157.
[6] — *Changes of sign of error terms related to Euler's function and to divisor functions*, Comm. Math. Helv. 61(1986), 84–101.
[7] S. S. Pillai and S. D. Chowla, *On the error terms in some asymptotic formulae in the theory of numbers I*, J. London Math. Soc. 5 (1930), 95–101.
[8] — *On the error terms in some asymptotic formulae in the theory of numbers II*, J. Indian Math. Soc. 18 (1930), 181–184.
[9] J. H. Proschan, *On the changes of sign of a certain class of error functions*, Acta Arith. 17 (1971), 407–430.
[10] W. Recknagel, *Über eine Vermutung von S. Chowla und H. Walum*, Arch. Math. 44 (1985), 348–354.

[11] D. Suryanarayana, *On the average order of the function* $E(x) = \sum_{n\leqslant x}\varphi(n) - 3x^2/\pi^2$ *(II)*, J. Indian Math. Soc. 42 (1978), 179–195.

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